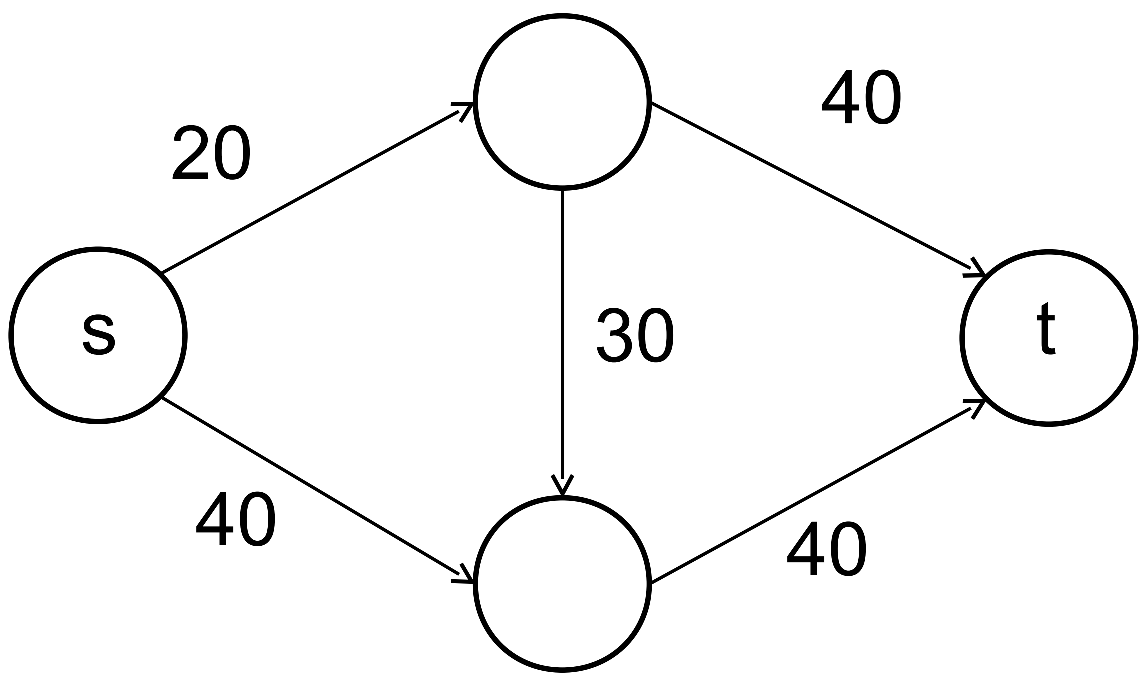
Network flow

Definition: A flow network is a directed graph $G = (V,E)$ with two nodes s and t, and a function $c(u,v) \geq 0$ on each directed edge (u,v)

- s is called the source
- t is called the sink
- $c: E \rightarrow R^+$ is called the capacity function

- Example



- A flow $f : V \times V \rightarrow R$ satisfies:

Skew symmetry: $f(u,v) = - f(v,u)$ for every pair (u,v)

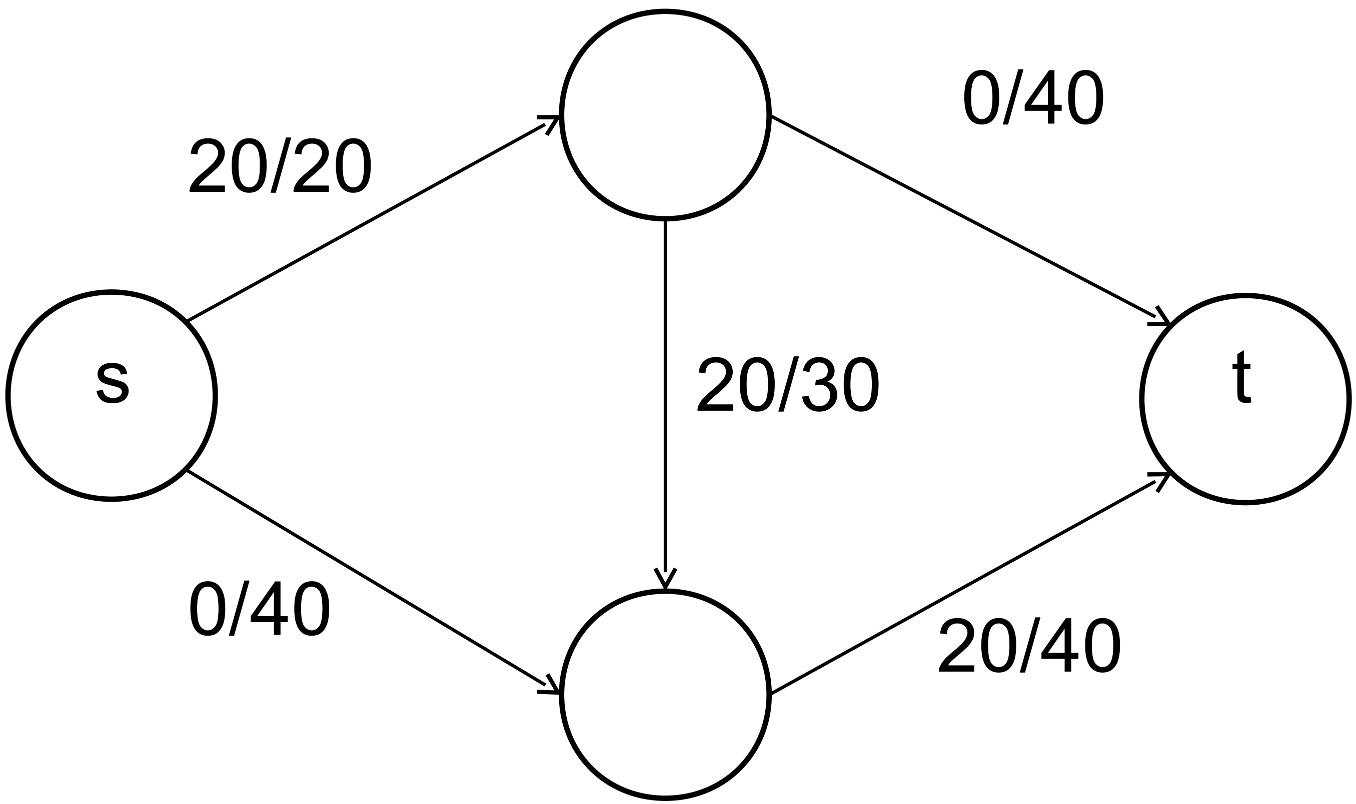
Capacity constraint $f(u,v) \le c(u,v)$ for each $(u,v) \in E$

Conservation of flows: $f(u,V) = 0$ for every $u \notin \{s,t\}$,
Where we define $f(X,Y) := \sum f(x,y)$ over $x \in X$ and $y \in Y$

- The value of flow f is $| f | = f(s,V)$

It represents the amount of flow passing from the source to the sink.

- Example



| f | = 20

Maximum flow problem

Input: A flow network G with s and t, a capacity function c
Output: A flow f so that | f | is maximum.

Applications: railway traffic, food supply, airline scheduling, image segmentation, baseball elimination...

Residual network

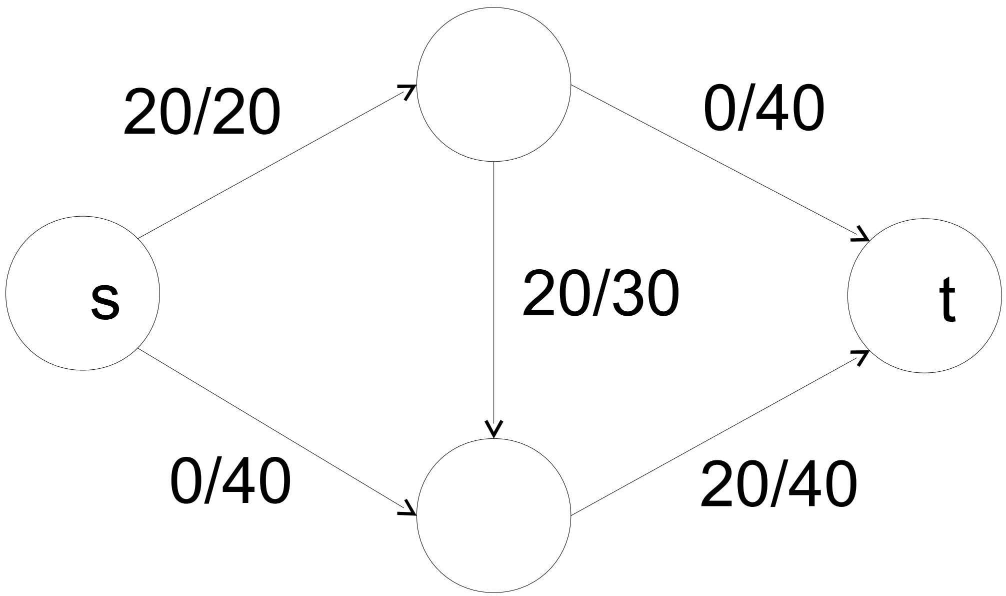
- A flow f induces a residual network G_f , consisting of the original graph G, and residual capacity function c_f :

For every (u,v) such that (u,v) or (v,u) $\in$ E we set $c_f(u,v) := c(u,v) – f(u,v) \geq 0$.

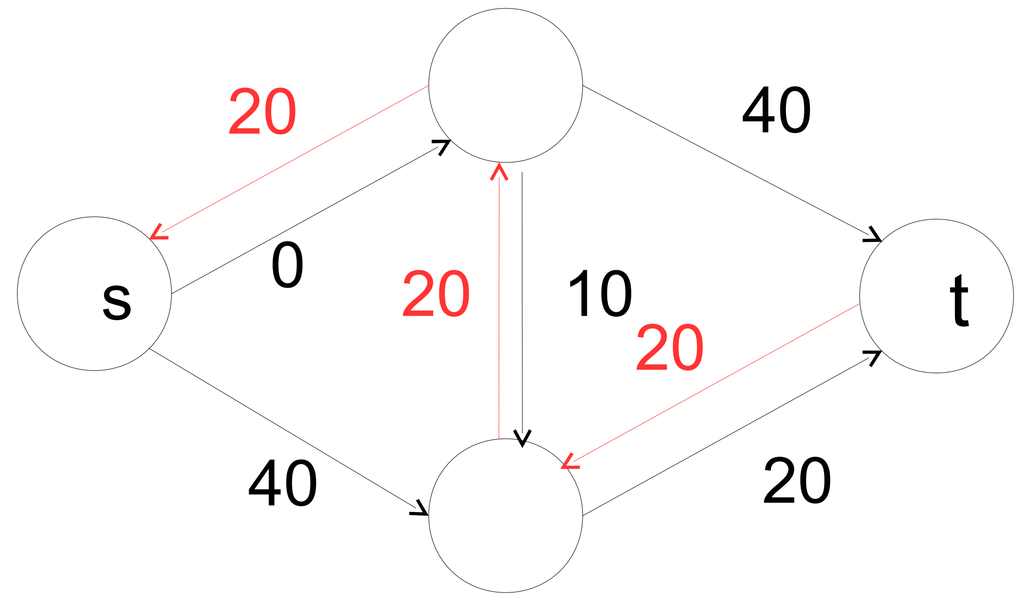
Note: the residual network may put non-zero capacity on edges which were non-existing or had zero capacity.

- An augmenting path is a path from s to t in the residual network

- Example

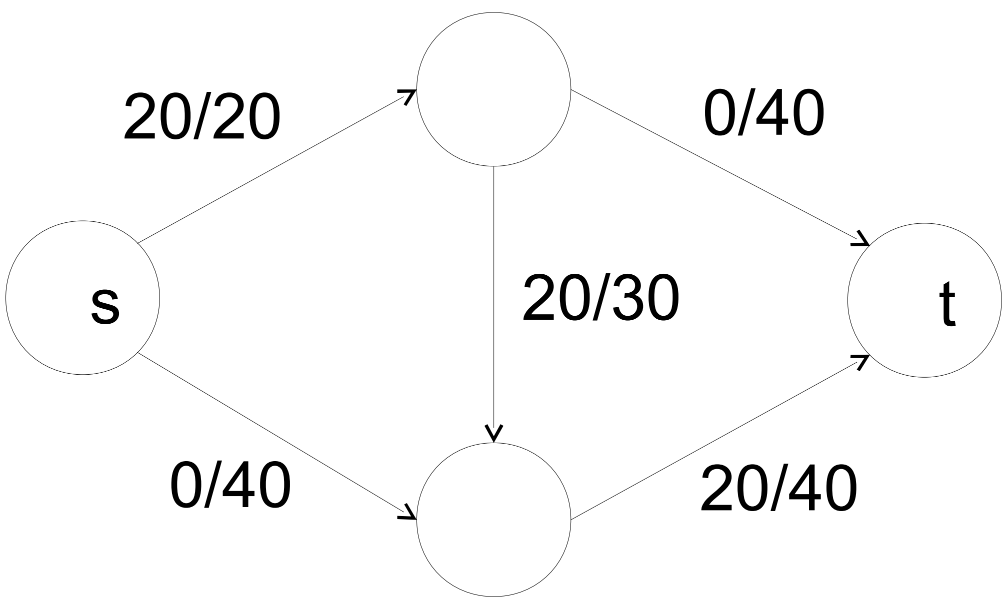


f



G_f

- Example



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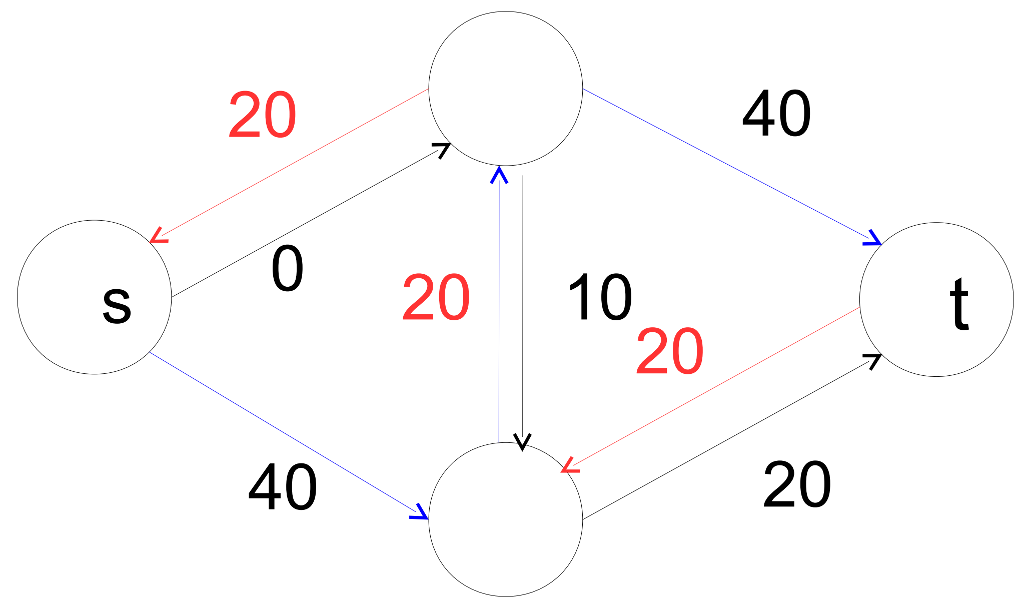


G_f

An augmenting path

Ford—Fulkerson Algorithm

Given G, s, t, $c(\cdot,\cdot)$. Start with $f \equiv 0$

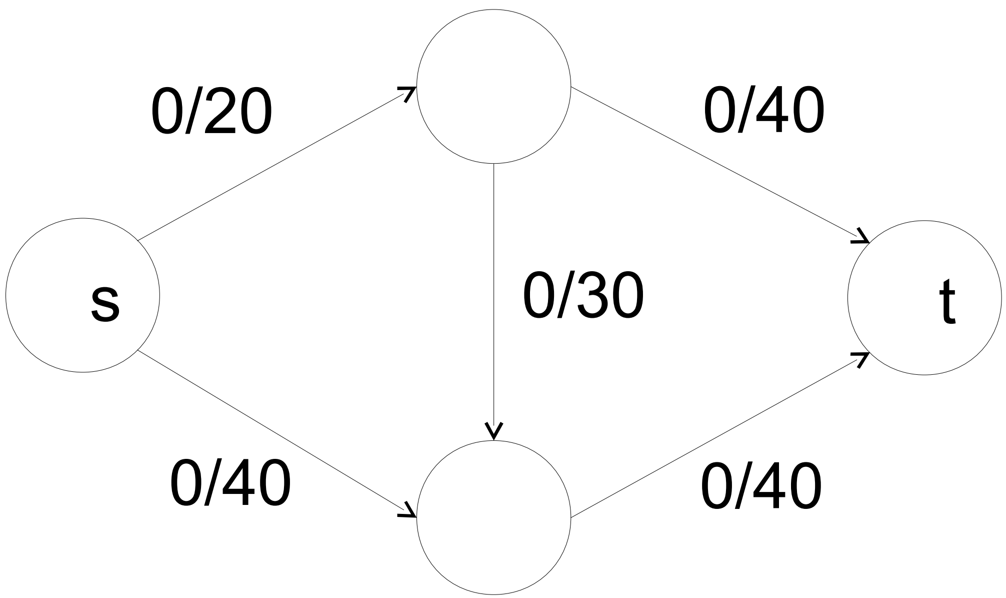
Repeat while there is an augmenting path P in G_f

Let $m = \min_{(u,v)\in P} c_f(u,v)$.

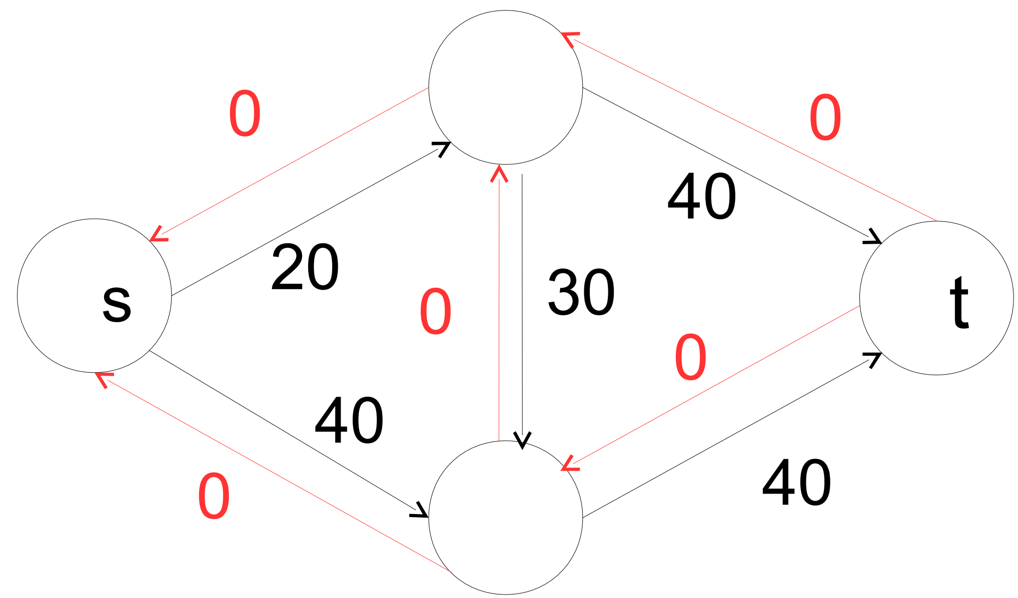
Define $f'(u,v) = m$ if (u,v) in P, $f'(u,v) = 0$ otherwise.

Augment the flow by setting $f = f + f'$

- Demo



| f | = 0



G_f

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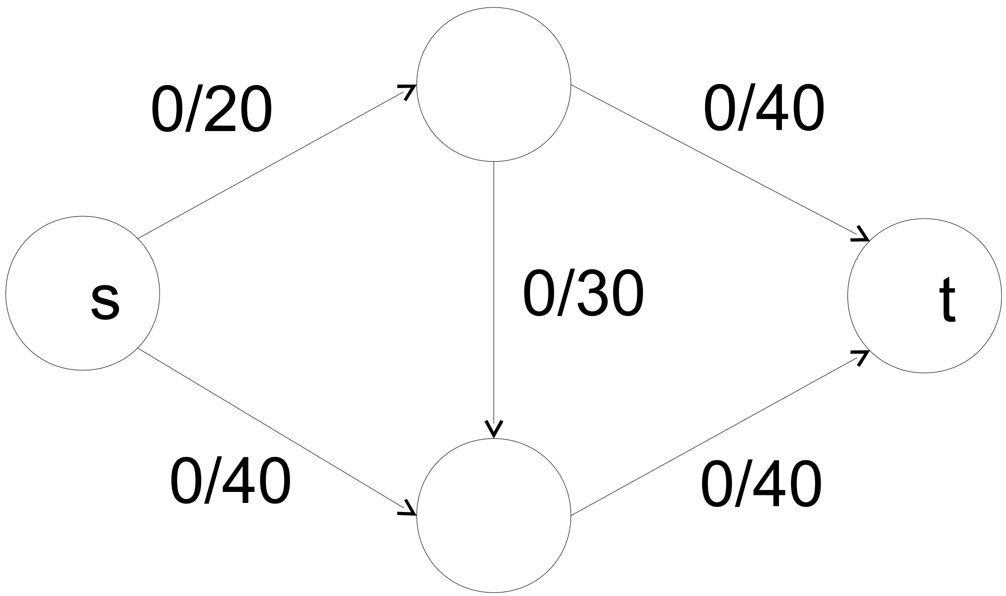

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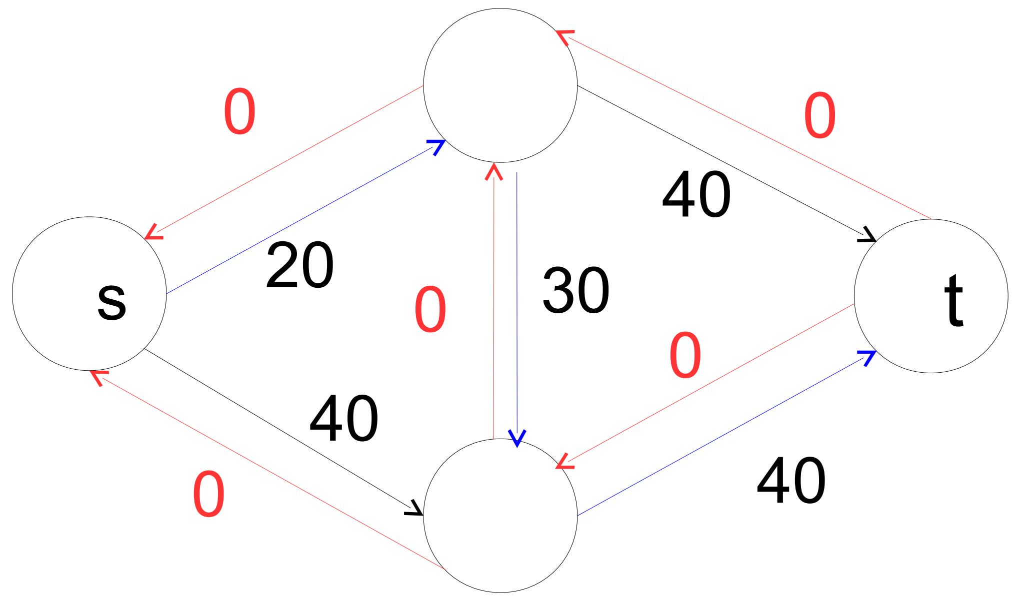
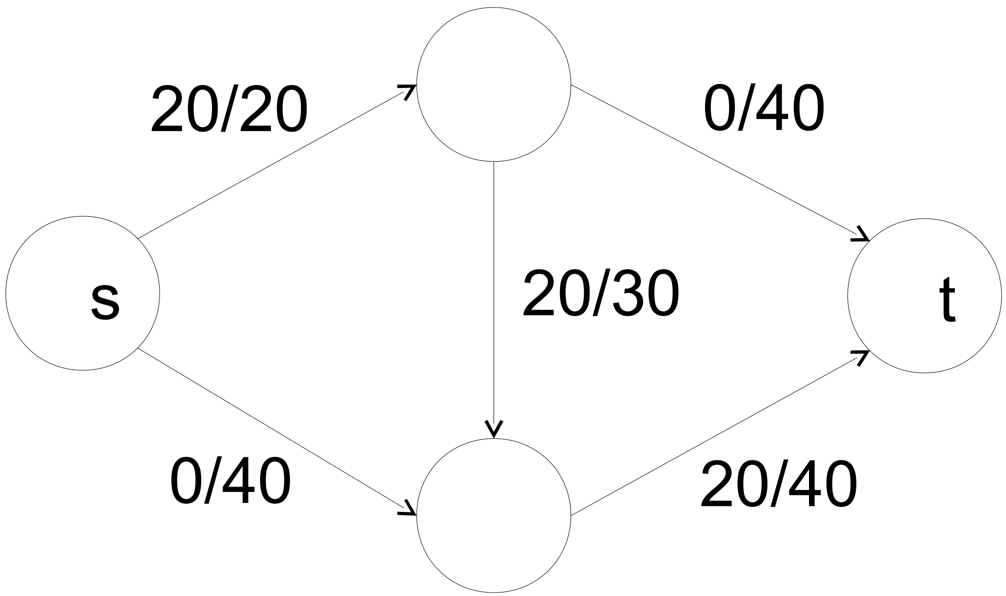


G_f

$\min_{(u,v)\in P} c_f(u,v) = 20$

- Demo



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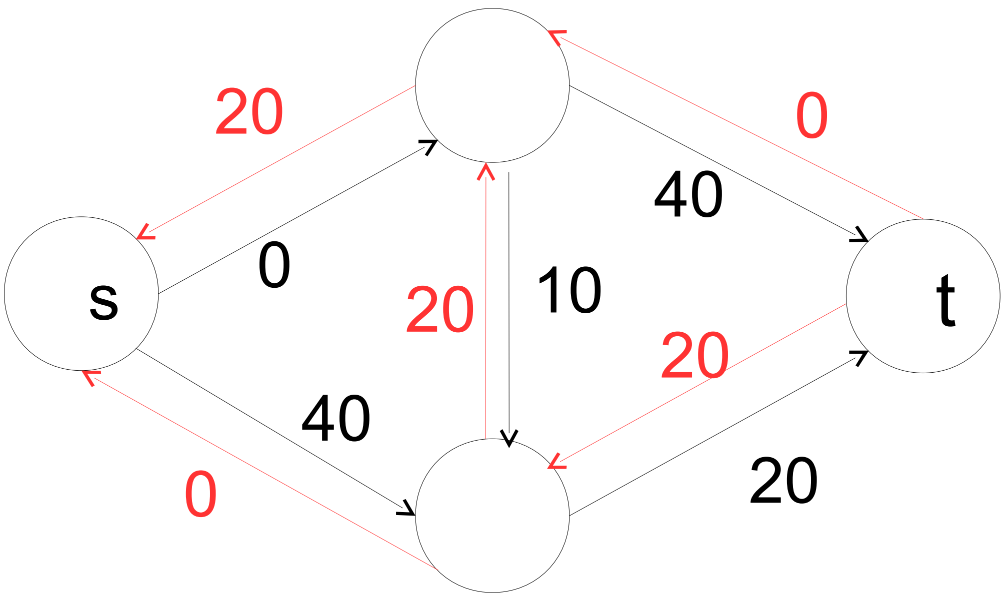

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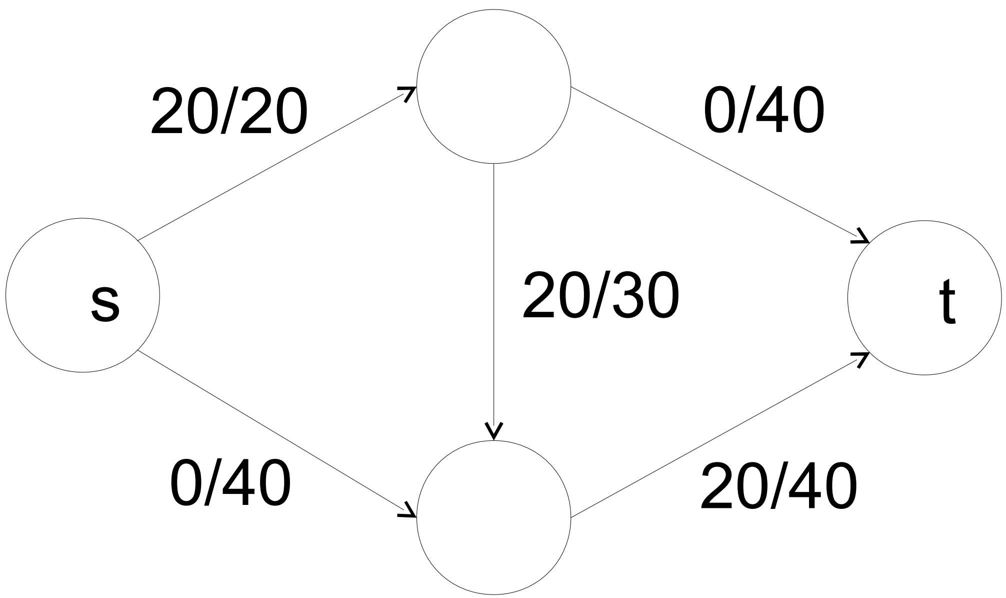

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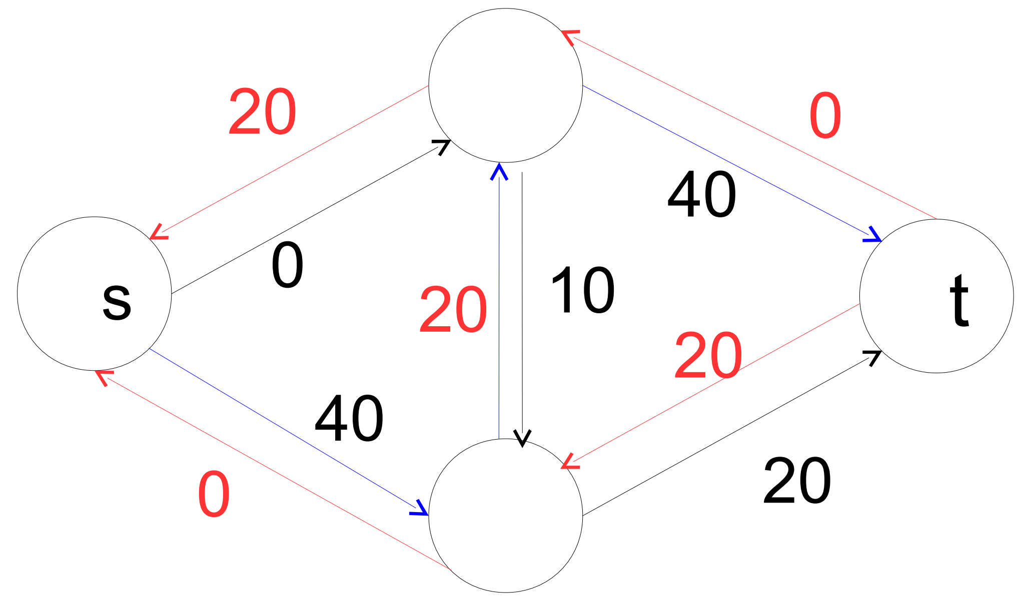

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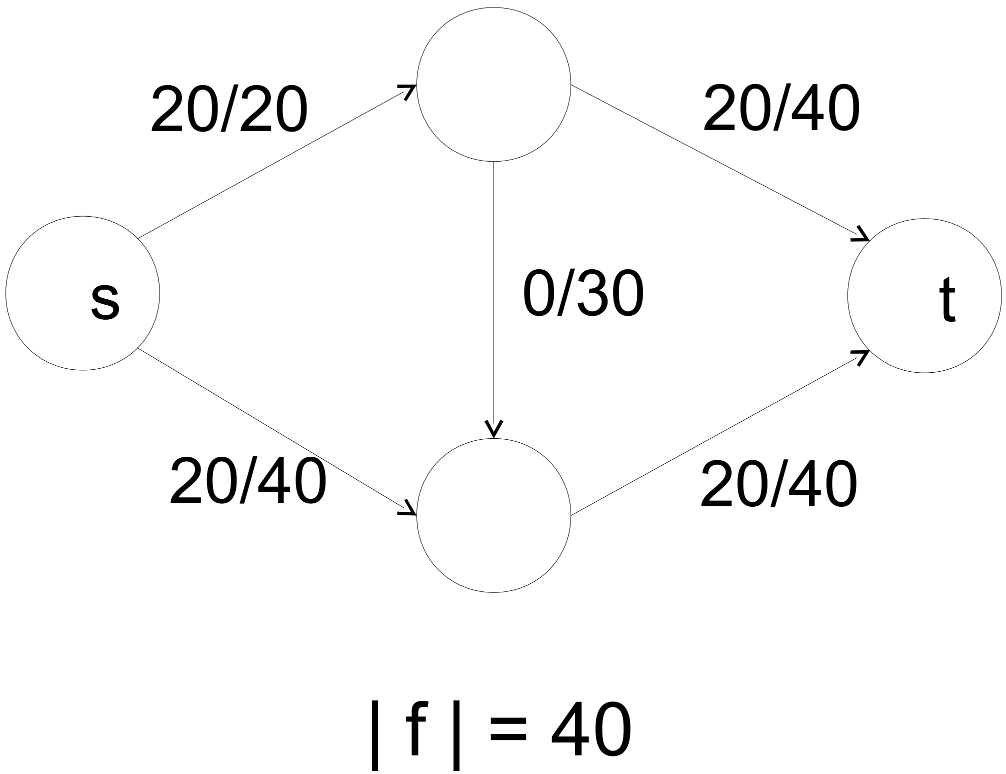

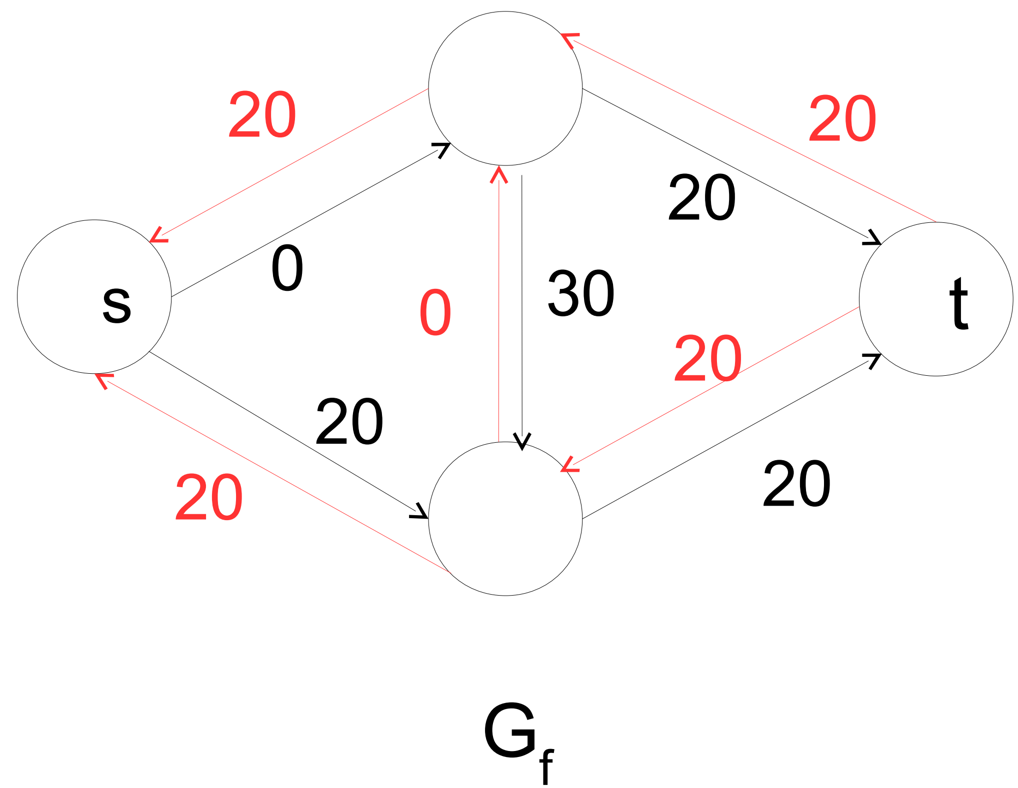

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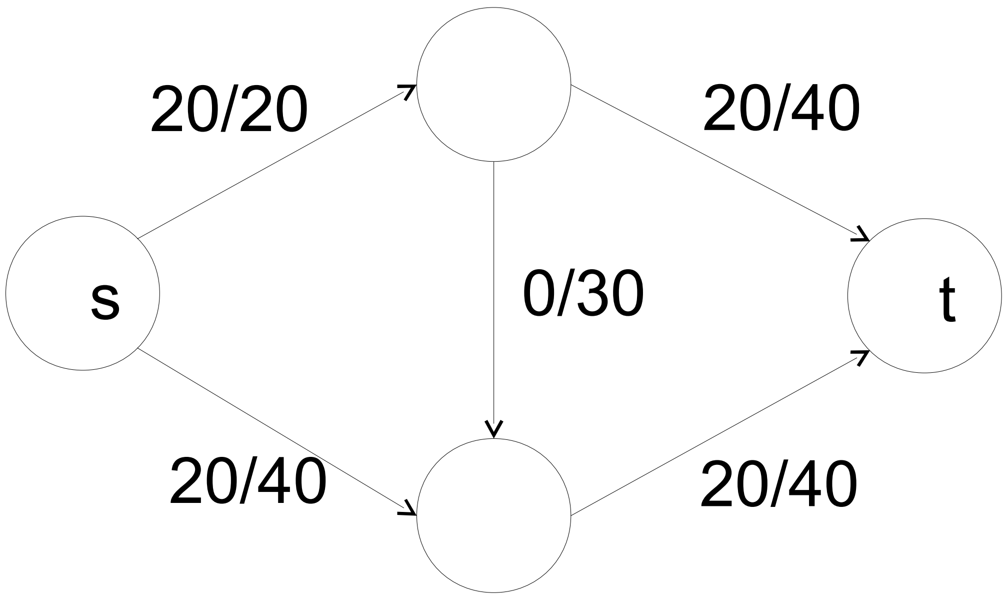
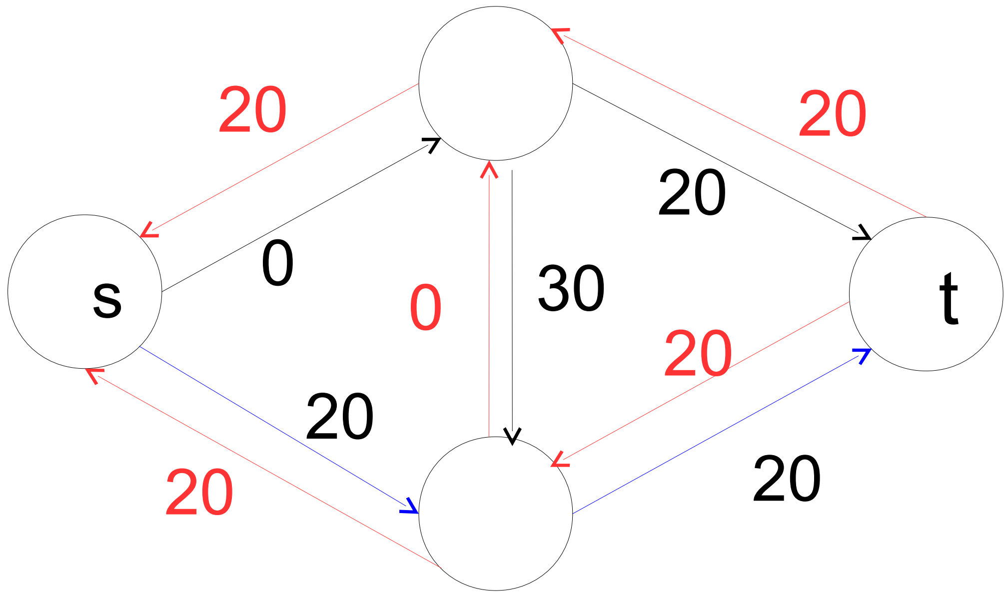


$|f| = 40$



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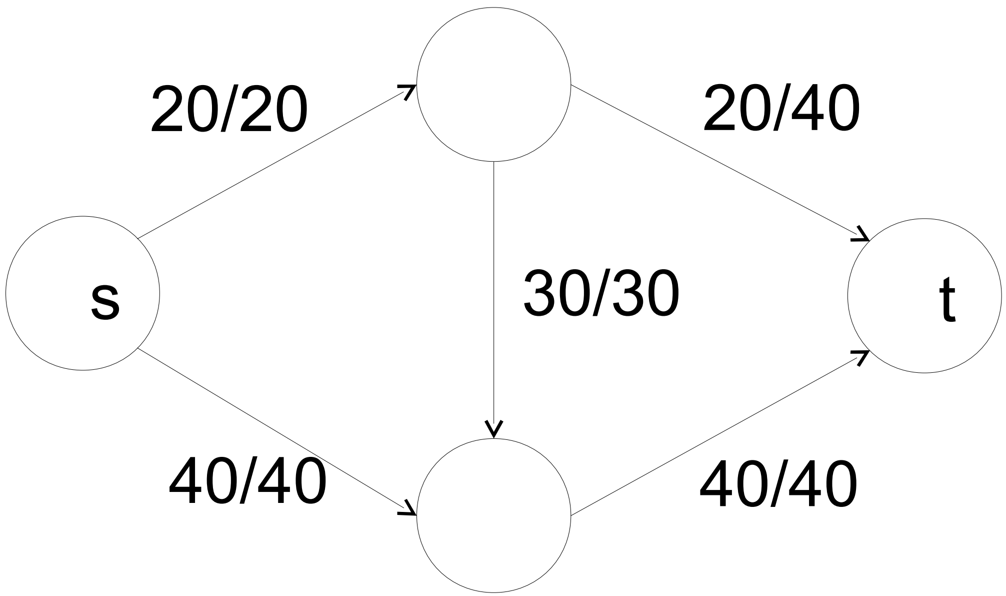
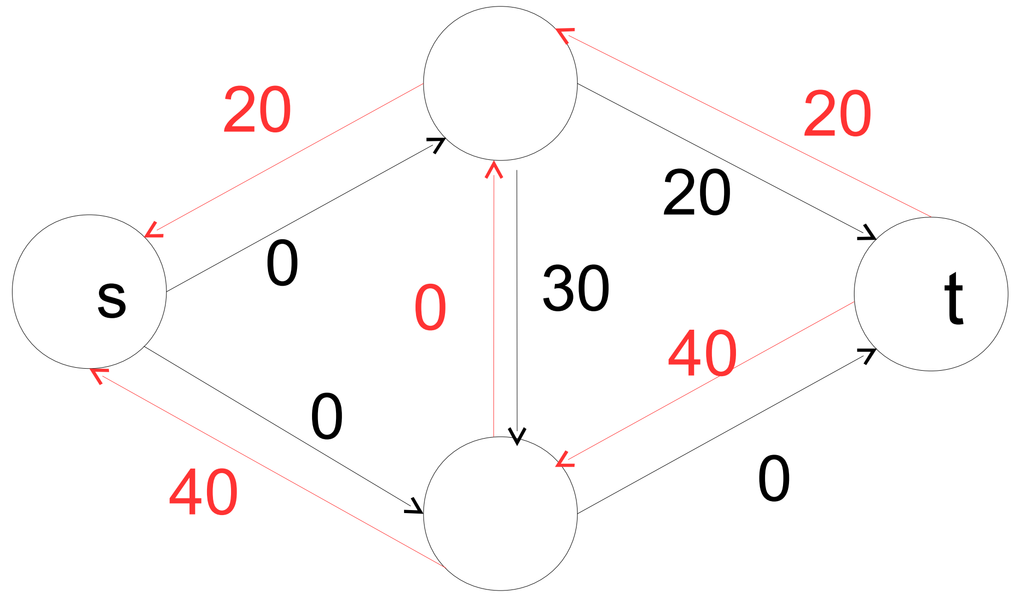


| f | = 60



G_f

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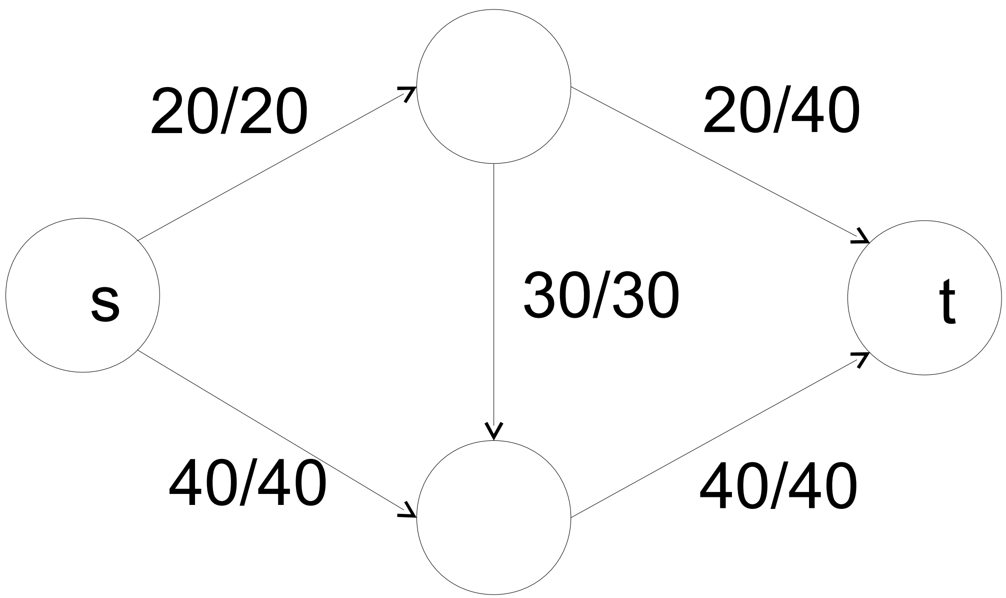

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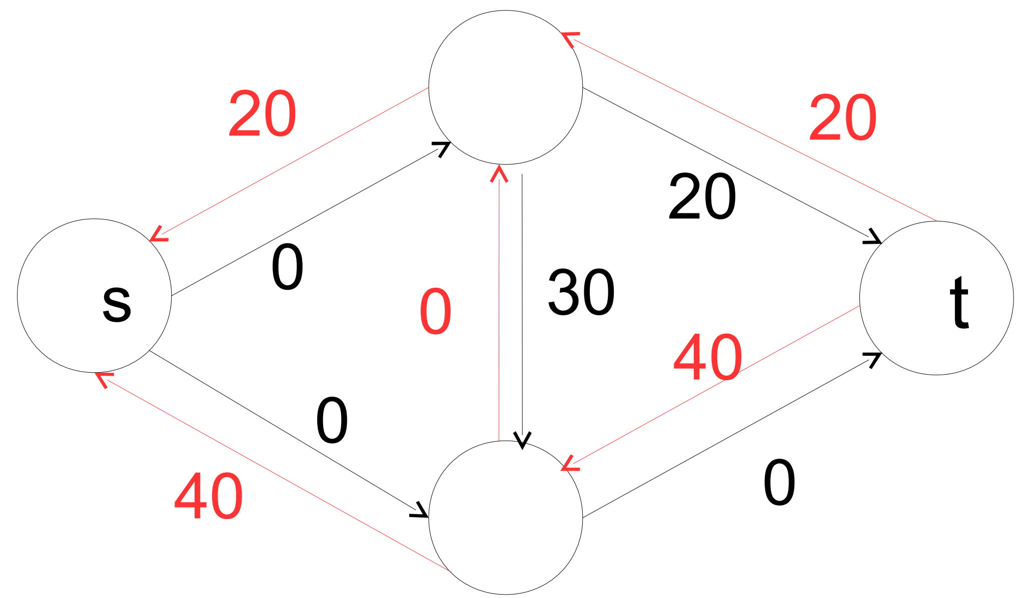


G_f

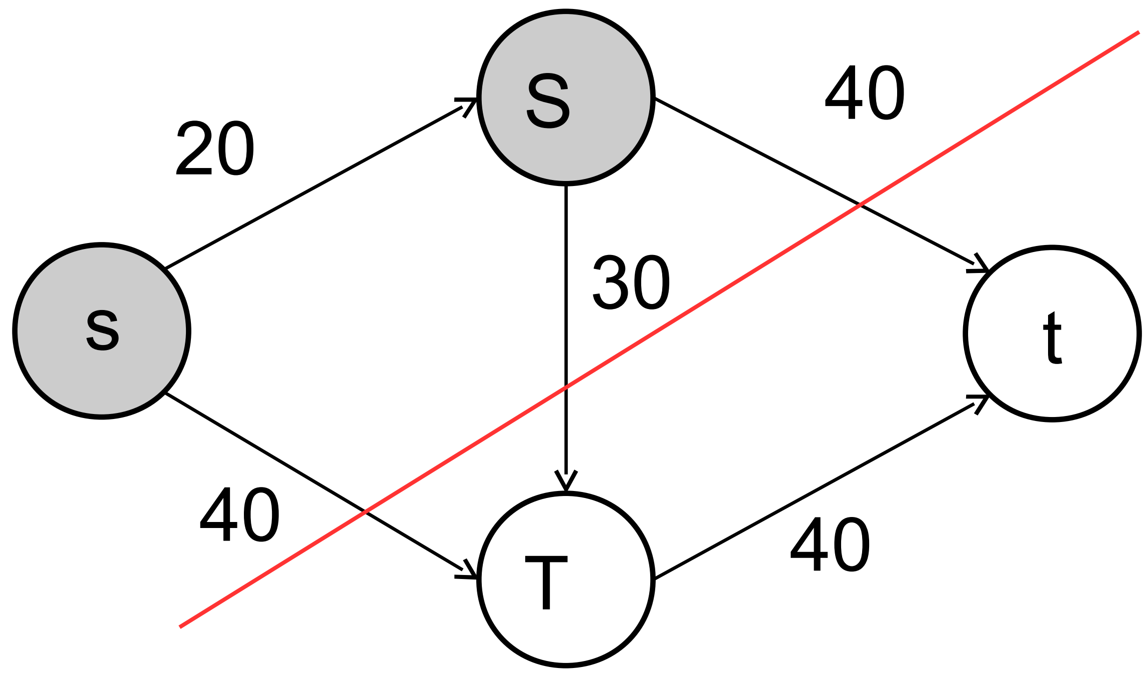
No augmenting path

max | f | = 60

Definition: An s-t cut (S,T) is a partition S, T = V – S such that s in S and t in T.

Meaning: removing the edges between S and T disconnects s and t

Example:



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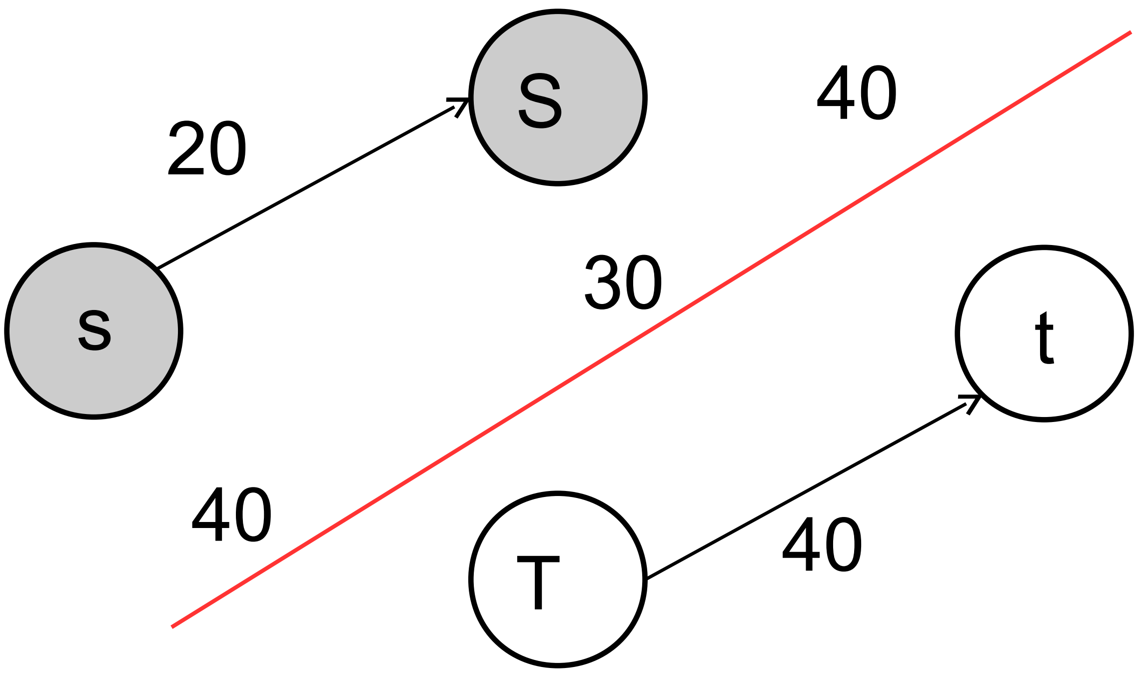

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The capacity of an s-t cut (S,T) is $c(S,T) := \sum_{u \in S,\, v \in T} c(u,v)$

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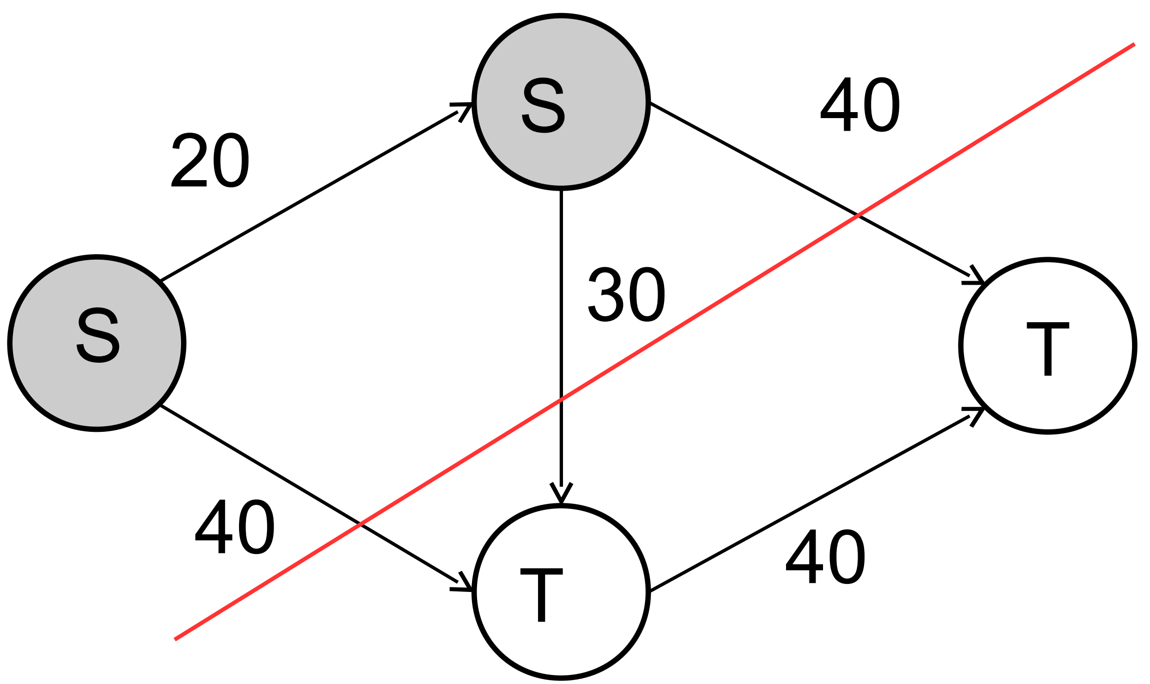

$c(S,T) := 40 + 30 + 40 = 110$

Analysis of Ford—Fulkerson algorithm:

Lemma: Let f be a flow. For any cut (S,T), f(S,T) = | f |

Proof:

Let's move x from S to T.

We lose f(x,T), and we gain f(S,x).

But f(x,T) = - f(x,S) because f(x,V) = 0. qed

Theorem (Max flow-min cut): The following are equivalent:

1. |f| is maximum
2. the residual network has no augmenting paths
3. | f | = c(S,T) for some cut (S,T)

Proof:

1 → 2: otherwise could increment the flow as said before.

2 → 3: define S := vertices reachable from s on residual network. Note $t \notin S$. By previous lemma, | f | = f(S,T).

Now note for each edge (u,v) in S×T, f(u,v) = c(u,v), otherwise v would be in S.

3 → 1: if f is not maximum, could have a better flow. But by lemma it would augment the flow on this cut, thus violate capacity constraints. □

Analysis of running time

Fact: Let f be a flow in G. Let f' be a flow on residual network G_f. Then f + f' is a flow on G_f with $|f + f'| = |f| + |f'| > |f|$.

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In each iteration,

- finding an augmentation path takes time $O(|E|)$
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Running time $O(|E| \max |f|)$

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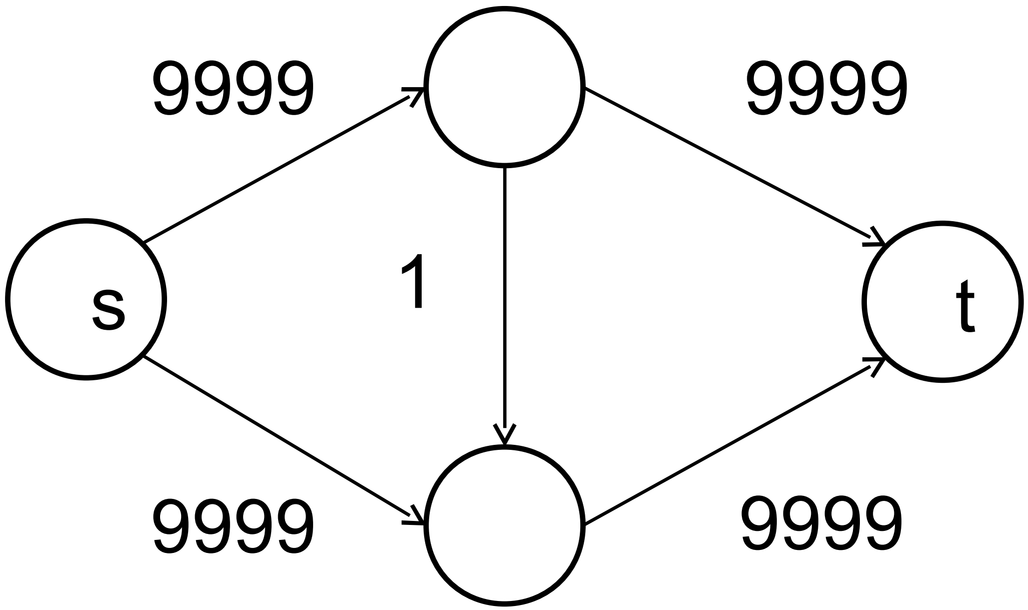

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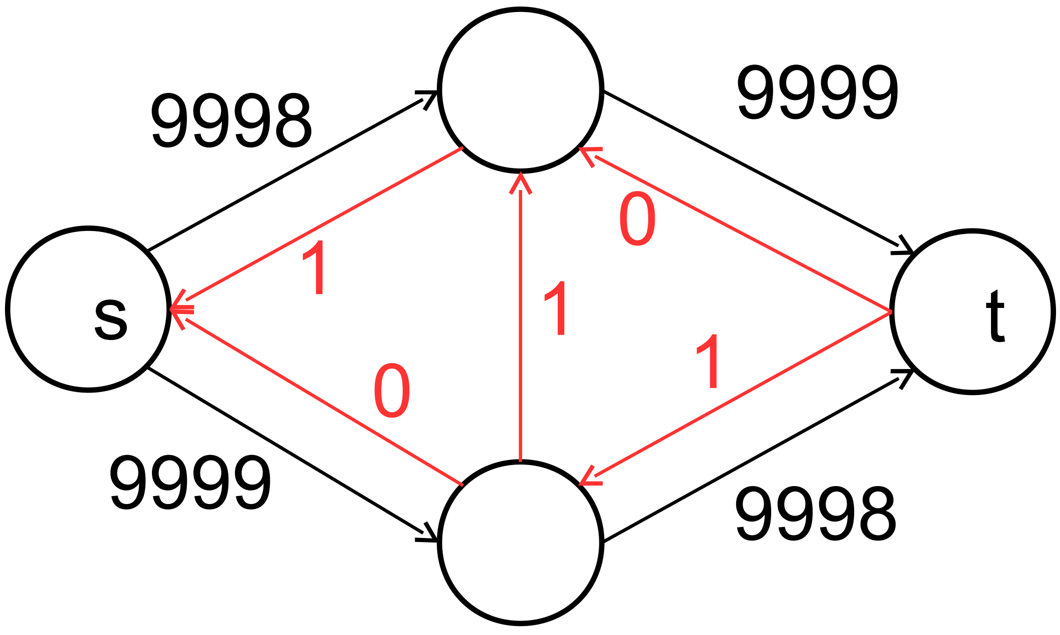

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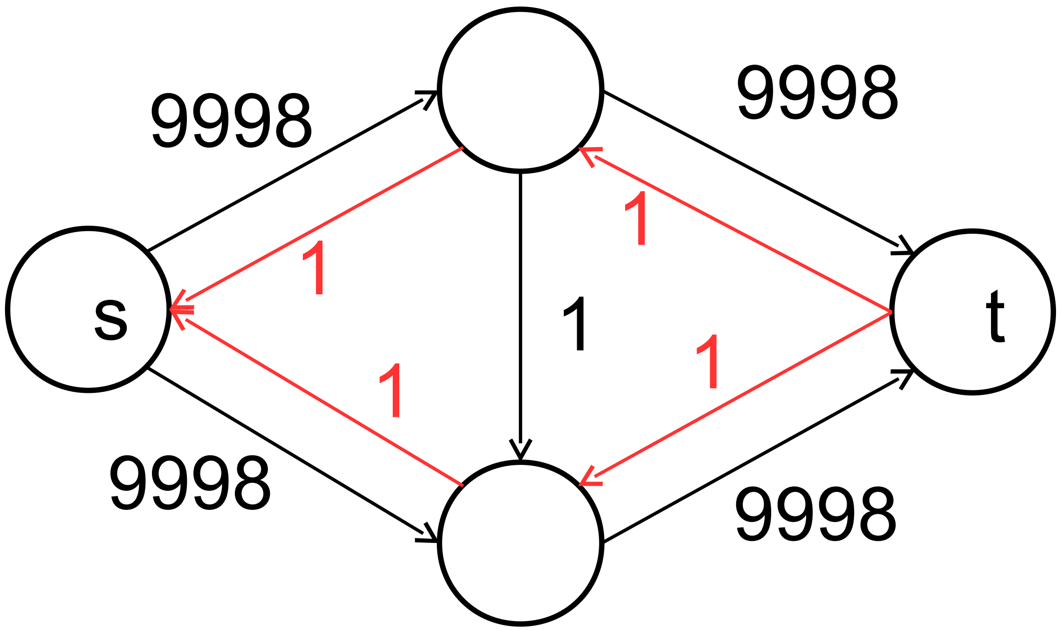

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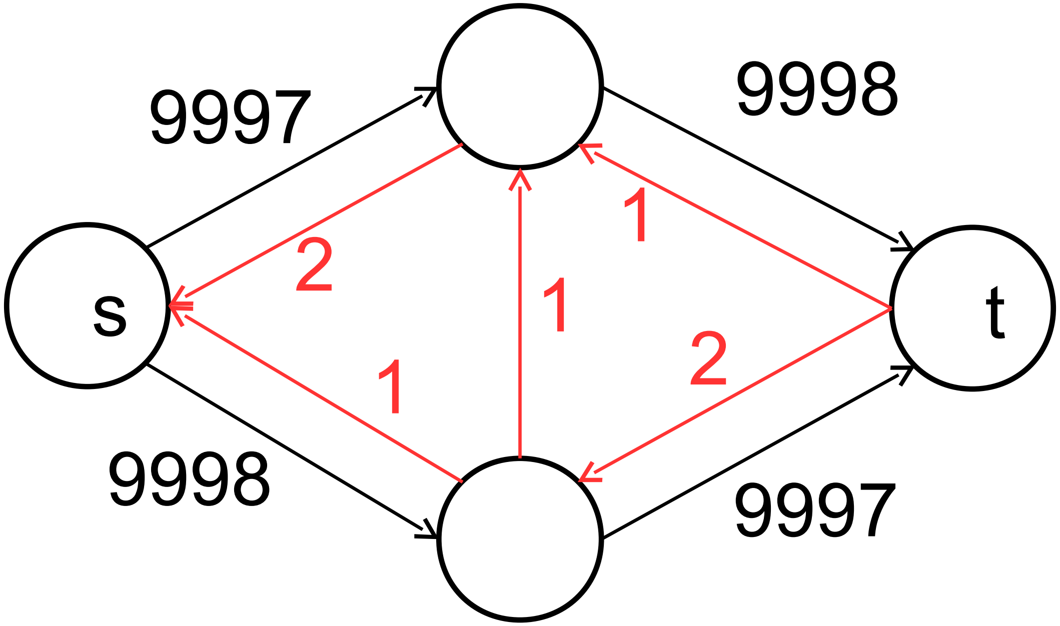

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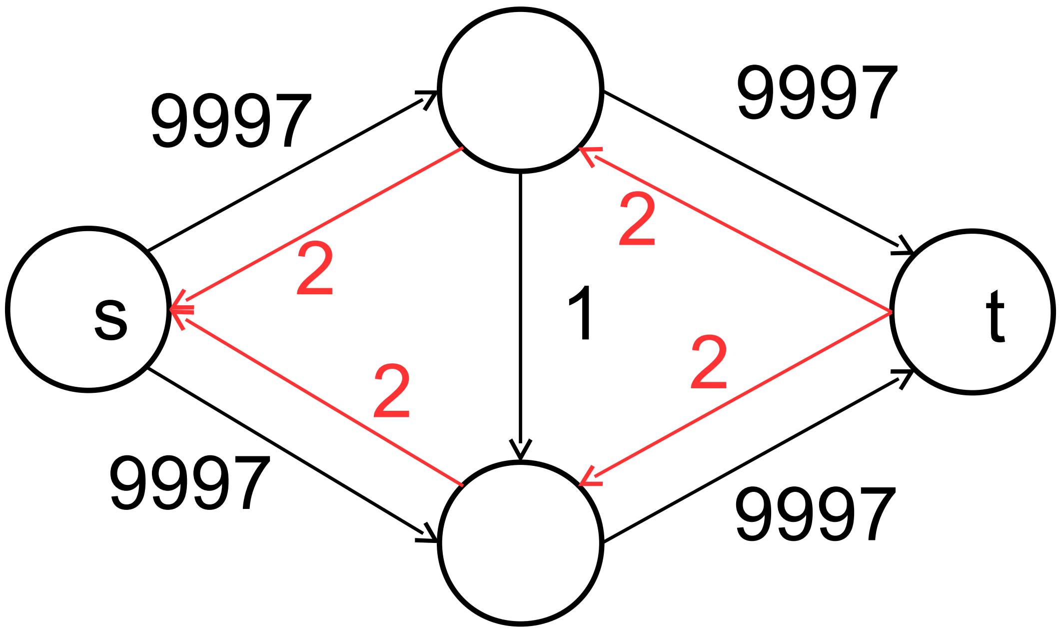


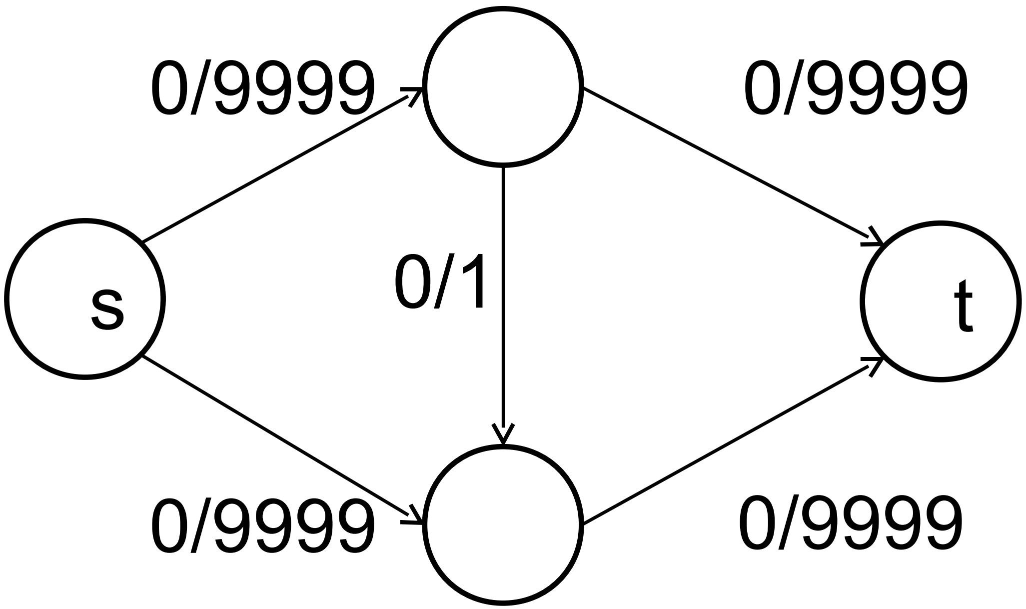
Edmonds—Karp algorithm

- Same as Ford-Fulkerson, but each time use a shortest path in residual network

Let's run it on the previous example.

Edmonds—Karp algorithm

Edmonds—Karp on previous example:



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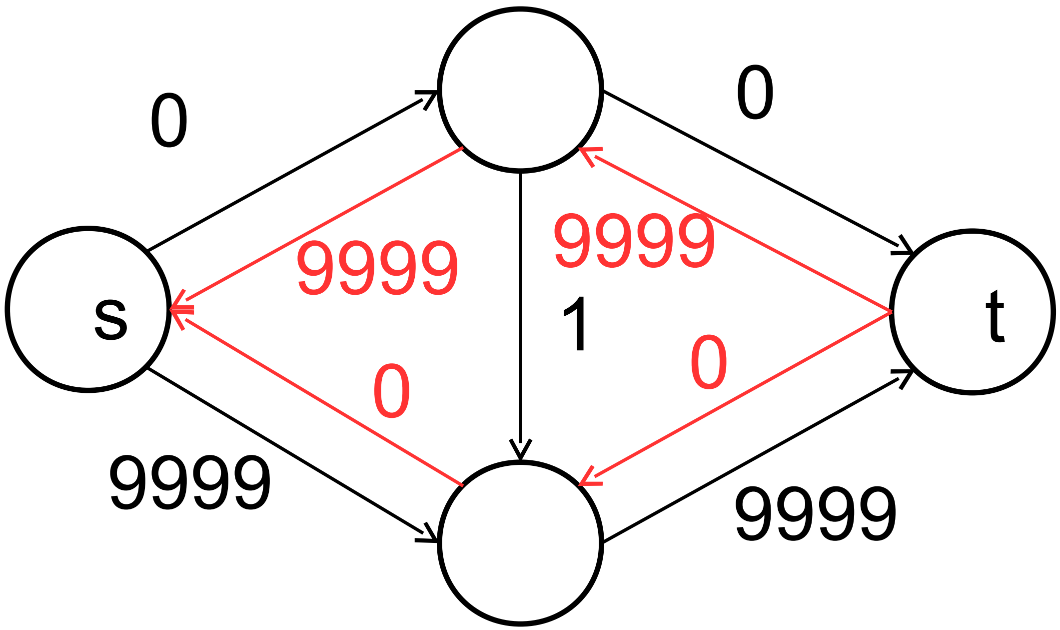

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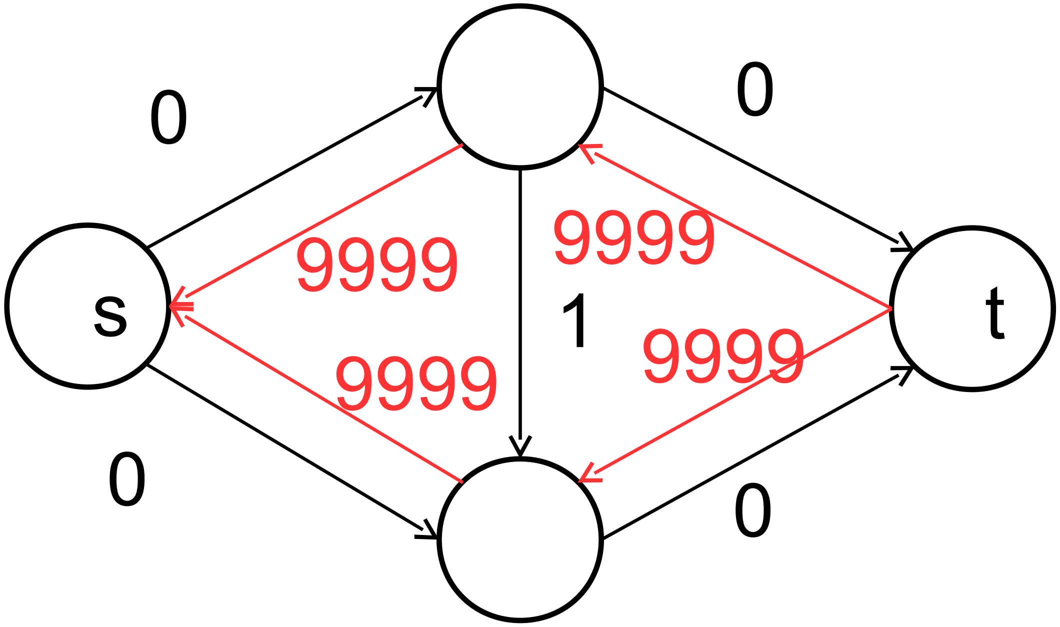

Analysis of Edmonds—Karp algorithm

Correctness: ???

Analysis of Edmonds—Karp algorithm

Correctness: Follows from previous analysis.

Running time:

Analysis of Edmonds—Karp algorithm

Correctness: Follows from previous analysis.

Running time:

Let $\delta_f(s,v)$ be the distance from s to v in G_f

Lemma: Each time we update the flow, $\delta_f(s,v)$ does not decrease

i.e. $\delta_{f'}(s,v) \geq \delta_f(s,v)$ for every v, for every f' after f

Meaning: shortest path distances increase after each iteration.

Proof: We show that $\delta_{f'}(s,v) \geq \delta_f(s,v)$ if f' is right after f.

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Suppose not. Let v be the vertex v among $B:=\{v: \delta_{f'}(s,v) < \delta_f(s,v)\}$ such that $\delta_{f'}(s,v)$ is minimal.

Take shortest path s ~~~> u ~> v in $G_{f'}$.

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We claim that $(u,v) \notin G_f$.

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$$\delta_f(s,v) \quad \leq \delta_f(s,u) + 1$$

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Contradicting our assumption. So we have (u,v) in $G_{f'}$ but $(u,v) \notin G_f$

That means ???

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That means the augmentation from f to f' must have (v, u) on the augmented path.

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But augmentations are along shortest paths, so

$$\delta_f(s,v) \quad = \delta_f(s,u) - 1$$

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which contradicts our assumption. □

Theorem: Total # of flow augmentations in Edmond—Karp is O(|V| |E|)

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Proof: Call (u,v) critical in residual network G_f if $c_f(u,v)$ is minimial among all edges on an augmenting path.

(i.e. (u,v) is the bottleneck edge of the augmenting path).

Theorem: Total # of flow augmentations in Edmond—Karp is $O(|V| |E|)$

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Note there is always a critical edge. After the flow augmentation, the edge will be ???

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Note there is always a critical edge. After the flow augmentation, the edge will be saturated and thus will disappear from the network.

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(again because you augment along shortest path).

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Remark

This is NOT saying every augmentation increases the distance of some node

This is saying every 2 augmentations of same edge increase distance of starting point.

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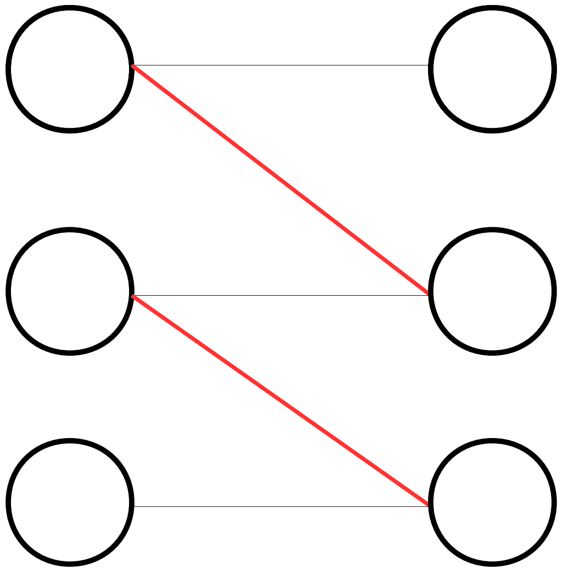
Example:



|M| = ???

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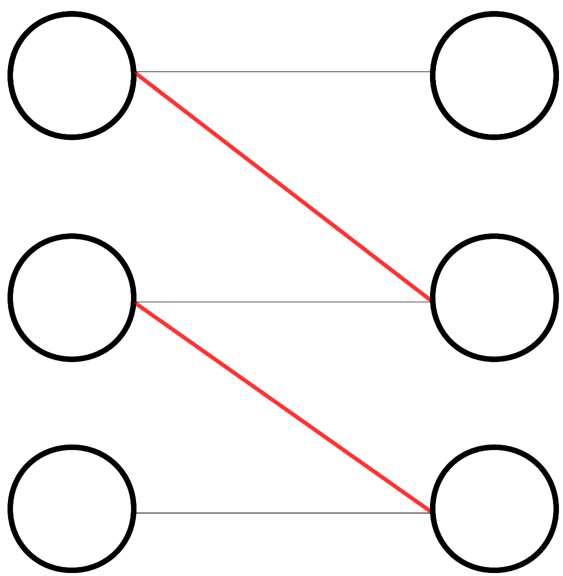
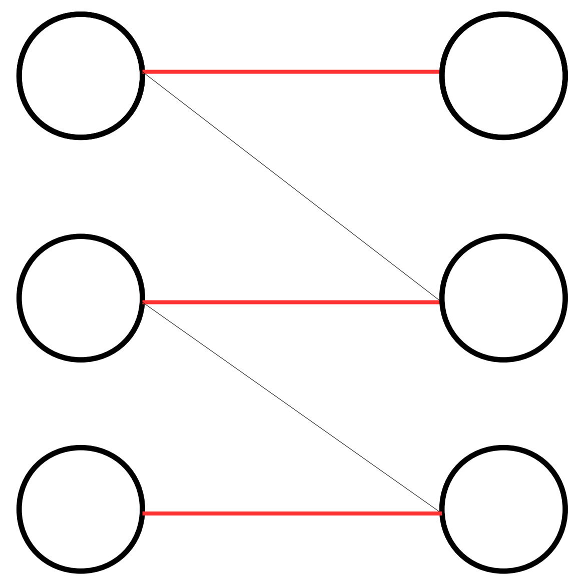


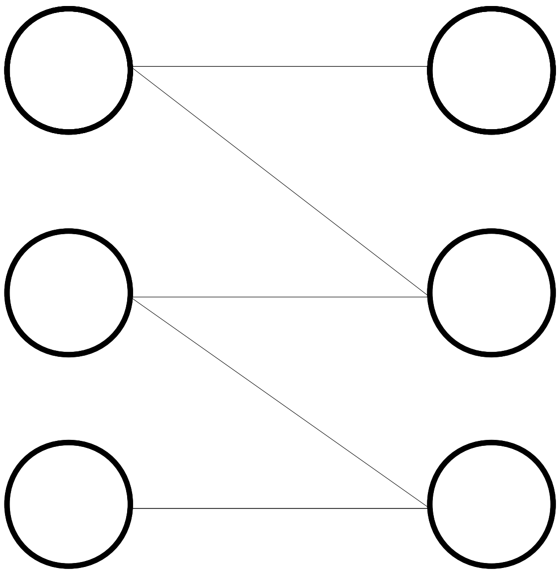
|M| = 2



|M| = 3

Application: Maximum matching

We turn it into a maximum flow problem



Application: Maximum matching

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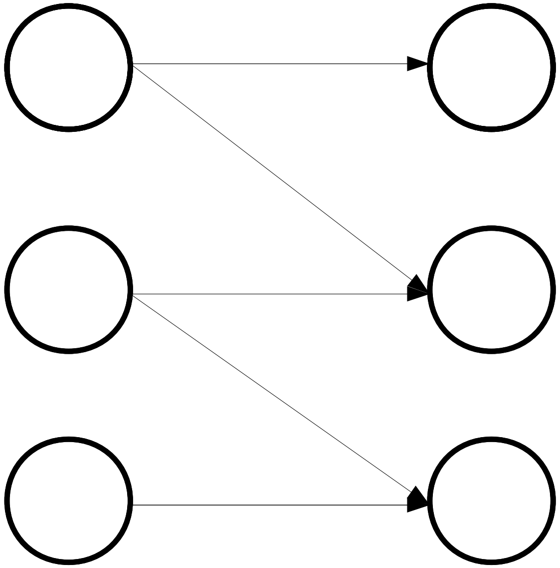
- Direct every edge from L to R



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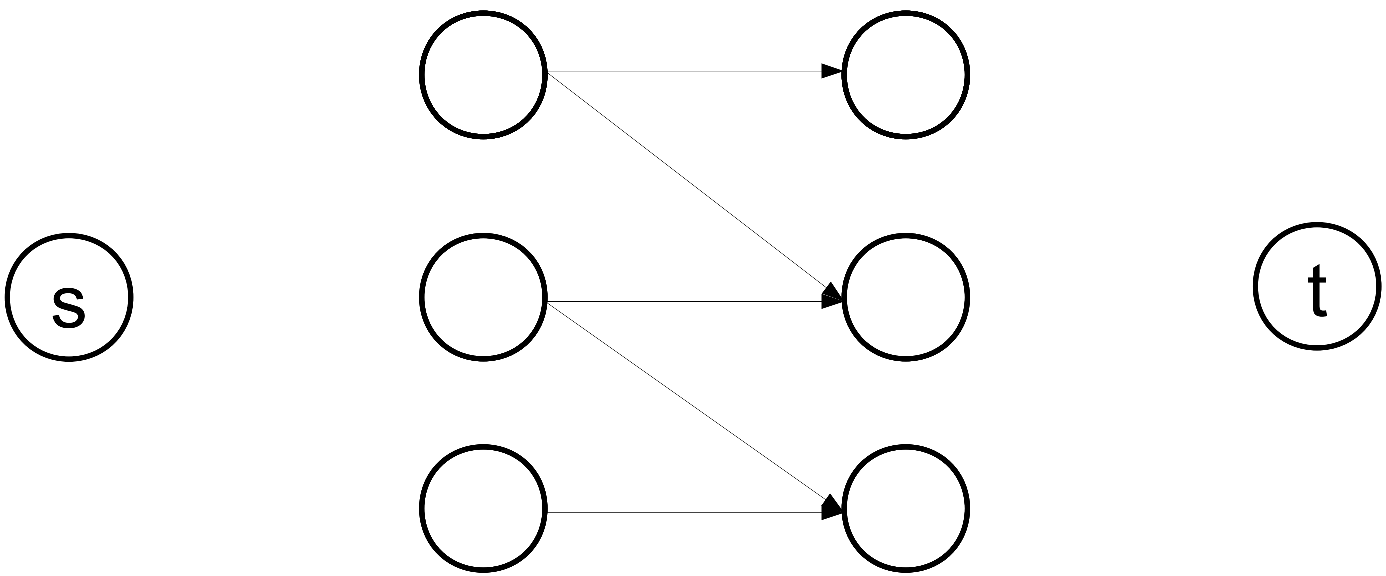
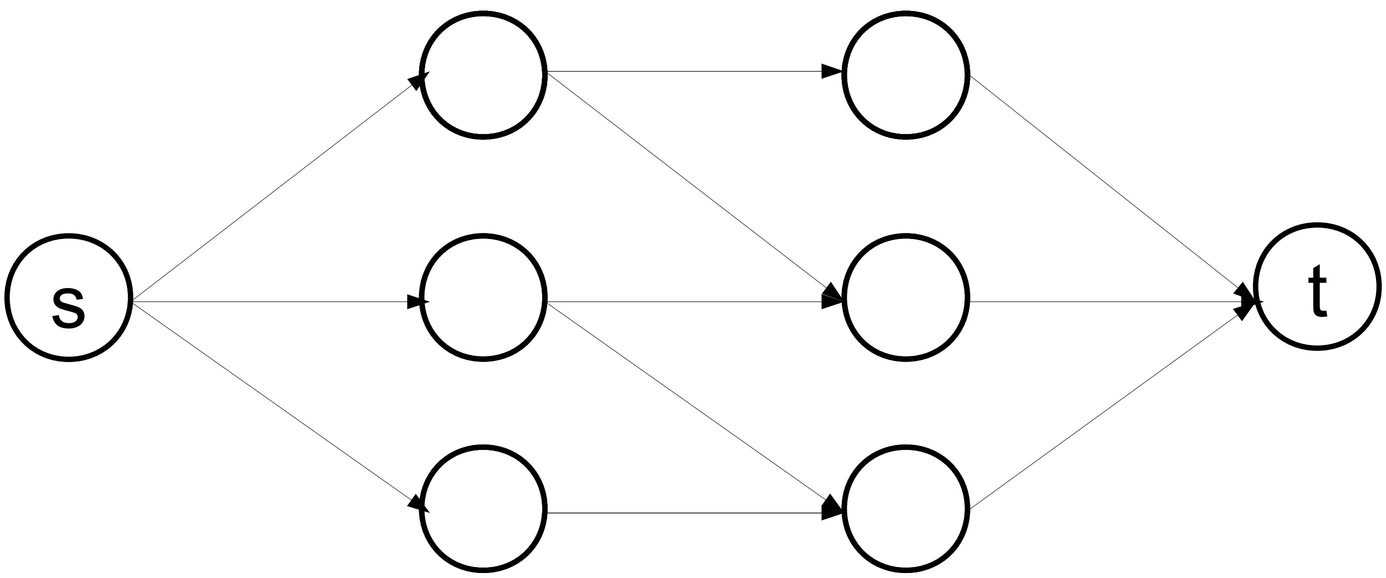
- Direct every edge from L to R
- Add a source and a sink



Application: Maximum matching

We turn it into a maximum flow problem

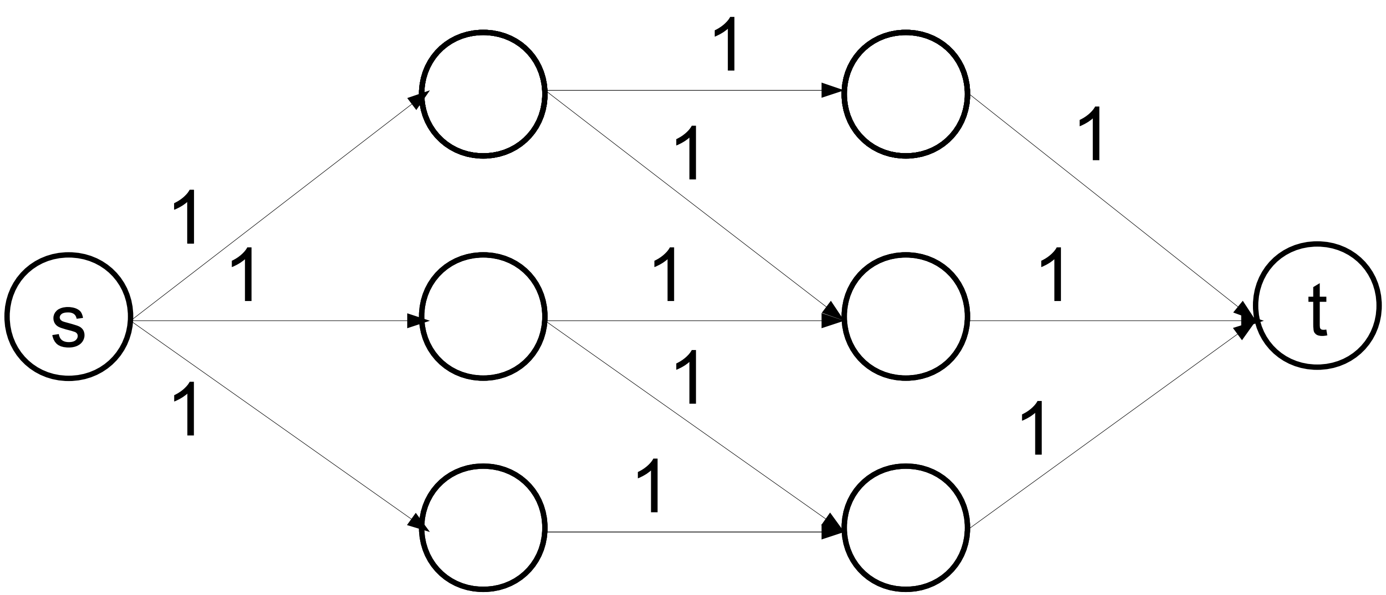
- Direct every edge from L to R
- Add a source and a sink
- Add edges between s and vertices in L, and between t and the vertices in R



Application: Maximum matching

We turn it into a maximum flow problem

- Direct every edge from L to R
- Add a source and a sink
- Add edges between s and vertices in L, and between t and the vertices in R
- Set capacities of all edges to 1



Application: Maximum matching

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Claim: $\exists$Matching of size M $\Leftrightarrow$ $\exists$flow of value M

Proof:

$\Rightarrow$:

Application: Maximum matching

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Claim: ∃Matching of size M ⇔ ∃flow of value M

Proof:

⇒: Clear

⇐: Let f be a flow. There are |f| units.

The |f| units form |f| edge-disjoint paths from s to t because ???

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$\Leftarrow$: Let f be a flow. There are $|f|$ units.

The $|f|$ units form $|f|$ edge-disjoint paths from s to t because all edges have capacity 1.

So there are exactly $|f|$ edges (u,v) in $(L \cup R, E)$ with $f(u,v) = 1$.

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No two vertices in L and R shares these edges because ???

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So there are exactly |f| edges (u,v) in $(L \cup R, E)$ with f(u,v) = 1.

No two vertices in L and R shares these edges because each edge touching s or t has capacity 1. So the |f| edges form a matching.□