# Pseudorandomness: New Results and Applications


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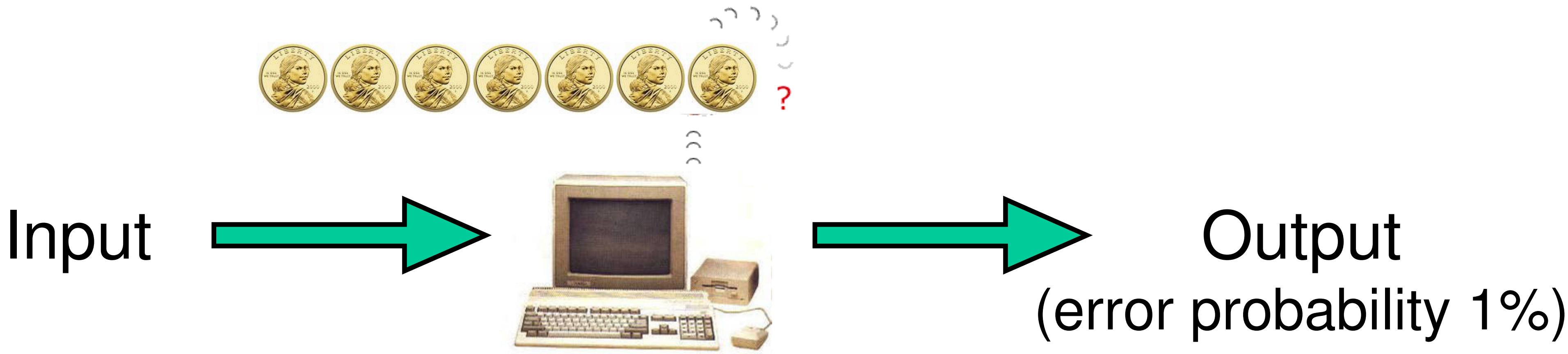
# Randomness in Computation




- Useful throughout Computer Science
  - Algorithms
  - Cryptography
  - Complexity Theory

- Question: Is “true” randomness necessary?

# Pseudorandomness


- Goal: low-entropy distributions
  that ``look random''

- Why study pseudorandomness?

- Basis for most cryptography [S 49]

- Algorithmic breakthroughs:
  Connectivity in logarithmic space [R 04]
  Primality in polynomial time [AKS 02]

# Pseudorandom Generator (PRG) [BM,Y]

$$\overbrace{010110}^{n} \longrightarrow \boxed{\textbf{PRG}} \longrightarrow \overbrace{100110\cdots01100}^{n+s(n)}$$

- Poly(n)-time Computable
- Stretch $s(n) \geq 1$ (e.g., $s(n) = 1$, $s(n) = n^2$)
- Output ``looks random”

# Outline

- Overview of pseudorandomness
- Cryptographic pseudorandom generators
  - Complexity vs. stretch
- Specialized pseudorandom generators
  - Constant-depth, with application to NP
  - Polynomials

# Cryptographic PRG

$$\overbrace{010110}^{n} \longrightarrow \boxed{\textbf{PRG}} \longrightarrow \overbrace{100110\cdots01100}^{n+s(n)}$$

- "Looks random": $\forall$ efficient adversary A : $\{0,1\}^{n+s(n)} \to \{0,1\}$

  $$\Pr_U[A(U) = 1] \approx \Pr_X[A(PRG(X)) = 1]$$

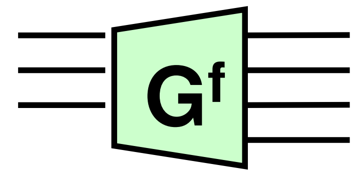
- Cryptography: sym. encryption(m) := m $\oplus$ G(X) [S49]

  need big stretch s >> n

- PRG $\Leftrightarrow$ One-Way Functions (OWF) [BM,Y,GL,…,HILL]
  - OWF: easy to compute but hard to invert

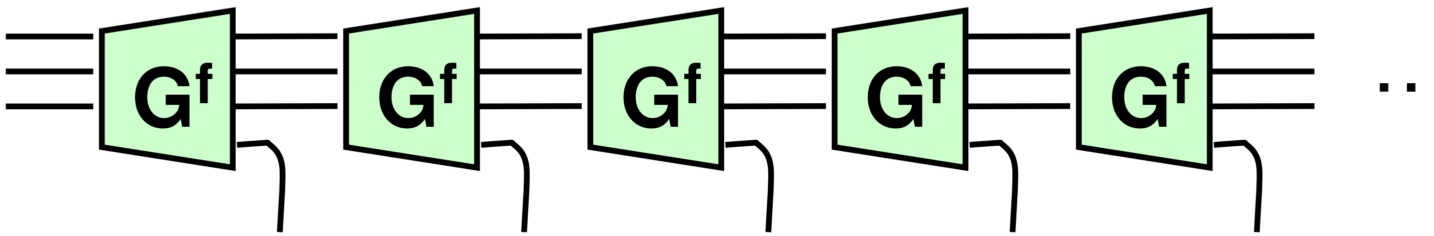
# Standard Constructions w/ big stretch

- STEP 1: OWF f $\Rightarrow G^f : \{0,1\}^n \to \{0,1\}^{n+1}$
  - Think e.g. $f : \{0,1\}^{n^a} \to \{0,1\}^{n^b}$




- STEP 2: $G^f \Rightarrow$ PRG with stretch s(n) = poly(n) [GM]

Input X




. . . . . . . . . . Output . . . . . . . . . .

- Stretch s $\Rightarrow$ s adaptive queries to f $\Rightarrow$ circuit depth $\geq$ s

- Question [this work]: stretch s vs. adaptivity & depth? E.g., can have s = n, circuit depth O(log n)?

# Previous Results

- [AIK] Log-depth OWF/PRG $\Rightarrow$ O(1)-depth PRG (!!!)

  However, any stretch $\Rightarrow$ stretch s = 1

- [GT] s vs. *number* q of queries to OWF (Thm: $q \geq s$)

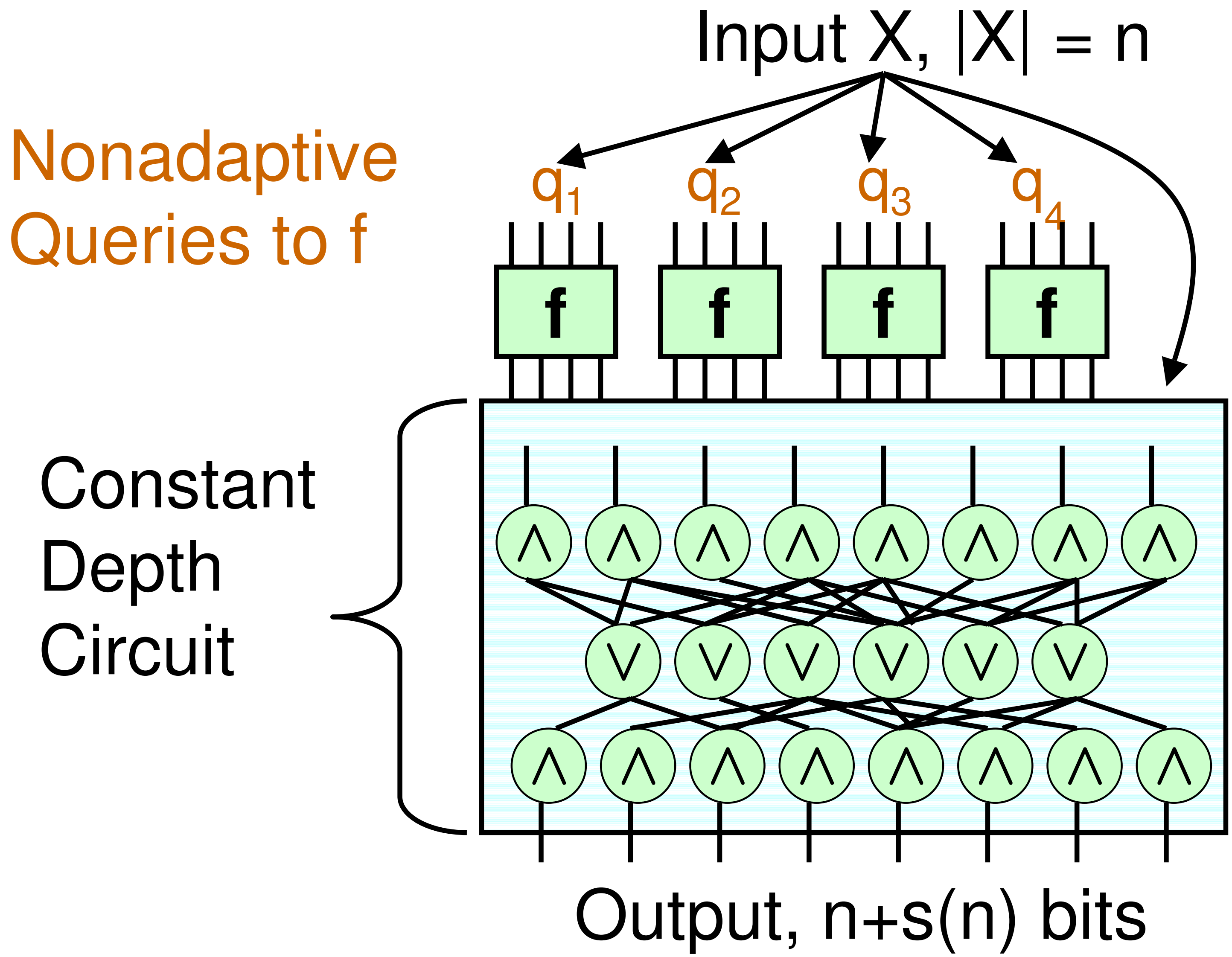
  [This work] s vs. *adaptivity & circuit depth*

- [...,IN,NR] O(1)-depth PRG from *specific* assumptions

  [This work] *general* assumptions

# Our Model of PRG construction

- **Parallel** PRG $G^f : \{0,1\}^n \to \{0,1\}^{n+s(n)}$ from OWF f



# Our Results on PRG Constructions

- Theorem [V] Parallel $G^f : \{0,1\}^n \to \{0,1\}^{n+s(n)}$ from OWF ( e.g. $f : \{0,1\}^{n^a} \to \{0,1\}^{n^b}$ ) must have:

| | **f arbitrary** | **f one-to-one** | **f permutation** |
|---|---|---|---|
| **Neg.** | $s(n) \leq o(n)$ | $s(n) \leq o(n)$ | ? |
| **Pos.** | ? | $s(n) \geq 1$ | $s(n) \geq 1$ |

- Proof idea (f = permutation $\pi$, s = 1)

  [GL] $G^f(x,r) := \pi(x),r,\langle x,r\rangle$ $\quad(\langle x,r\rangle := \sum_i x_i r_i)$

  Problem: can't compute $\langle x,r\rangle$ in constant-depth [GNR]

  Solution: don't have to! $G^f(x,r) := \pi(x),r',\langle x,r'\rangle$

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# Specialized PRG

$$\overbrace{010110}^{n} \longrightarrow \boxed{\textbf{PRG}} \longrightarrow \overbrace{100110\cdots01100}^{n+s(n)}$$

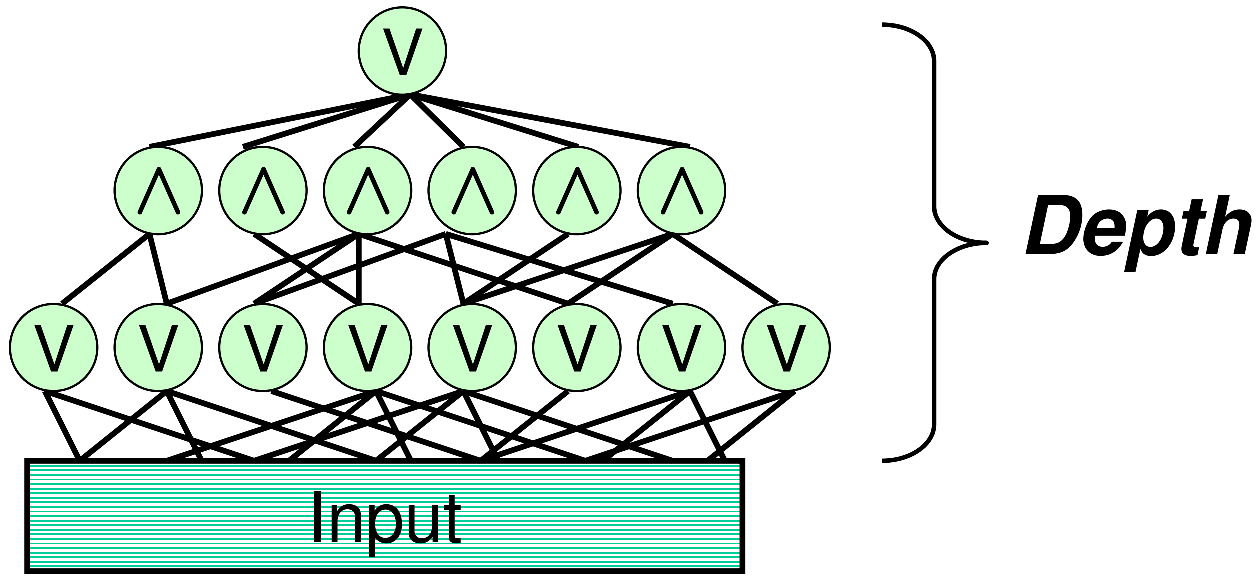
- "looks random": $\forall$ **restricted** A : $\{0,1\}^{n+s(n)} \rightarrow \{0,1\}$

$$\Pr_U[A(U) = 1] \;\approx\; \Pr_X[A(PRG(X)) = 1]$$

- Sometimes known unconditionally!

# PRG for Constant-Depth Circuits

- Constant-depth circuit:




- Theorem [N '91]: PRG with stretch $s(n) = 2^{n^{\Omega(1)}}$

  output looks random to constant-depth circuits

# Application: Avg-Case Hardness of NP

- Study hardness of NP on random instances
  - Natural question, essential for cryptography

- Currently cannot relate to P $\neq$ NP [FF,BT,V]

- Hardness amplification

  Definition: $f : \{0,1\}^n \to \{0,1\}$ is $\varepsilon$-hard if

  $\forall$ efficient algorithm $M$ : $\Pr_x[M(x) \neq f(x)] \geq 1/2 - \varepsilon$

$$\underset{0.1\text{-hard}}{f} \longrightarrow \boxed{\textbf{Hardness Amplification}} \longrightarrow \underset{\varepsilon\text{-hard}}{f\,'}$$

# Previous Results

- Yao's XOR Lemma: $f'(x_1, \ldots, x_n) := f(x_1) \oplus \cdots \oplus f(x_n)$
  $f' \approx 2^{-n}$ -hard, almost optimal

- Cannot use XOR in NP: $f \in NP \not\Rightarrow f' \in NP$

- Idea: $f'(x_1, \ldots, x_n) = C( f(x_1), \ldots, f(x_n) )$, C monotone
  - e.g. $f(x_1) \wedge ( f(x_2) \vee f(x_3) )$. $f \in NP \Rightarrow f' \in NP$

- Theorem [O'D]: There is C s.t. $f' \approx (1/n)$-hard

- Barrier: No monotone C can do better!

# Our Result on Hardness Amplification

- Theorem [HVV]: Amplification in NP up to $\approx\ 2^{-n}$
  - Matches the XOR Lemma

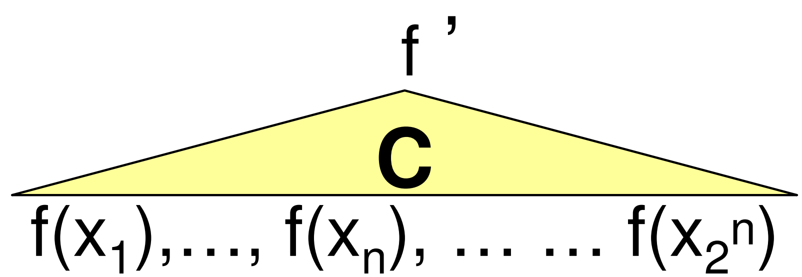
- Technique: Pseudorandomness!

  Intuitively, $f' := C(\ f(x_1),\ldots,\ f(x_n),\ \ldots\ \ldots\ f(x_{2^n})\ )$

  $f'$ $(1/2^n)$-hard by previous result

  Problem: Input length $= 2^n$

  Note C is constant-depth

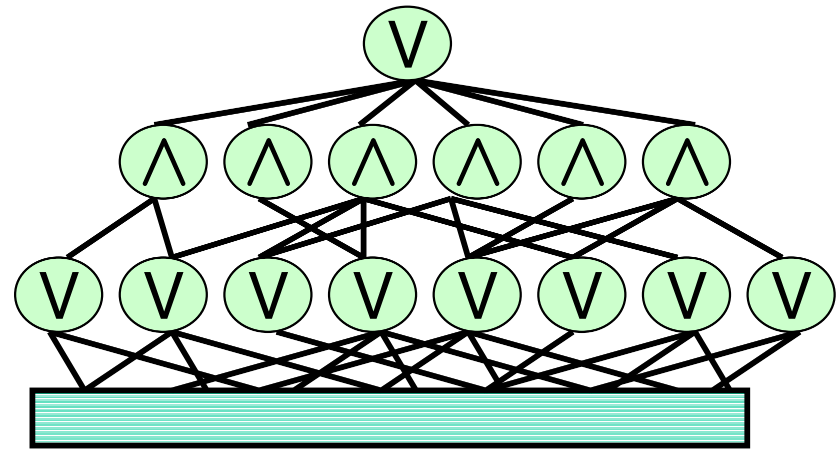



  Use PRG: input length $\rightarrow$ n, keep hardness

# Previous Results

- Recall Theorem [N]:

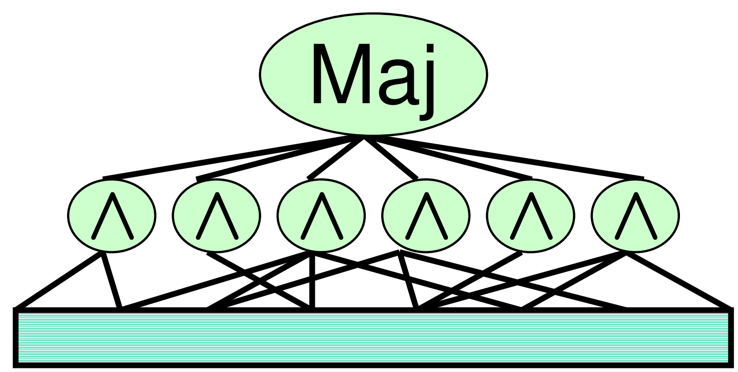
  PRG with stretch $s(n) = 2^{n^{\Omega(1)}}$




- But constant-depth circuits are weak:
  – Cannot compute $\text{Majority}(x_1,\ldots,x_n) := \sum_i x_i > n/2$ ?

- Theorem [LVW]:

  PRG with stretch $s(n) = n^{\log n}$

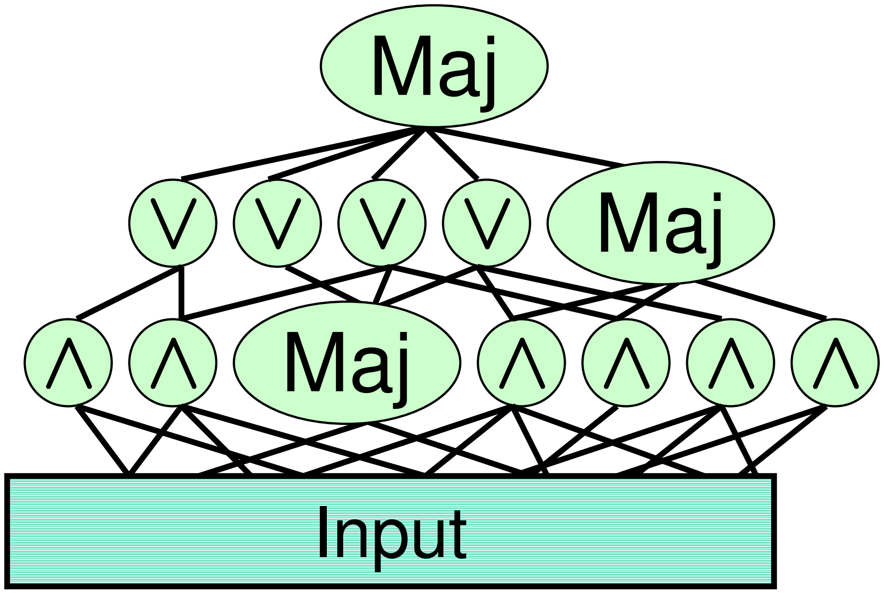



- PRG's for incomparable classes

# Our New PRG

- Constant-depth circuits with few Majority gates

- Theorem **[V] :** PRG with $s(n) = n^{\log n}$




- Improves on [LVW]; worse stretch than [N]
  Richest class for which PRG is known

- Techniques: Communication complexity + switching lemma [BNS,HG,H,HM,CH]

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# $F_2$ polynomials

- Field $F_2 = GF(2) = \{0,1\}$

- $F_2$-polynomial $p : F_2^n \to F_2$ of degree d

  E.g., $p = x_1 + x_5 + x_7$ $\quad$ $d = 1$

  $p = x_1 \cdot x_2 + x_3$ $\quad$ $d = 2$

- Theorem[NN90]: PRG for d=1 with stretch $s(n)=2^{\Omega(n)}$
  - Applications to algorithm design, PCP's,…

# Hardness for $F_2$ polynomials

- Want: explicit $f : \{0,1\}^n \to \{0,1\}$ $\varepsilon$-hard for degree d:
  $\forall$ p of degree d : $\Pr[f(x) \neq p(x)] \geq ½ - \varepsilon$
  $\varepsilon = \varepsilon(n,d)$ small

- Implies PRG with s=1. $G(X) := X\ f(X)$

- Interesting beyond PRG
  - Coding theory
  - $d = \log n$, $\varepsilon = 1/n^{10} \Rightarrow$ complexity breakthrough

# Previous Results

- Want: explicit $f : \{0,1\}^n \to \{0,1\}$ $\varepsilon$-hard for degree d:
  $\forall$ p of degree d : $\Pr[f(x) \neq p(x)] \geq 1/2 - \varepsilon$
  $\varepsilon = \varepsilon(n,d)$ small

- [Razborov 1987] Majority: $(1/n)$-hard ($d \leq \text{polylog}(n)$)

- [Babai et al. 1992] Explicit f: $\exp(-n/d \cdot 2^d)$-hard

- [Bourgain 2005] Mod 3: $\exp(-n/8^d)$-hard
  - Mod 3 $(x_1,\ldots,x_n) := 1$ iff $3 \mid \sum_i x_i$

# Our Results

- New approach based on ``Gowers uniformity''

- Theorem [V,VW] :

  Explicit f: $\exp(-n/2^d)$-hard ([BNS] $\exp(-n/d \cdot 2^d)$ )

  Mod 3: $\exp(-n/4^d)$-hard ([Bou] $\exp(-n/8^d)$)

  – Also arguably simpler proof

- Theorem [BV, unpublished] :

  PRG with stretch $s(n) = 2^{\Omega(n)}$ for d = 2,3,…(?)

  – Even for d=2, previous best was $s(n) = n^{\log n}$ [LVW]

# Gowers uniformity

- Idea: Measure closeness to degree-d polynomials by checking if d-th derivative vanishes
  - [G98] combinat., [A+,J+,…] testing

- Derivative $D_y\, p(x) := p(x+y) + p(x)$
  - E.g. $D_y\, (x_1 x_2 + x_3) = (y_1+x_1)(y_2+x_2)+(x_3+y_3)+x_1x_2+x_3$
    $= y_1x_2 + x_1y_2 + y_1y_2 + y_3$
  - p degree d $\Rightarrow$ $D_y\, p(x)$ degree d-1
  - Iterate: $D_{y,y'}\, p(x) := D_y( D_{y'}\, p(x))$

- d-th Gowers uniformity of f:

  $U_d(f) := E_{x,y^1,\ldots,y^d}[e(D_{y^1,\ldots,y^d}\, f(x))]$ $(e(X):=(-1)^X)$
  - $U_d(p) = 1$ if p degree d

# Main lemma

- Lemma [Gow,GT]:
  Hardness of f for degree-d polynomials $\leq U_d(f)^{1/2^d}$
  – Property of f only!

- Proof sketch: Let p have degree d.
  Hardness of f for p

  $= |\ \Pr[f(x) = p(x)] - \Pr[f(x) \neq p(x)]\ |$

  $= E_X[e(f(x)+p(x))] = U_0(f+p)$

  $\leq U_1(f+p)^{1/2} \leq \ldots \leq U_d(f+p)^{1/2^d}$ (Cauchy-Schwartz)

  $= U_d(f)^{1/2^d}$ (d-th derivative of p = 1)

Q.E.D.

# Establishing hardness

- Consider $f := x_1 \cdots x_{d+1} + x_{d+2} \cdots x_{2d+2} + \cdots$
  - not best parameters, but best to illustrate

- Theorem [V] f is $\exp(-n/c^d)$-hard for degree d

- Proof:
  Hardness of $f \leq U_d(f)^{1/2^d}$ (by lemma)

  $= U_d(x_1 \cdots x_{d+1} + x_{d+2} \cdots x_{2d+2} + \cdots)^{1/2^d}$

  $= U_d(x_1 \cdots x_{d+1})^{n/(d+1)2^d}$ (by property of U)

  $= \exp(-n/c^d)$ (by calculation)

Q.E.D.

# Conclusion

- Pseudorandom generators (PRG's): powerful tool

- Cryptographic PRG's
  - Tradeoff between stretch and parallel complexity [V]

- Specialized PRG's
  - Application: Hardness Amplification in NP [HVV]
  - PRG for const.-depth circuits with few Maj gates [V]
  - PRG for low-degree polynomials over $F_2$ using Gowers uniformity [V, VW,BV]

# Thank you!