On the complexity of distributions

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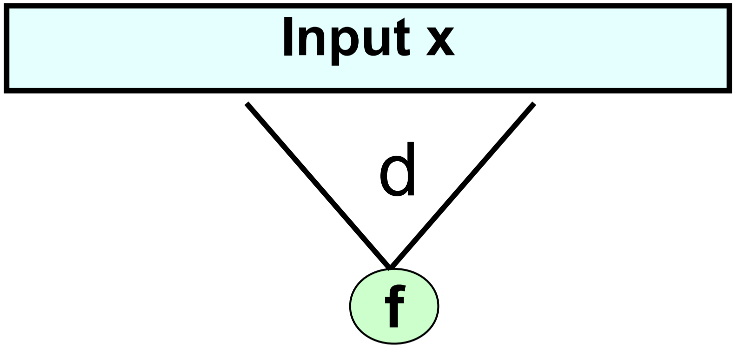
Local functions

- $f : \{0,1\}^n \rightarrow \{0,1\}$ d-local : output depends on d input bits



- Fact: $\text{Parity}(x) = 1 \Leftrightarrow \sum x_i = 1 \bmod 2$ is not n-1 local

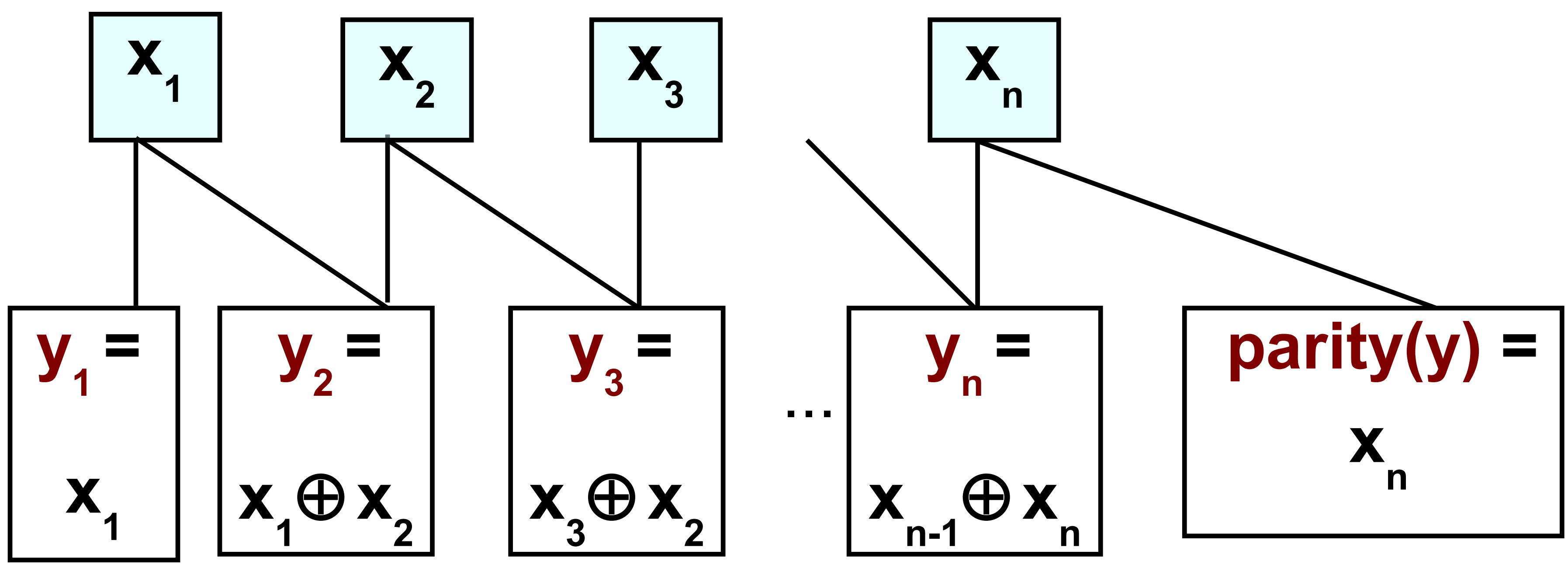
- Proof: Flip any input bit $\Rightarrow$ output flips ♦

Local generation of (Y, parity(Y))

- Theorem [Babai '87]

  There is $f : \{0,1\}^n \to \{0,1\}^{n+1}$, each bit is 2-local

  Distribution $f(X) \equiv$ (Y, parity(Y)) ($X, Y \in \{0,1\}^n$ uniform)



Message

- Complexity theory of distributions (as opposed to functions)

  How hard is it to generate distribution D given random bits ?

  E.g., D = (Y, parity(Y)), D = W_k := uniform n-bit with k 1's

Rest of this talk

- Connection with succinct data structures

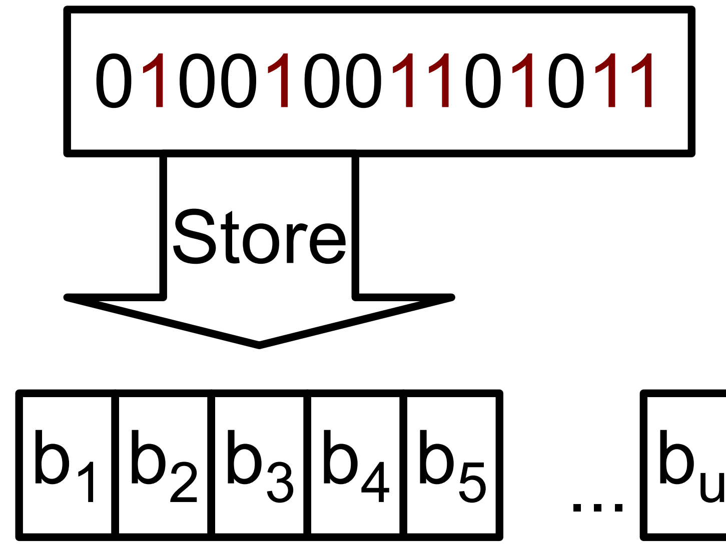
- Lower bound for generating $W_{n/2}$ = uniform n-bit with n/2 1's

- Other results and conclusion

Succinct data structures for sets

- Store $S \subseteq \{1, 2, \ldots, n\}$ of size $|S| = k$

  In u bits $b_1, \ldots, b_u \in \{0,1\}$



- Want:

  Small space u $\left(\text{optimal} = \lceil \lg_2 (n \text{ choose } k) \rceil\right)$

  Answer "$i \in S$?" by probing few bits $(\text{optimal} = 1)$

- In combinatorics: Nešetřil Pultr, …, Körner Monti

Previous results

- Store $S \subseteq \{1, 2, \dots, n\}$, $|S| = k$, in bits, answer "$i \in S$?"

- [Minsky Papert '69, Buhrman Miltersen Radhakrishnan Venkatesh; Pagh; ...; Pătraşcu; V. '09]

- Surprising upper bounds
  space = optimal + $o(n)$, probe $O(\log n)$

- No lower bounds for $k = n / 2^a$

General connection

- Claim: If store $S \subseteq \{1, 2, \ldots, n\}$, $|S| = k$ in u = optimal + r bits answer "$i \in S$?" by (non-adaptively) probing d bits.

Then $\exists f : \{0,1\}^u \rightarrow \{0,1\}^n$, d-local

Distance($f(X)$, W_k = uniform set of size k) $< 1 - 2^{-r}$

$$\left(\text{distance}(A, B) := \max_T \left| \Pr[A \in T] - \Pr[B \in T] \right| \right)$$

- Proof: f_i := "$i \in S$?"

$f(X) = W_k$ with probability (n choose k) / $2^u = 2^{-r}$ ♦

Our result

- **Theorem**: $f : \{0,1\}^{\text{optimal} + n^{O(1)}} \to \{0,1\}^n$, $(d < \varepsilon \log(n))$-local.

  Distance$(f(X), W_k$ = uniform set of size $k = \Theta(n)) > 1 - n^{-\Omega(1)}$

- Tight up to $\Omega()$ if $k = n/2$: $f(x) = x$, $(n \text{ choose } n/2) = O(2^n/\sqrt{n})$

- **Corollary:** To store $S \subseteq \{1, 2, \ldots, n\}$, $|S| = k = n / 2^a$

  answer "$i \in S$?" probing $d < \varepsilon \log(n)$ bits:

  Need space > optimal + $\Omega(\log n)$

Rest of this talk

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Our result

- Theorem: Let $f : \{0,1\}^n \to \{0,1\}^n$: $(d=O(1))$-local.

  There is $T \subseteq \{0,1\}^n$: $\left| \Pr[f(x)\in T] - \Pr[W_{n/2} \in T] \right| > 1 - n^{-\Omega(1)}$

- Warm-up scenarios:

- $f(x) = 000111$ Low-entropy $T := \{ 000111 \}$

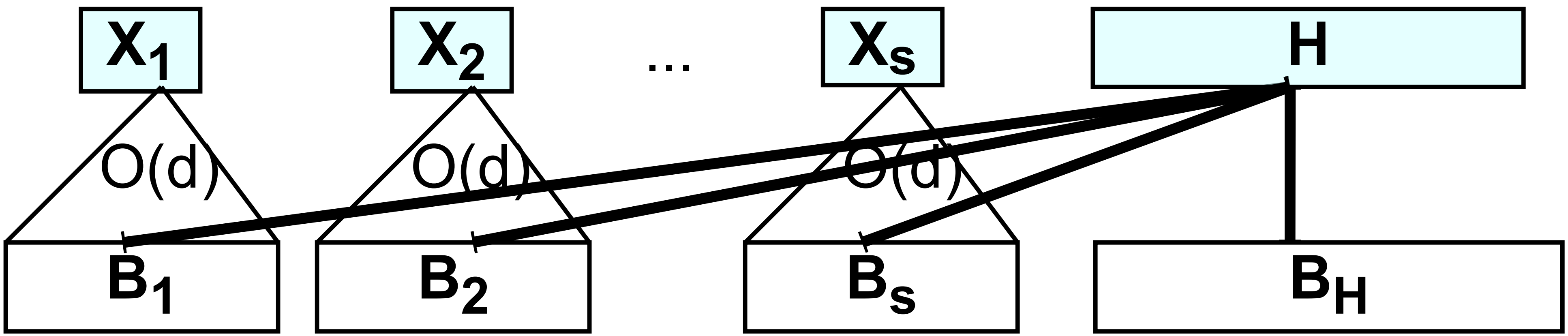
  $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| = \left| 1 - |T| / (n \text{ choose } n/2) \right|$

- $f(x) = x$ "Anti-concentration" $T := \{ z : \sum_i z_i = n/2 \}$

  $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| = \left| 1/\sqrt{n} - 1 \right|$

Proof

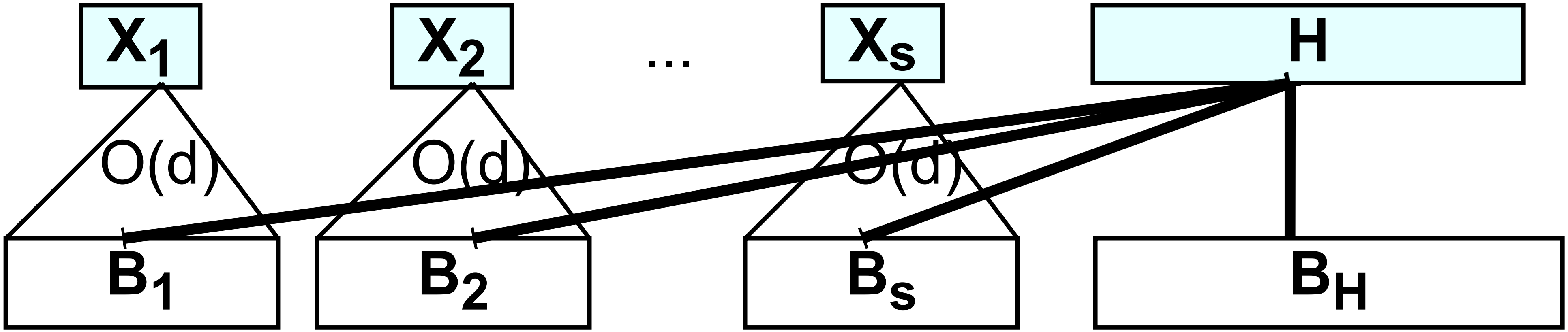
- Partition input bits $X = (X_1 , X_2 , \dots , X_s , H)$



- Fix H. Output block B_i depends only on bit X_i

- Many B_i constant ($B_i(0,H) = B_i(1,H)$) $\Rightarrow$ low-entropy

- Many B_i depend on X_i ($B_i(0,H) \neq B_i(1,H)$)
  Intuitively, anti-concentration: output bits can't sum to n/2



- If many $B_i(0,H)$, $B_i(1,H)$ have different sum of bits, use

Anti-concentration Lemma [Littlewood Offord]

For $a_1, a_2, \ldots, a_s \neq 0$, any c, $\Pr_{X \in \{0,1\}^s} \left[\sum_i a_i X_i = c\right] < 1/\sqrt{n}$

- Problem: $B_i(0,H) = 100$, $B_i(1,H) = 010$
  high entropy but no anti-concentration

- Fix: want many blocks 000, so high entropy $\Rightarrow$ different sum

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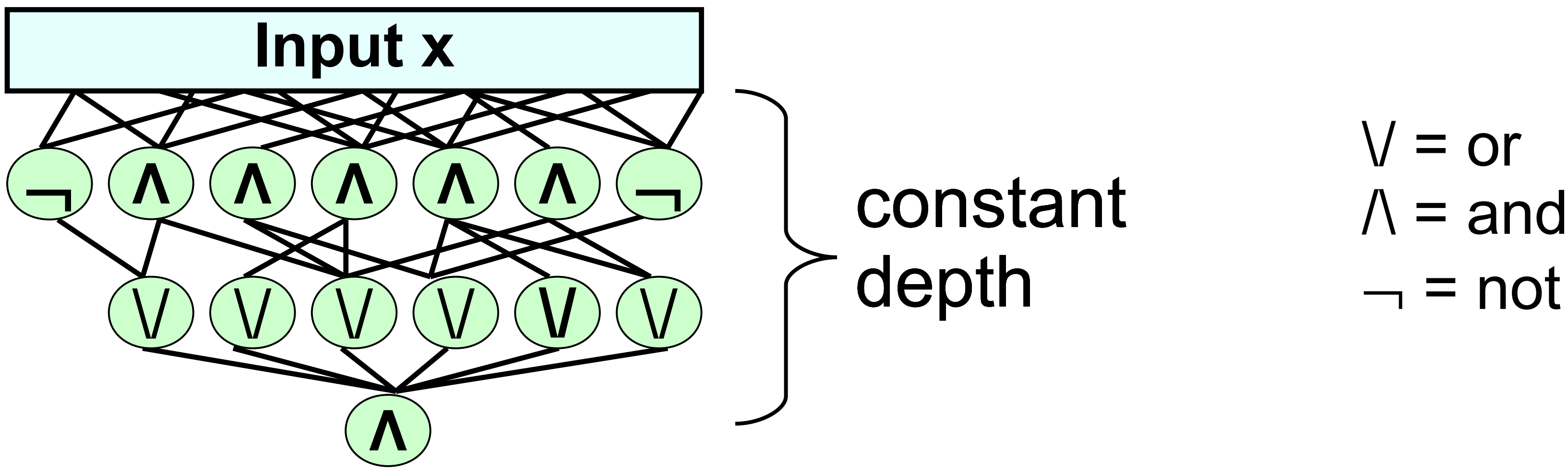
Other directions and results

- More general model: small bounded-depth circuits (AC^0)



- Challenge: ∃ explicit boolean f : cannot generate (Y, f(Y)) ?

- Theorem[Matias Vishkin, Hagerup, Czumaj Kanarek Lorys Kutyłowski, V.]
  Can generate (Y, majority(Y)) (exp. small error)

- Theorem [Lovett V.] Cannot generate error-correcting code

Conclusion

- Complexity of distributions = uncharted territory
- Lower bound for generating W_k locally
- $\Rightarrow$ lower bound for succinct data structures for storing sets of size $n / 2^a$

- $\sum\prod\surd\cap\notin\cup\supset\supseteq\not\subset\subset\subseteq\in\Downarrow\Rightarrow\Uparrow\Leftarrow\Leftrightarrow\vee\wedge\geq\leq\forall\exists\Omega\alpha\beta\varepsilon\gamma\delta\rightarrow$
- $\neq\approx$

More connections

- More uses of generating W_k := uniform n-bit string with k 1's
- McEliece cryptosystem
- Switching networks, …

Previous results

- Store S $\subseteq$ {1, 2, … , n}, |S| = k, in bits, answer "i $\in$ S?"
- [Minsky Papert '69] Average-case study
- [Buhrman Miltersen Radhakrishnan Venkatesh; Pagh '00]

  Space O(optimal), probe O(1) when $k = \Theta(n)$

  Lower bounds for $k < n^{1-\varepsilon}$
- [..., Pagh, Pătraşcu] space = optimal + o(n), probe O(log n)
- [V. '09] lower bounds for $k = \Omega(n)$, except $k = n / 2^a$