Polynomials

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Polynomials

- Polynomials:

  degree d, n variables over $F_2 = \{0,1\}$

  E.g., $p = x_1 + x_5 + x_7$ degree d = 1

  $p = x_1 \cdot x_2 + x_3$ degree d = 2

- Computational model: $p : \{0,1\}^n \to \{0,1\}$

  Sum (+) = XOR, Product ($\cdot$) = AND

  $x^2 = x$ over $F_2 \Rightarrow$ Multilinear

- Complexity = degree

Motivation

- Coding theory

  Hadamard, Reed-Muller codes based on polynomials

- Circuit lower bounds [Razborov '87; Smolensky '87]

  Lower bound on polynomials $\Rightarrow$ circuit lower bound

- Pseudorandomness [Naor & Naor '90]

  Useful for algorithms, PCP, expanders, learning…

Outline

- Overview
- Correlation bounds
- Pseudorandom generators

Lower bound

- Question: Which functions cannot be computed by low-degree polynomials?

- Answer:

  $x_1 \cdot x_2 \cdots x_d$ requires degree d

  Majority$(x_1,\ldots,x_n)$:= 1 $\Leftrightarrow \sum x_i > n/2$
  requires degree n/2

Correlation bound

- Question: Which functions do not correlate with low-degree polynomials?

- $\mathrm{Cor}(f,\text{degree } d) := \max_{\text{degree-}d\ p} \mathrm{Bias}(f+p) \quad \in [0,1]$

  $\mathrm{Bias}(f+p) := |\ \Pr_{U \in \{0,1\}^n} [f(U)=p(U)] - \Pr_U [f(U) \neq p(U)]\ |$

  E.g. Cor(deg. d, deg. d) = 1; Cor(random f, deg. d) ~ 0

- Want: correlation small, degree large. Motivation: pseudorandomness [Y,NW], lower bounds [R,S,H+]

- Barrier: $\exists$ explicit n-bit f : $\mathrm{Cor}(f, \text{degree } \log_2 n) \leq 1/n$?

A sample of correlation bounds

- $\mathrm{Cor}(f,\text{degree } d) := \max_{\text{degree-}d\ p} \mathrm{Bias}(f+p)$

- [Babai, Nisan and Szegedy '92, Bourgain '05]

  Explicit f: $\mathrm{Cor}(f,\text{degree } 0.1 \log n) \leq \exp(-n)$

- [Razborov '87]: $\mathrm{Cor}(\text{Majority},\text{degree } \log n) \leq 1/\sqrt{n}$

- Hardness amplification question:

  Can we amplify Razborov's bound and break the ``$\mathrm{Cor}(f, \text{degree } \log n) \leq 1/n$'' barrier?

Yao's XOR lemma

- Generic way to boost correlation bound

  M = computational model (e.g. M = degree log n)

- $f^{\oplus k}(x_1,\ldots, x_k) := f(x_1) \oplus \cdots \oplus f(x_k)$

  Hope: $\mathrm{Cor}(f^{\oplus k},M) \leq \mathrm{Cor}(f,M)^{\Omega(k)}$

- Theorem [Yao, Levin, Goldreich Nisan Wigderson, Impagliazzo,…]

  XOR lemma for M = circuits

- Question: [Razborov] bound for Maj + XOR lemma $\Rightarrow$

  $\mathrm{Cor}(\ \mathrm{Maj}(x_1) \oplus \cdots \oplus \mathrm{Maj}(x_k)$, degree $\log n\ ) \ll 1/n$?

XOR lemma proofs require majority

- XOR lemma proofs [L,GNW,I,…] are code-theoretic

- Theorem [Shaltiel V. '07]: Code-theoretic proofs of XOR lemma require model to compute majority

- Since polynomials cannot compute majority, no code-theoretic proof of XOR lemma for polynomials

- [Shaltiel V.] + [Razborov Rudich] + [Naor Reingold]: "prove XOR lemma $\Rightarrow$ can't prove correlation bounds"

Where we are

- Theorem[Shaltiel V. '07]: Code-theoretic proofs of XOR lemma do not work for polynomials

- Open: XOR lemma for degree log n

- Note: XOR lemma trivially true for degree 0, 1

- Next[V. Wigderson]: XOR lemma for any constant degree
  Proof not code-theoretic

XOR lemma for constant degree

- Theorem[V. Wigderson]: XOR lemma for degree O(1)
- Technique: Use norm $N(f) \in [0,1]$:

  (I) $\mathrm{Cor}(f,\text{degree } d) \approx N(f)$

  (II) $N(f^{\oplus k}) = N(f)^k$

- Proof of the XOR lemma:

$$\mathrm{Cor}(f^{\oplus k},\text{degree } d) \approx N(f^{\oplus k}) = N(f)^k \approx \mathrm{Cor}(f,\text{degree } d)^k$$

Q.e.d.

Gowers norm

[Gowers '98; Alon Kaufman Krivelevich Litsyn Ron '03]

- Measure correlation with degree-d polynomials: check if random d-th derivative is biased

- Derivative in direction $y \in \{0,1\}^n$: $D_y\, p(x) := p(x+y) - p(x)$
  - E.g. $D_{y_1\, y_2\, y_3}(x_1\, x_2 + x_3) = y_1 x_2 + x_1 y_2 + y_1 y_2 + y_3$

- Norm $N_d(p) := E_{Y^1 \ldots Y^d \in \{0,1\}^n} \mathrm{Bias}_U[D_{Y^1 \ldots Y^d}\, p(U)] \in [0,1]$

  (Bias [Z] := | Pr[Z = 0] - Pr[Z = 1] |)

  $N_d(p) = 1 \Leftrightarrow$ p has degree d

- From combinatorics [Gowers; Green Tao], to PCP [Samorodnitsky Trevisan], to correlation bounds [**V.** Wigderson]

Properties of norm

- $N_d(p) := E_{Y^1 \ldots Y^d \in \{0,1\}^n} \, \mathrm{Bias}_U[D_{Y^1 \ldots Y^d} \, p(U)]$

(I) $N_d(f) \approx \mathrm{Cor}(f, \text{degree } d)$:

Lemma[Gowers, Green Tao]:
$\mathrm{Cor}(f, \text{degree } d) \le N_d(f)^{1/2^d}$

Lemma[Alon Kaufman Krivelevich Litsyn Ron]:
(Gowers inverse conjecture, $N \approx 1$ case)
$\mathrm{Cor}(f, \text{degree } d) \le 1/2 \Rightarrow N_d(f) \le 1 - 2^{-d}$

(II) $N(f^{\oplus k}) = N(f)^k$
Follows from definition

Proof of XOR lemma

(I) $N_{d+1}(f) \approx Cor(f, \text{degree } d)$:

Lemma[G, GT]: $Cor(f, \text{degree } d) \leq N_d(f)^{1/2^d}$

Lemma[AKKLR]: $Cor(f, \text{degree } d) \leq 1/2 \Rightarrow N_d(f) \leq 1 - 2^{-d}$

(II) $N(f^{\oplus k}) = N(f)^k$

- Theorem[V. Wigderson]:
  $Cor(f, \text{deg. } d) \leq 1/2 \Rightarrow Cor(f^{\oplus k}, \text{deg. } d) \leq \exp(-k/4^d)$
- Proof:
  $Cor(f^{\oplus k}, \text{deg. } d) \leq N_d(f^{\oplus k})^{1/2^d} = N_d(f)^{k/2^d} \leq (1-2^{-d})^{k/2^d}$

  Q.e.d.
- More[VW]: Best known bound for degree 0.5 log n, …

Outline

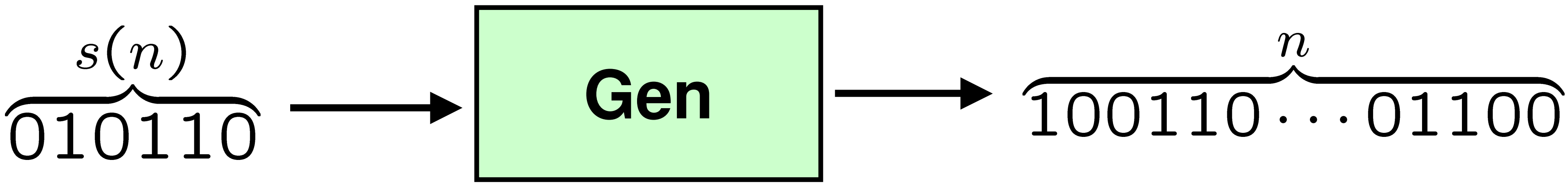
- Overview

- Correlation bounds

- Pseudorandom generators

Pseudorandom generator

[Blum Micali; Yao; Nisan Wigderson]



- Efficient

- Short seed s(n) << n

- Output ``fools'' degree-d polynomial p

$| \, \text{Bias}_{X\in\{0,1\}^s}[p(\text{Gen}(X))] - \text{Bias}_{U\in\{0,1\}^n}[p(U)] \, | \leq \varepsilon$

Previous results

- Th.[Naor & Naor '90]: Fools linear, seed = O(log n/ε)
  - Applications: derandomization, PCP, expanders, learning…
- Th.[Luby Velickovic Wigderson '93]:
  Fools constant degree, seed = exp(√log n/ε)
  - [V. '05] gives modular proof of more general result
- Th.[Bogdanov '05]: Any degree, but over large fields
- Over small fields such as {0,1}:

  no progress since 1993, even for degree d=2

Our results

- New approach based on Gowers norm

- Theorem[Bogdanov V.]:
  Unconditionally:
  Fool degree d=2 with seed = $2 \cdot \log(n) + O(\log(1/\varepsilon))$
  Fool degree d=3 with seed = $3 \cdot \log(n) + f(\varepsilon)$

- Theorem[Bogdanov V.]:
  Under "d vs. d-1 Gowers inverse conjecture":
  Fool any degree d with seed = $d \cdot \log(n) + f(d,\varepsilon)$

- Results apply to any prime field.
  Focus on {0,1} for simplicity

Our generator

- Generator that fools degree d:

  Let $L \in \{0,1\}^n$ fool linear polynomials [NN]

  bit-wise XOR d independent copies of L:

  Generator := $L^1 + \ldots + L^d$

- Seed length $d \cdot \log(n) + f(d,\varepsilon)$ optimal for fixed d, ε

  $\Rightarrow$ XORing d-1 copies is not enough.

Recent developments after [BV]

- Th.[Lovett]: The sum of 2^d generators for degree 1 fools degree d, without using Gowers norm.
  - Recall [BV] sums d copies

- Progress on "d vs. d-1 Gowers inverse conjecture":

- Th.[Green Tao]: True when $|F| > d$
  Proof uses techniques from [BV]
  [BV] works when $|F| > d$ or $d = 2,3$

- Th. [Green Tao], [Lovett Meshulam Samorodnitsky]:
  False when $F = \{0,1\}$, $d = 4$

Our latest result

- Theorem[V.]:
  The sum of d generators for degree 1
  fools polynomials of degree d.
  For every d and over any field.

  (Despite the Gowers inverse conjecture being false)

- Improves on both [Bogdanov V.] and [Lovett]

- Also simpler proof

Proof idea

- Recall: want to show the sum of d generators for degree 1 fools degree-d polynomial p

- Induction: Fool degree d $\Rightarrow$ fool degree-(d+1) p

  Inductive step: Case-analysis based on
  Bias(p) := $| \Pr_{U \in \{0,1\}^n} [p(U)=1] - \Pr_U [p(U)=0] |$

  Cases:
- Bias(p) negligible $\Rightarrow$ Fool p using extra copy of generator for degree 1

- Bias(p) noticeable $\Rightarrow$ p close to degree-d polynomial $\Rightarrow$ fool p by induction

Case Bias(p) negligible

- Hypothesis: $L^1, \ldots, L^d, L$ over $\{0,1\}^n$ fool degree 1
  $W := L^1 + \cdots + L^d$ fools degree d
- Goal: For degree-(d+1) p: Bias (p(W+L)) $\approx$ Bias (p(U))
- Lemma[V]: Bias (p(W + L)) $\leq$ Bias (p(U)) ≈ 0
- Proof: $\text{Bias}_{W,L}\,[p(W + L)]^2 = E_W\,[\,\text{Bias}_L\,(p(W+L))\,]^2$

$$\leq E_W\,[\,\text{Bias}_{L,L'}\,(\underbrace{p(W+L) + p(W+L')}_{\text{degree } d \text{ in } W})\,]$$

$$\approx E_U\,[\,\text{Bias}_{L,L'}\,(p(U+L) + p(U+L'))\,] \approx \text{Bias}_U(p)^2$$

Q.e.d.

Case Bias(p) noticeable

- Bias(p) noticeable

⇕

- p noticeably correlates with constant (51 %)

⇓

Self-correction [Bogdanov V.]
This result used in [Green Tao]

- p highly correlates with (function of) degree-d polynomials (99 %)

- Apply induction

Conclusion

- Computational model: degree-d polynomials over F_2
  Arises in codes, lower bounds, pseudorandomness

- Correlation bounds
  Barrier: correlation 1/n for degree log n
  Standard XOR lemma does not work [Shaltiel V.]
  XOR lemma for constant degree [V. Wigderson]

- Pseudorandom generators
  Recent developments [BV,L,GT,LMS]
  Sum of d generators for degree 1 fools degree d [V.]