Divide and conquer

Philip II of Macedon

Divide and conquer

1) Divide your problem into subproblems

2) Solve the subproblems recursively, that is,
run the same algorithm on the subproblems
(when the subproblems are very small, solve them from scratch)

3) Combine the solutions to the subproblems into a solution of the original problem

Divide and conquer

Recursion is “top-down” start from big problem, and make it smaller

Every divide and conquer algorithm can be written without recursion, in an iterative “bottom-up” fashion:
solve smallest subproblems, combine them, and continue

Sometimes recursion is a bit more elegant

Merge sort

```
Mergesort (low, high) {
   if (high – low < 1) return;  //Smallest subproblems

   //Divide into subproblems low..split and split..high
   split = (low+high) / 2;

   MergeSort(low, split);        //Solve subproblem recursively
   MergeSort(split+1, high);  //Solve subproblem recursively

   //Combine solutions
   merge sorted sequences low..split and split+1..high  into
   the single sorted sequence low..high
}
```

Merge example

Merge sorted sequences A1 and A2 into B

A1 = [3 8 10 21 57]

A2 = [7 13 14 17]

B =   [

```
Mergesort (low, high) {
  if (high-low < 1) return;
  split = (low+high) / 2;
  MergeSort(low, split);
  MergeSort(split+1, high);

  Merge
}
```

```
Merge A1[1..s1], A2[1..s2]
into B[1..(s1+s2)]

i1=i2=j=1;

while i1 < s1 and i2 < s2
  if (A1[i1] < A2[i2])
    B[j++] = A1[i1++])
  else
    B[j++] = A2[i2++])
end while;

Put what left in A1 or A2 in B
```

Analysis of running time

Merging A1[1..s1], A2[1..s2]

into B[1..(s1+s2)] takes time ?

```
MergeSort(low, high) {
 if (high-low < 1) return;
 split = (low+high) / 2;
 MergeSort(low, split);
 MergeSort(split+1, high);
 Merge low..split and
            split+1 ..high
}
```

Analysis of running time

Merging A1[1..s1], A2[1..s2]

into B[1..(s1+s2)] takes time

c•(s1+s2) for some constant c

```
MergeSort(low, high) {
  if (high-low < 1) return;
  split = (low+high) / 2;
  MergeSort(low, split);
  MergeSort(split+1, high);
  Merge low..split and
             split+1 ..high
}
```

Let T(n) be time for merge sort on A[1..n]

Recurrence relation T(n) = ?

Analysis of running time

Merging A1[1..s1], A2[1..s2]

into B[1..(s1+s2)] takes time

c•(s1+s2) for some constant c

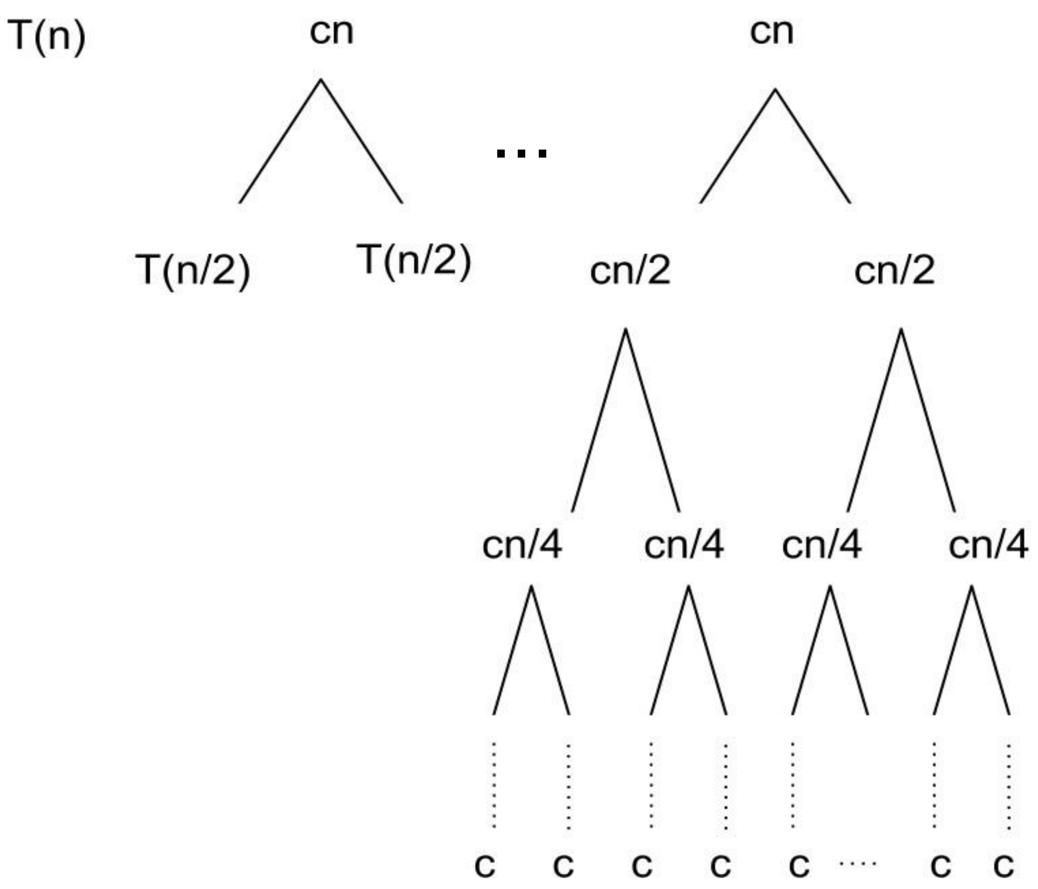
```
MergeSort(low, high) {
 if (high-low < 1) return;
 split = (low+high) / 2;
 MergeSort(low, split);
 MergeSort(split+1, high);
 Merge low..split and
           split+1 ..high
}
```

Let T(n) be time for merge sort on A[1..n]

Recurrence relation T(n) = 2 T(n/2) + c•n

Solving recurrence $T(n) = 2\, T(n/2) + c\, n$

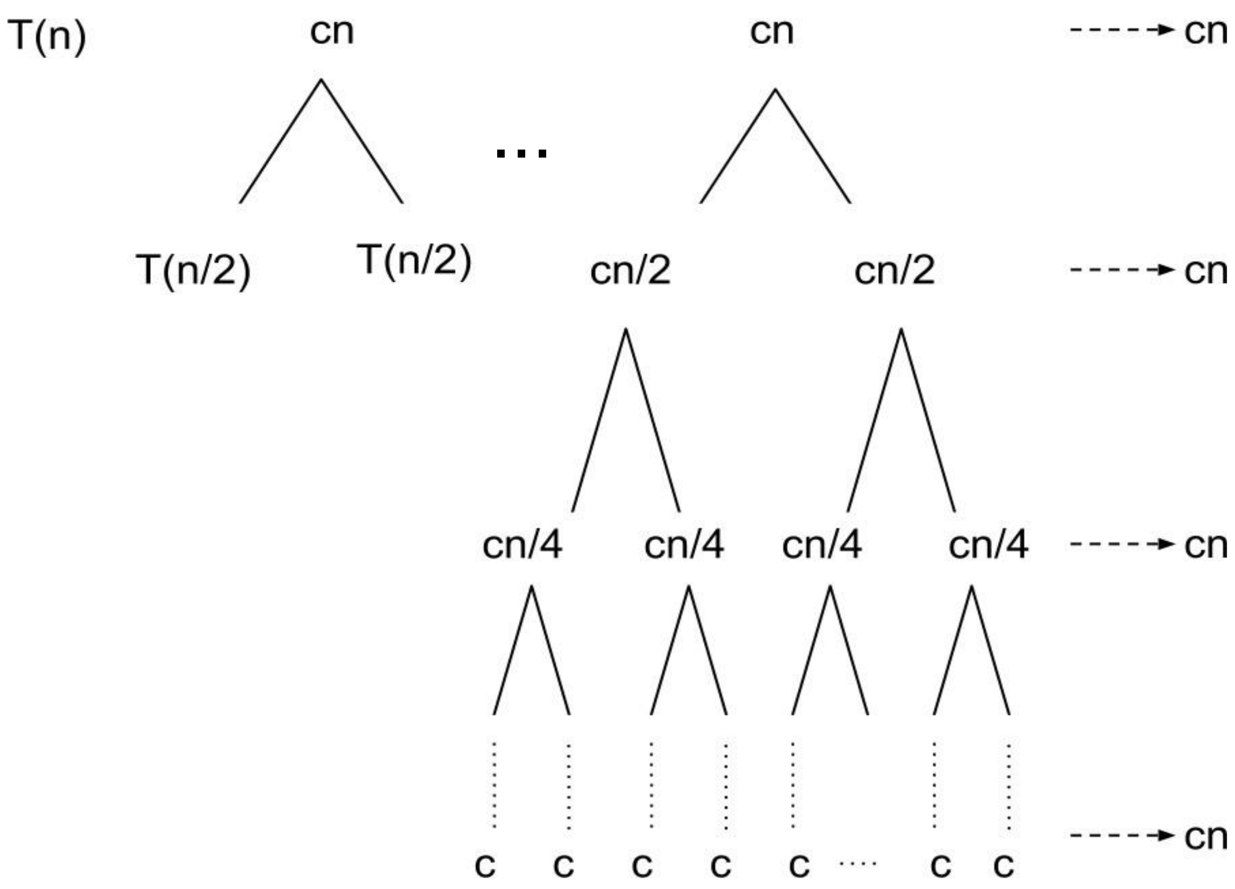
Expand recurrence to obtain recursion tree



Sum of costs at level i is ?

Solving recurrence T(n) = 2 T(n/2) + c n

Expand recurrence to obtain recursion tree



Sum of costs at level i is $2^i\ cn/2^i \quad = cn$

Numbers of levels is ?

Analysis of space

How many extra array elements we need?

At least n to merge

It can be implemented to use O(n) space.

Quick sort

```
QuickSort(lo, hi) {   // Sorts array A
  if (hi-lo < 1) return;
  partition(lo, hi) and return split;
  QuickSort(lo, split-1);
  QuickSort(split+1, hi);
}
```

Partition permutes A[lo..hi] so that

each element in A[lo.. split] is ≤ A[split],

each element in A[split+1.. hi] is > A[split].

Partition(A[lo.. hi]) For simplicity, assume distinct elements

```
Pick pivot index p.  // We will explain later how
Swap A[p] and A[hi]; i = lo-1; j = hi;
Repeat {  //Invariant: A[lo.. i] < A[hi], A[j.. hi-1] >A[hi]
  Do i++ while A[i] < A[hi];
  Do j-- while A[j]  > A[hi] and i < j;
  If i < j then swap A[i] and A[j]
  Else {
    swap A[i] and A[hi]; return i
  }
}
```

Running time: O(hi – lo)

Analysis of running time

T(n) = number of comparisons on an array of length n.

T(n) depends on the choice of the pivot index p

- Choosing pivot deterministically
- Choosing pivot randomly

```
QuickSort(lo, hi)  {
  if (hi-lo <= 1) return;
  partition(lo, hi) and  return split,
  QuickSort(lo, split-1);
  QuickSort(split+1, hi);
  }
```

Analysis of running time

T(n) = number of comparisons on an array of length n.

- Choosing pivot deterministically:

  the worst case happens when one sub-array is empty and the other is of size n-1, in this case :

  $T(n) = T(n-1) + T(0) + c\,n$

  $= ?$

Analysis of running time

$T(n)$ = number of comparisons on an array of length n.

- Choosing pivot deterministically:

  the worst case happens when one sub-array is empty
  and the other is of size n-1, in this case :

  $T(n)= T(n-1) + T(0) + c\,n$

  $= \Theta(n^2)$.

- Choosing pivot randomly we can guarantee

  $T(n) = O(n \log n)$ with high probability

Randomized-Quick sort:

```
R-QuickSort(low, high) {
  if (high-low < 1) return;
  R-partition(low, high) and return split,
  R-QuickSort(low, split-1);
  R-QuickSort(split+1, high);
}
```

```
R-partition(low, high)
   Pick pivot index p uniformly in {low, low+1, … high}
   Then partition as before
```

We bound the total time spent by Partition

- Definition: X is the number of comparisons

- Next we bound the expectation of X, E[X]

- Rename array A as $z_1, z_2, \ldots, z_n$, with z_i being the i-th smallest

- Note: each pair of elements z_i, z_j is compared at most once.

  Why?

- Rename array A as $z_1, z_2, \ldots, z_n$, with z_i being the i-th smallest

- Note: each pair of elements z_i, z_j is compared at most once.

  Elements are compared with the pivot.

  An element is a pivot at most once.

- Define indicator random variables

  X_{ij}:= 1 if { z_i is compared to z_j }

  X_{ij}:= 0 otherwise

- Note: X = ?

- Rename array A as $z_1, z_2, \ldots, z_n$, with z_i being the i-th smallest

- Note: each pair of elements z_i, z_j is compared at most once.

  Elements are compared with the pivot.

  An element is a pivot at most once.

- Define indicator random variables

  $X_{ij} := 1$ if { z_i is compared to z_j }

  $X_{ij} := 0$ otherwise

- Note: $X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij}$.

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \,.$$

Taking expectation, and using linearity:

$$E[X] = E\left(\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{ij} \right)$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E\,[X_{ij}\,]$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\,\{z_i \text{ is compared to } z_j\}$$

- Pr {z_i is compared to z_j}=?

- If some element y, $z_i < y < z_j$ chosen as pivot,

  z_i and z_j can not be compared.

  Why?

- Pr {z_i is compared to z_j}=?

- If some element y, $z_i < y < z_j$ chosen as pivot,

  z_i and z_j can not be compared.

  Because after partition z_i and z_j will be in two different parts.

- Definition: Z_{ij} is = { z_i , z_{i+1} , …, z_j }

- z_i and z_j are compared if

  first element chosen as pivot from Z_{ij} is either z_i or z_j.

Pr {z_i is compared to z_j} = Pr [z_i or z_j is first pivot chosen from Z_{ij}]

Pr {z_i is compared to z_j} = Pr [z_i or z_j is first pivot chosen from Z_{ij}]
= Pr [z_j is first pivot chosen from Z_{ij}]
+ Pr [z_i is first pivot chosen from Z_{ij}]

$$\begin{aligned}\Pr\{z_i \text{ is compared to } z_j\} &= \Pr[z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}] \\ &= \Pr[z_i \text{ is first pivot chosen from } Z_{ij}] \\ &\quad + \Pr[z_j \text{ is first pivot chosen from } Z_{ij}] \\ &= 1/(j-i+1) + 1/(j-i+1) = 2/(j-i+1)\ .\end{aligned}$$

$$\Pr\{z_i \text{ is compared to } z_j\} = \Pr[z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}]$$

$$= \Pr[z_i \text{ is first pivot chosen from } Z_{ij}]$$

$$+ \Pr[z_j \text{ is first pivot chosen from } Z_{ij}]$$

$$= 1/(j-i+1) + 1/(j-i+1) = 2/(j-i+1) .$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2/(j-i+1) \quad .$$

$$\begin{aligned}\Pr\{z_i \text{ is compared to } z_j\} &= \Pr[z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}] \\ &= \Pr[z_i \text{ is first pivot chosen from } Z_{ij}] \\ &\quad + \Pr[z_j \text{ is first pivot chosen from } Z_{ij}] \\ &= 1/(j-i+1) + 1/(j-i+1) = 2/(j-i+1)\ .\end{aligned}$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2/(j-i+1) \quad = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} 2/(k+1)$$

$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} 2/k$$

$$\Pr\{z_i \text{ is compared to } z_j\} = \Pr[z_i \text{ or } z_j \text{ is first pivot chosen from } Z_{ij}]$$

$$= \Pr[z_i \text{ is first pivot chosen from } Z_{ij}]$$

$$+ \Pr[z_j \text{ is first pivot chosen from } Z_{ij}]$$

$$= 1/(j-i+1) + 1/(j-i+1) = 2/(j-i+1) .$$

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\}$$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2/(j-i+1) = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} 2/(k+1)$$

$$< \sum_{i=1}^{n-1} \sum_{k=1}^{n} 2/k = \sum_{i=1}^{n-1} O(\log n) = O(n \log n).$$

Expected running time of Randomized-QuickSort is O(n log n).

An application of Markov's inequality

Let T be the running time of Randomized Quick sort.

We just proved $E[T] \leq c\ n \log n$, for some constant c.

Hence, $\Pr[\ T > 100\ c\ n \log n] <$?

An application of Markov's inequality

Let T be the running time of Randomized Quick sort.

We just proved E[T] ≤ c n log n, for some constant c.

Hence, Pr[T > 100 c n log n] < 1/100

Markov's inequality useful to translate bounds on the expectation in bounds of the form: “It is unlikely the algorithm will take too long.”

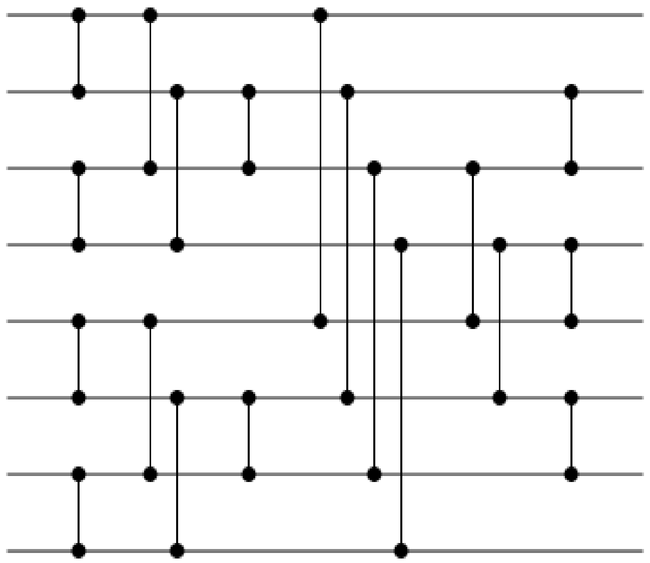
Oblivious Sorting

Want an algorithm that only accesses the input via

Compare-exchange(x,y)

Compares A[x] and A[y] and swaps them if necessary

We call such algorithms oblivious. Useful if you want to sort with a (non-programmable) piece of hardware



Did we see any oblivious algorithms?

Oblivious Mergesort

This is just like Merge sort except that the merge subroutine is replaced with a subroutine whose comparisons do not depend on the input.

Assumption:

Size of the input sequence, n, is a power of 2.

Convenient to index from 0 to n-1

```
Oblivious-Mergesort (A[0..n-1])
{  if n > 1 then
   Oblivious-Mergesort(A[0.. n/2-1]);
   Oblivious-Mergesort(A [n/2 .. n-1]);
   odd-even-Merge(A[0..n-1]);
}
```

Same structure as Mergesort

But Odd-even-merge is more complicated, recursive

```
odd-even-merge(A[0..n-1]); {
 if n = 2 then compare-exchange(0,1);
 else {
   odd-even-merge(A[0,2 .. n-2]); //even subsequence

   odd-even-merge(A[1,3,5 .. n-1]); //odd subsequence

   for i ∈ {1,3,5, … n-1} do
       compare-exchange(i, i +1);
}
```

Compare-exchange(x,y) compares A[x] and A[y] and
swaps them if necessary

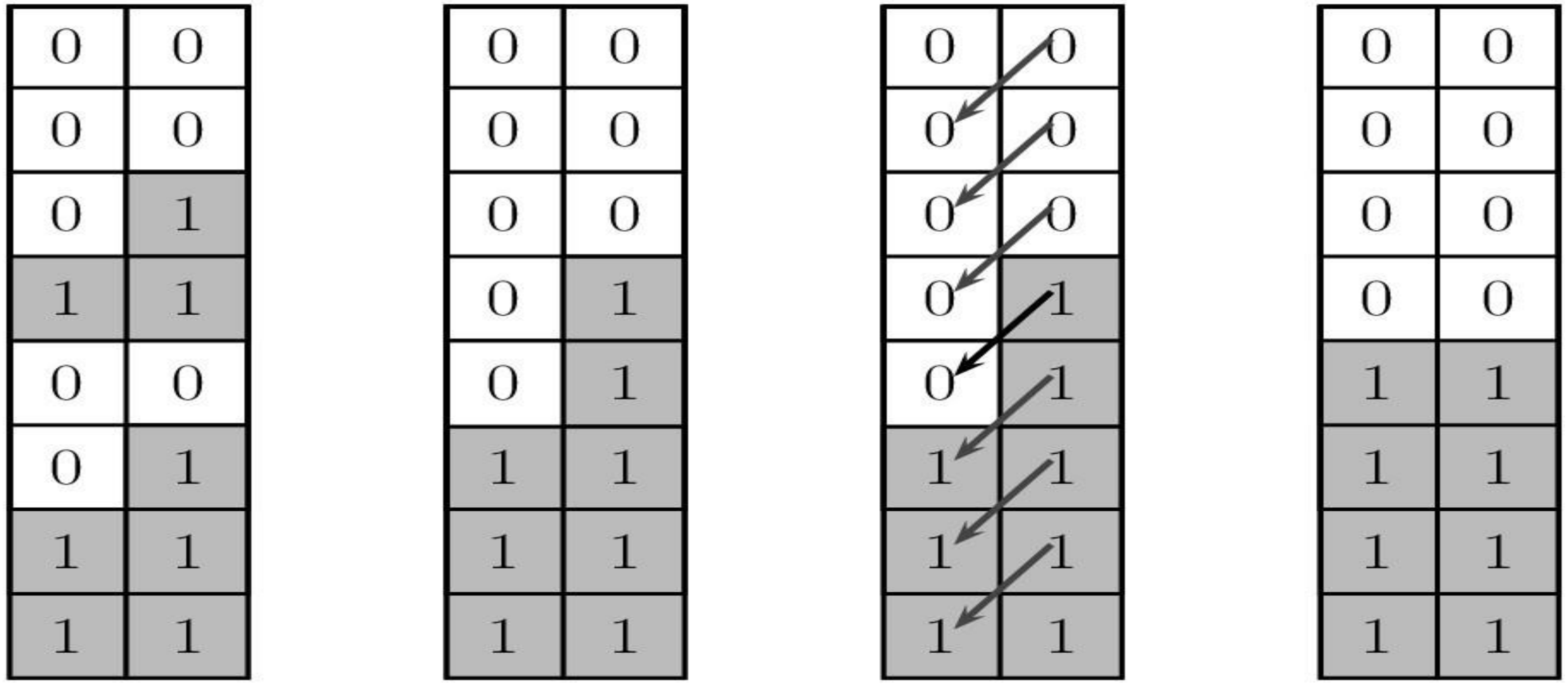
Merges correctly if A[0.. n/2-1] and A[n/2 .. n-1] are sorted

```
odd-even-merge(A[0..n-1]);
  if n = 2 then compare-exchange(0,1);
  else
     odd-even-merge(A[0,2 .. n-2]);
     odd-even-merge(A[1,3,5 .. n-1]);
     for $i \in \{1,3,5, \dots n-1\}$ do
       compare-exchange(i, i +1);
```

0-1 principle: If algorithm works correctly on sequences of 0 and 1, then it works correctly on all sequences

True when input only accessed through compare-exchange

```
odd-even-merge(A[0..n-1]);
  if n = 2 then compare-exchange(0,1);
  else
     odd-even-merge(A[0,2 .. n-2]);
     odd-even-merge(A[1,3,5 .. n-1]);
     for i ∈ {1,3,5, … n-1} do
        compare-exchange(i, i +1);
```



0	0
0	0
0	1
1	1
0	0
0	1
1	1
1	1

0	0
0	0
0	0
0	1
0	1
1	1
1	1
1	1

0	0
0	0
0	0
0	1
0	1
1	1
1	1
1	1

0	0
0	0
0	0
0	0
1	1
1	1
1	1
1	1

Analysis of running time

$T(n)$ = number of comparisons.

$= 2T(n/2) + T'(n)$.

$T'(n)$ = number of operations in odd-even-merge

$= 2T'(n/2) + c\,n = ?$

```
Oblivious-Mergesort(A[0..n-1])
if n > 1 then
  Oblivious-Mergesort(A[0.. n/2-1]);
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  Odd-even-merge(A[0..n-1]);
```

```
odd-even-merge(A[0..n-1]);
  if n = 2 then
    compare-exchange(0,1);
  else
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    odd-even-merge(A[1,3,5 .. n-1]);
    for i $\in$ {1,3,5, … n-1} do
      compare-exchange(i, i +1);
```

Analysis of running time

$T(n)$ = number of comparisons.

$= 2T(n/2) + T'(n)$

$= 2T(n/2) + O(n \log n)$.

= ?

$T'(n)$ = number of operations in odd-even-merge

$= 2T'(n/2) + c\,n = O(n \log n)$.

```
Oblivious-Mergesort(A[0..n-1])
if n > 1 then
  Oblivious-Mergesort(A[0.. n/2-1]);
  Oblivious-Mergesort(A [n/2 .. n-1]);
  Odd-even-merge(A[0..n-1]);
```

```
odd-even-merge(A[0..n-1]);
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    for i ∈ {1,3,5, … n-1} do
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```

Analysis of running time

$T(n)$ = number of comparisons.

$= 2T(n/2)+ T'(n)$

$= 2T(n/2)+ O(n \log n)$

$= O(n \log^2 n)$.

```
Oblivious-Mergesort(A[0..n-1])
if n > 1 then
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odd-even-merge(A[0..n-1]);
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    odd-even-merge(A[1,3,5 .. n-1]);
    for i ∈ {1,3,5, … n-1} do
      compare-exchange(i, i +1);
```

Sorting algorithm	Time	Space	Assumption/ Advantage
Bubble sort	$\Theta(n^2)$	$O(1)$	Easy to code
Counting sort	$\Theta(n+k)$	$O(n+k)$	Input range is [0..k]
Radix sort	$\Theta(d(n+k))$	$O(n+k)$	Inputs are d-digit integers in base k
Quick sort (deterministic)	$O(n^2)$	$O(1)$	
Quick sort (Randomized)	$O(n \log n)$	$O(1)$	
Merge sort	$O\ (n \log n)$	$O(n)$	
Oblivious merge sort	$O\ (n \log^2 n)$	$O(1)$	Comparisons are independent of input

Sorting is still open!

- Input: n integers in {0, 1, …, 2^w - 1}

- Model: Usual operations (+, *, AND, …) on w-bit integers in constant time

- Open question: Can you sort in time O(n)?

- Best known time: O(n log log n)

Next

- View other divide-and-conquer algorithms
- Some related to sorting

Selecting h-th smallest element

- Definition: For array A[1..n] and index h,

  S(A,h) := h-th smallest element in A,

  = B[h] for B = sorted version of A

- S(A,(n+1)/2) is the median of A, when n is odd

- We show how to compute S(A,h) with O(n) comparisons

Computing S(A,h)

- Divide array in consecutive blocks of 5:
  A[1..5], A[6..10], A[11..15] , ...

- Find median of each
  m_1 = S(A[1..5],3), m_2 = S(A[6..10],3), m_3 = S(A[11..15],3)

- Find median of medians, x= S([m_1, m_2, ..., $m_{n/5}$], (n/5+1)/2)

- Partition A according to x. Let x be in position k

- If h = k return x, if h < k return S(A[1..k-1],h),
  if h > k return S(A[k+1..n],h-k-1)

- Divide array in consecutive blocks of 5
- Find median of each

  m_1 = S(A[1..5],3), m_2 = S(A[6..10],3), m_3 = S(A[11..15],3)
- Find median of medians, x= S([m_1, m_2, ..., $m_{n/5}$], (n/5+1)/2)
- Partition A according to x. Let x be in position k
- If h = k return x, if h < k return S(A[1..k-1],h),

  if h > k return S(A[k+1..n],h-k-1)

- Running time:

  When partition, half the medians m_i will be ≥ x.

  Each contributes ≥ ? elements from their 5.

- Divide array in consecutive blocks of 5
- Find median of each

  m_1 = S(A[1..5],3), m_2 = S(A[6..10],3), m_3 = S(A[11..15],3)
- Find median of medians, x= S([m_1, m_2, ..., $m_{n/5}$], (n/5+1)/2)
- Partition A according to x. Let x be in position k
- If h = k return x, if h < k return S(A[1..k-1],h),

  if h > k return S(A[k+1..n],h-k-1)

- Running time:

  When partition, half the medians m_i will be ≥ x.

  Each contributes ≥ 3 elements from their 5.

  So we recurse on ≤ ??

- Divide array in consecutive blocks of 5
- Find median of each

  $m_1 = S(A[1..5],3)$, $m_2 = S(A[6..10],3)$, $m_3 = S(A[11..15],3)$
- Find median of medians, $x = S([m_1, m_2, ..., m_{n/5}], (n/5+1)/2)$
- Partition A according to x. Let x be in position k
- If h = k return x, if h < k return S(A[1..k-1],h),

  if h > k return S(A[k+1..n],h-k-1)

- Running time:

  When partition, half the medians m_i will be ≥ x.

  Each contributes ≥ 3 elements from their 5.

  So we recurse on ≤ 7n/10 elements

  $T(n) \leq T(n/5) + T(7n/10) + O(n)$

  This implies $T(n) = O(n)$

How to solve recurrence $T(n) \leq T(n/5) + T(7n/10) + cn$

Guess $T(n) \leq an$, for some constant a

Does guess hold for recurrence?

$an \geq an/5 + a7n/10 + cn$
$\Leftrightarrow$ (divide by an)
$1 \geq 1/5 + 7/10 + c/a$
$\Leftrightarrow$
$1/10 \geq c/a$

This is true for $a \geq 10c$. □

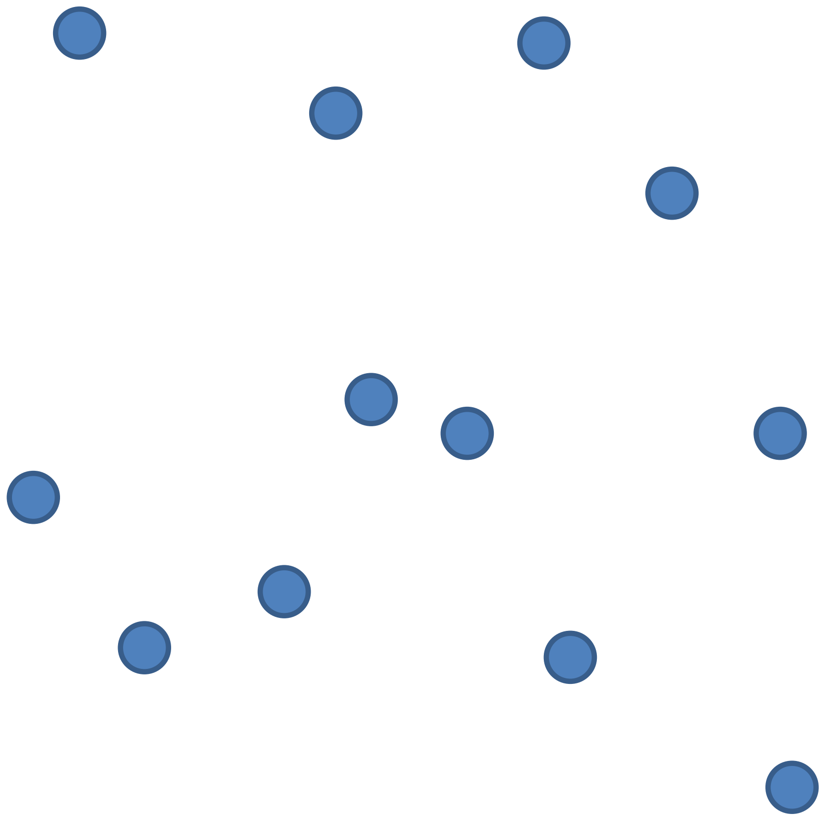
Closest pair of points

Input:

Set P of n points in the plane

Output:

Two points x_1 and x_2 with the shortest (Euclidean) distance from each other.



Trivial algorithm: Compute every distance: $\Omega(n^2)$ time

Next: Clever algorithm with $O(n \log(n))$ time

Closest pair of points

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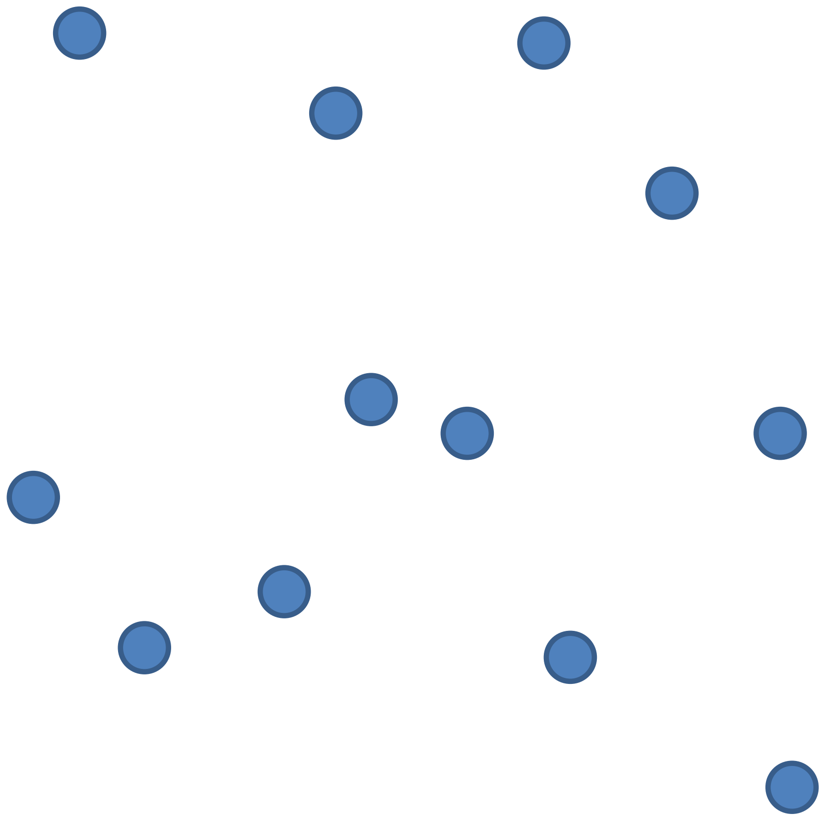
Set P of n points in the plane

Output:

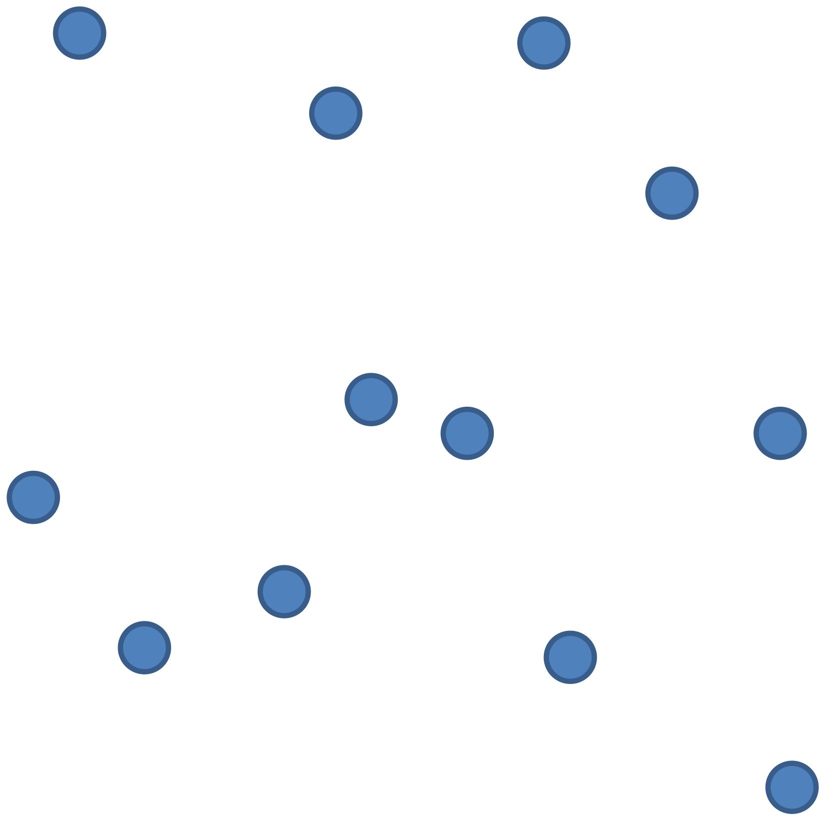
Two points x_1 and x_2 with the shortest (Euclidean) distance from each other.



- For the following algorithm we assume that we have two arrays X and Y, each containing all the points of P.
- X is sorted so that the x-coordinates are increasing
- Y is sorted so that y-coordinates are increasing.

Closest pair of points

Divide:



Closest pair of points

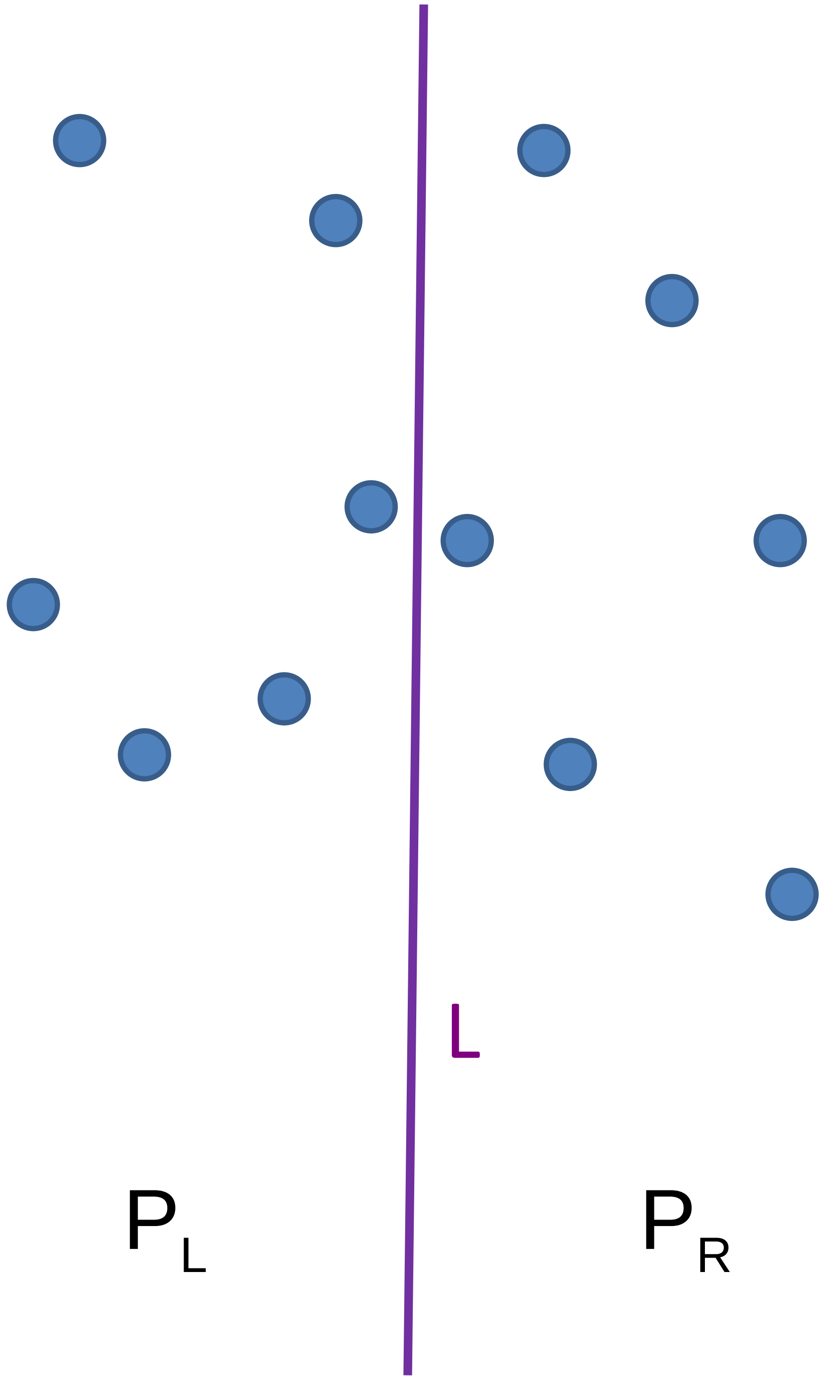
Divide: find a vertical line L that bisects P into two sets

P_L:= { points in P that are on L or to the left of L}.

P_R:= { points in P that are to the right of L}.

Such that $|P_L|$= n/2 and $|P_R|$= n/2 (plus or minus 1)

Conquer:



Closest pair of points

Divide: find a vertical line L that bisects P into two sets

P_L:= { points in P that are on L or to the left of L}.

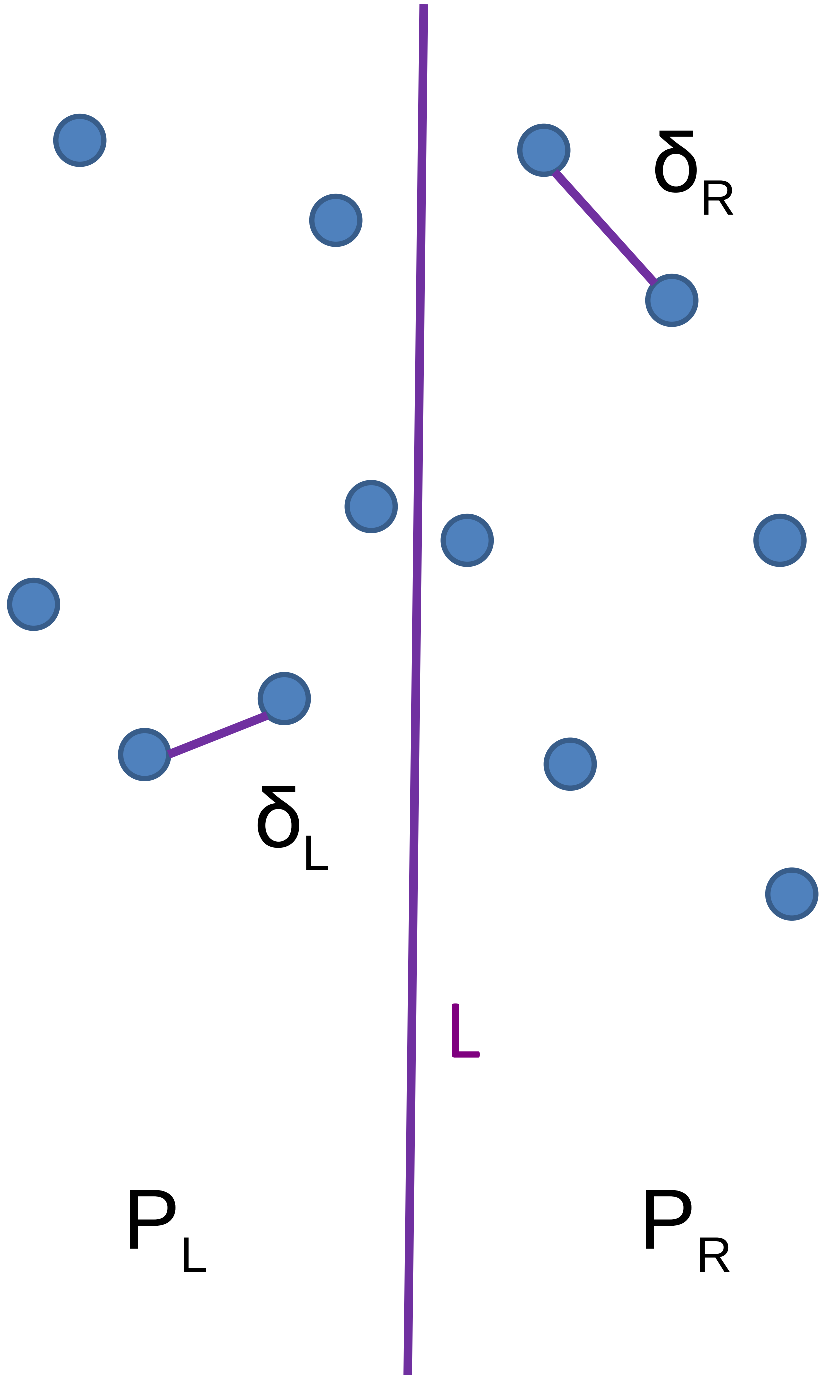
P_R:= { points in P that are to the right of L}.

Such that $|P_L|$= n/2 and $|P_R|$= n/2 (plus or minus 1)

Conquer: Make two recursive calls to find the closest pair of point in P_L and P_R.

Let the closest distances in P_L and P_R be δ_L and δ_R ,and let $\delta = \min(\delta_L , \delta_R)$.

Combine:



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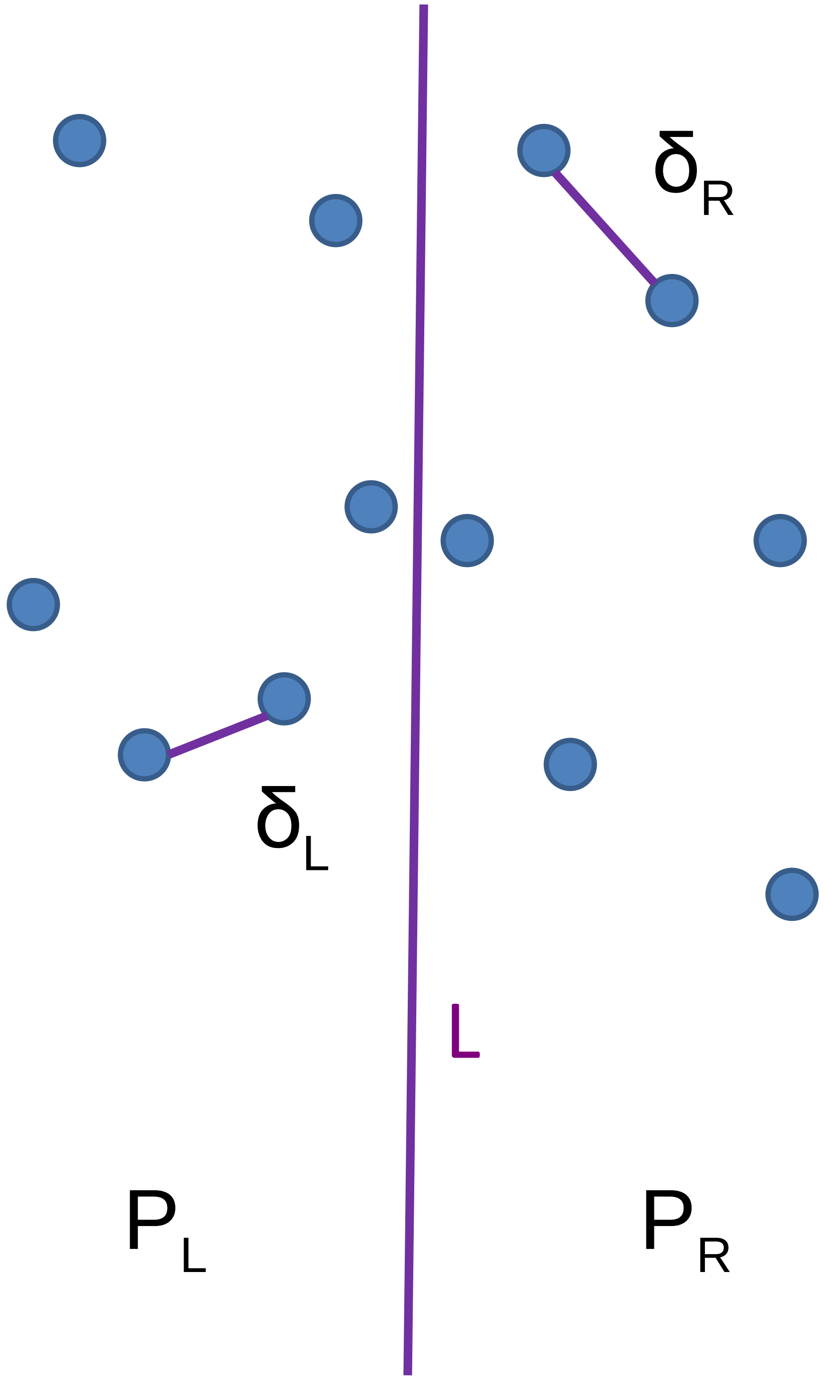
Combine: The closest pair is either the one with distance δ or it is a pair with one point in P_L and the other in P_R with distance less than δ, NO SAVING?



Closest pair of points

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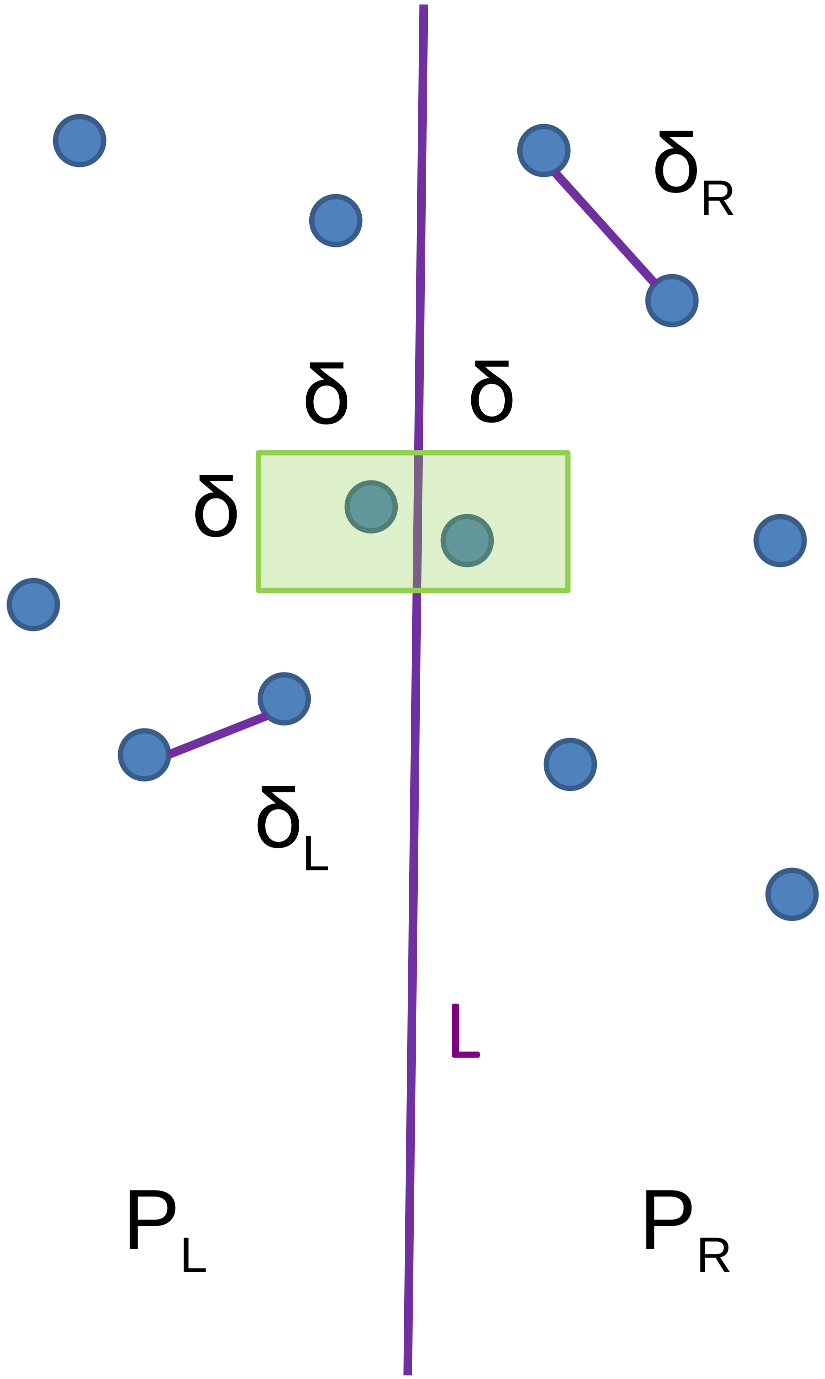
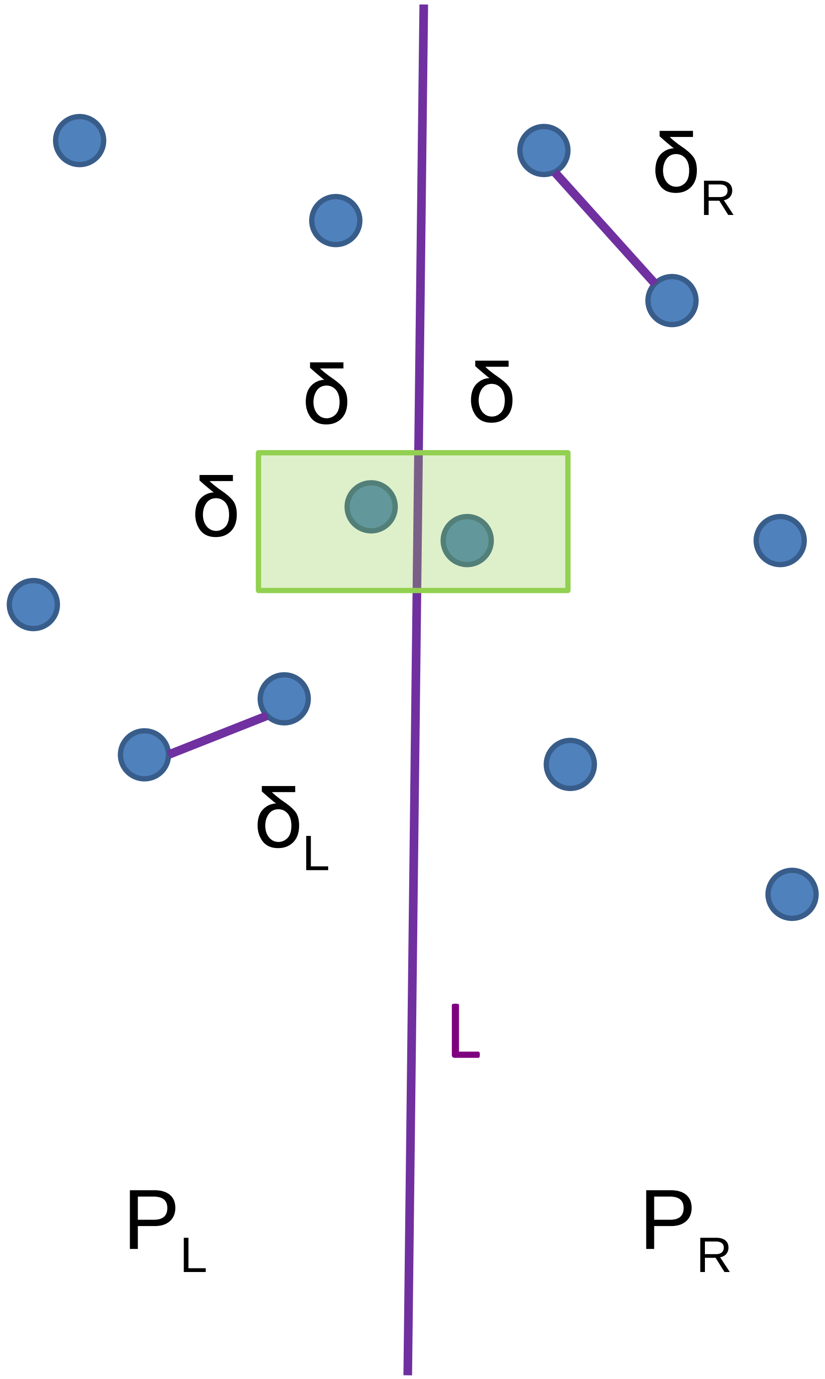
Combine: The closest pair is either the one with distance δ or it is a pair with one point in P_L and the other in P_R with distance less than δ, in a δ x 2δ box straddling L.



How to find points in the box

- Create Y' by removing from Y points that are not in 2δ-wide vertical strip.



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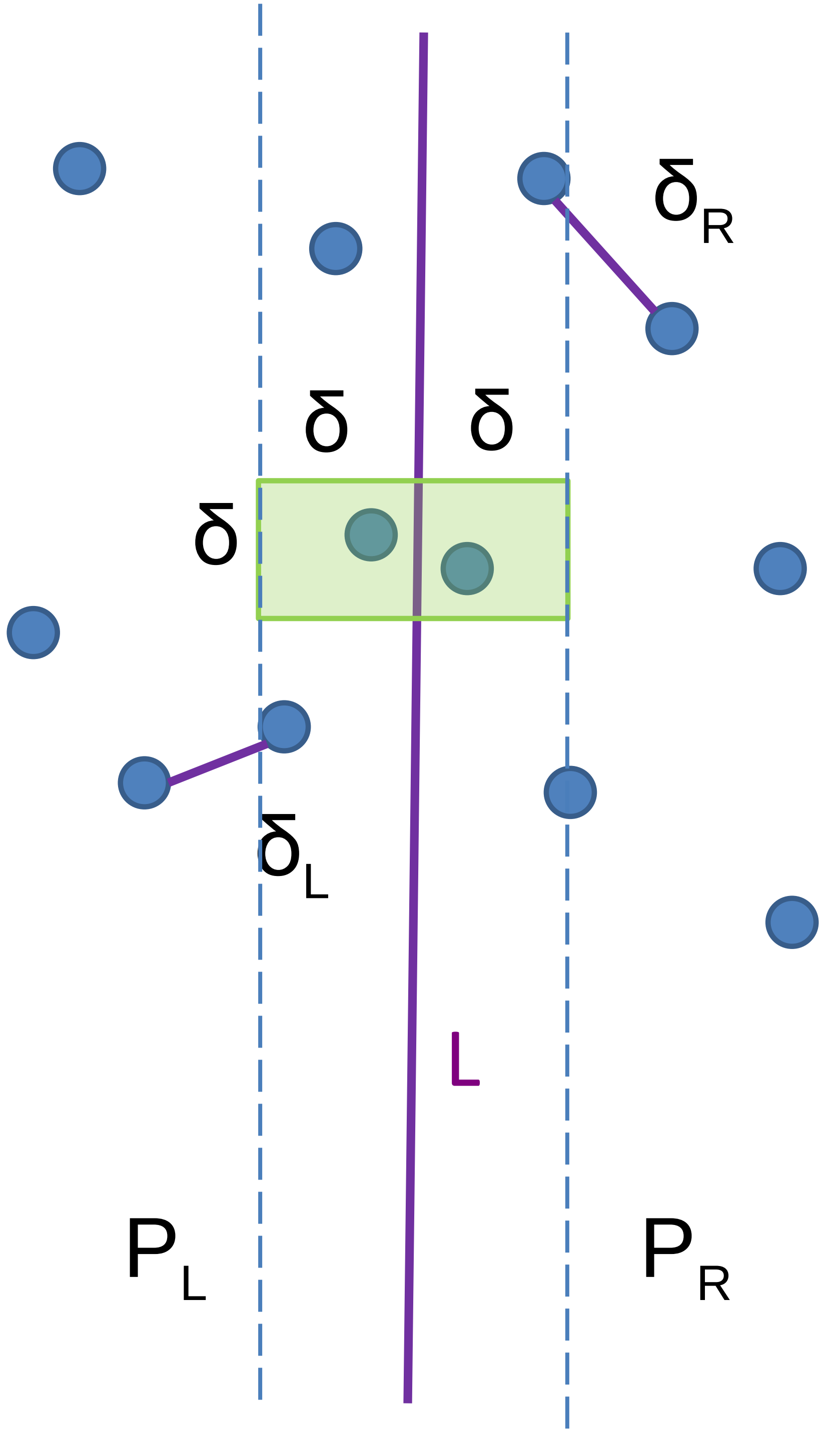

How to find points in the box

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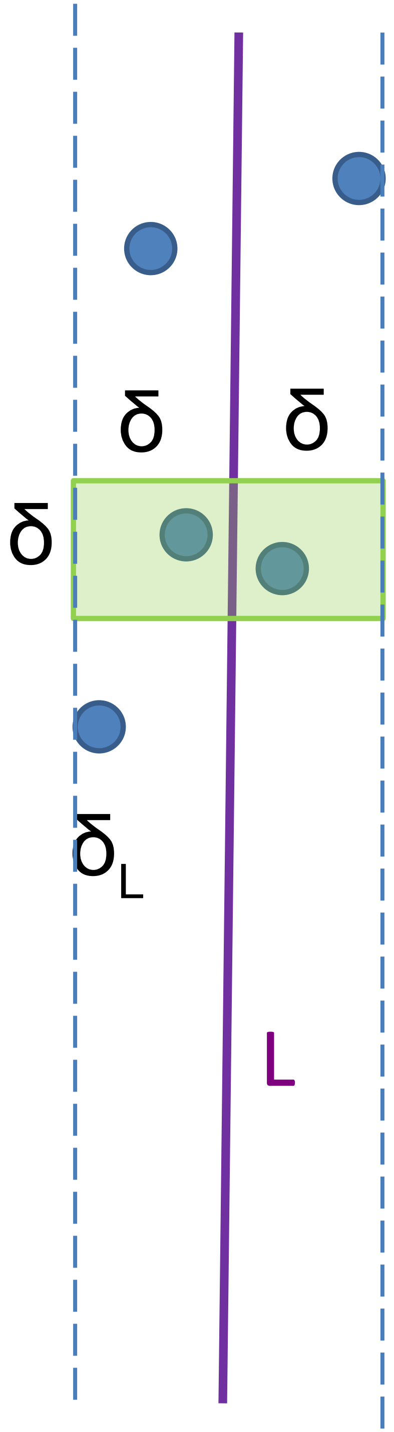


How to find points in the box

- Create Y' by removing from Y points that are not in 2δ-wide vertical strip.
- For each consecutive 8 points in Y'

  $p_1 , p_2 , \dots , p_8$

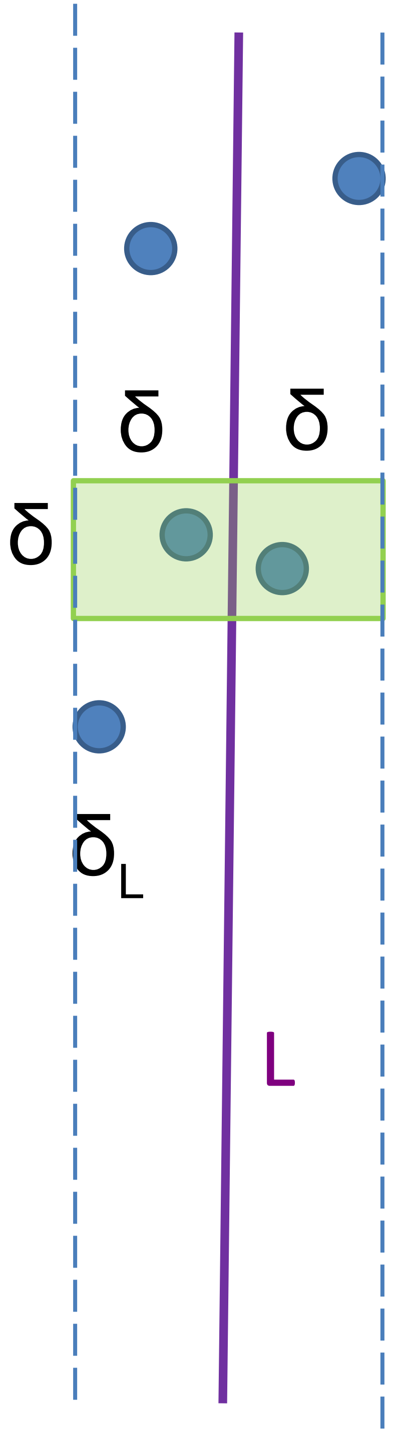
  compute all their distances.

- If any of them are closer than δ,

  update the closest pair

  and the shortest distance δ.

- Return δ and the closest pair.



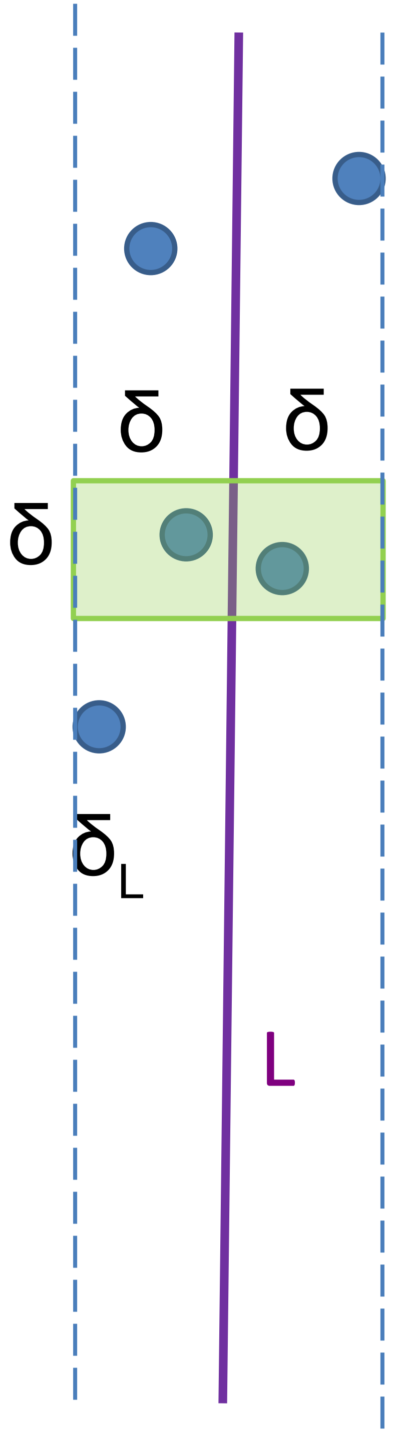
Why 8?

Fact: If there are 9 points in a δ x 2δ box straddling L.

⇒ there are 5 points in a δ x δ box on one side of L.

⇒ there are 2 points on one side of L with
distance less than δ.

This violates the definition of δ.



Analysis of running time

Same as Merge sort:

T(n) = number of operations

$T(n) = 2\ T(n/2) + c\ n$

$\quad = O(n \log n).$

Is multiplication harder than addition?

Alan Cobham, < 1964

Is multiplication harder than addition?

Alan Cobham, < 1964

We still do not know!

Addition

Input: two n-digit integers a, b in base w

(think w = 2, 10)

Output: One integer c=a + b.

Operations allowed: only on digits

The simple way to add takes ?

Addition

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(think w = 2, 10)

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The simple way to add takes O(n)

optimal?

Addition

Input: two n-digit integers a, b in base w

(think w = 2, 10)

Output: One integer c=a + b.

Operations allowed: only on digits

The simple way to add takes O(n)

This is optimal, since we need at least to write c

Multiplication

Input: two n-digit integers a, b in base w

(think w = 2, 10)

Output: One integer c=a·b.

Operations allowed: only on digits

Simple way takes ?

```
    23958233
       5830 ×
------------
    00000000 ( =     23,958,233 ×     0)
   71874699  ( =     23,958,233 ×    30)
  191665864   ( =     23,958,233 ×   800)
119791165    ( =     23,958,233 × 5,000)
------------
139676498390 ( = 139,676,498,390        )
```

Multiplication

Input: two n-digit integers a, b in base w

(think w = 2, 10)

Output: One integer c=a·b.

Operations allowed: only on digits

The simple way to multiply takes $\Omega(n^2)$

Can we do this any faster?

Can we multiply faster than n^2 ?

Feeling: “As regards number systems and calculation techniques, it seems that the final and best solutions were found in science long ago”

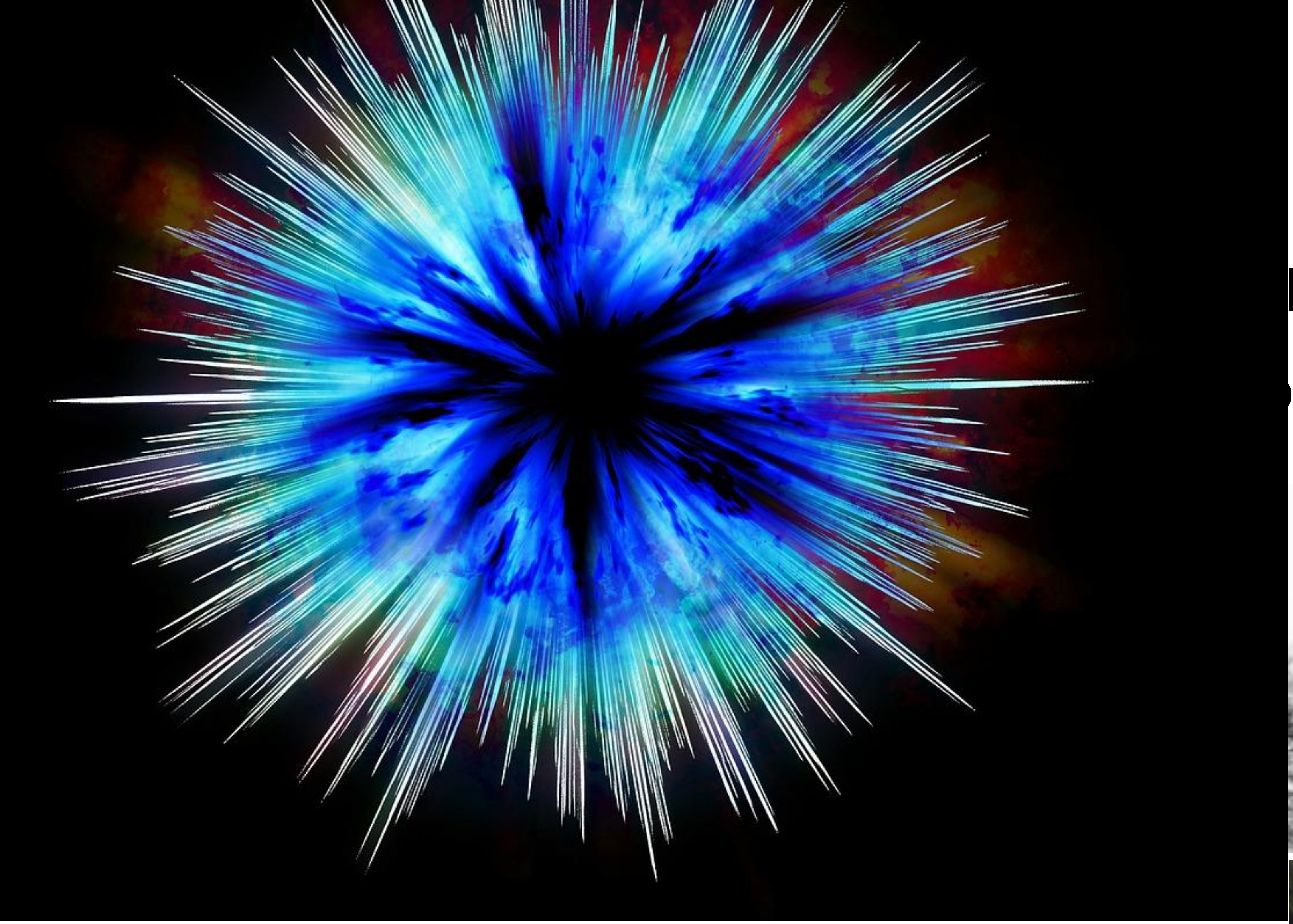
In 1950’s, Kolmogorov conjectured $\Omega(n^2)$

One week later, $O(n^{1.59})$ time by Karatsuba

See “The complexity of Computations”

lation techniques,

ound in science long ago"



One week later, $O(n^{1.59})$ time by Karatsuba

See "The complexity of Computations"

Multiplication

Example:

2-digit numbers N_1 and N_2 in base w.

$N_1 = a_0+a_1w$.

$N_2 = b_0+b_1w$.

For this example, think w very large, like $w = 2^{32}$

Multiplication

Example:

2-digit numbers N_1 and N_2 in base w.

$N_1 = a_0+a_1w.$

$N_2 = b_0+b_1w.$

$P = N_1N_2$

$= a_0b_0+(a_0b_1+a_1b_0)w+a_1b_1w^2$

$= p_0 + p_1w + p_2w^2.$

This can be done with ? multiplications

Multiplication

Example:

2-digit numbers N_1 and N_2 in base w.

$N_1 = a_0+a_1w.$

$N_2 = b_0+b_1w.$

$P \; = N_1N_2$

$= a_0b_0+(a_0b_1+a_1b_0)w+a_1b_1w^2$

$= p_0 + p_1w + p_2w^2.$

This can be done with 4 multiplications

Can we save multiplications, possibly increasing additions?

Compute

$q_0=a_0b_{0.}$

$q_1=(a_0+a_1)(b_1+b_0).$

$q_2=a_1b_1.$

$$P = a_0b_0+(a_0b_1+a_1b_0)w+a_1b_1w^2$$
$$= p_0 + p_1w + p_2w^2.$$

Note:

$q_0=p_0.$

$q_1=p_1+p_0+p_2.$

$q_2=p_2.$

⇨

$p_0=q_0.$

$p_1=q_1-q_0-q_2.$

$p_2=q_2.$

So the three digits of P are evaluated using 3 multiplications rather than 4.

What to do for larger numbers?

The Karatsuba algorithm

Input: two n-digit integers a, b in base w.

Output: One integer $c = a \cdot b$.

Divide:

How?

The Karatsuba algorithm

Input: two n-digit integers a, b in base w.

Output: One integer c = a·b.

Divide:

$m = n/2.$

$a = a_0 + a_1 w^m.$

$b = b_0 + b_1 w^m.$

$$
\begin{aligned}
a \cdot b &= a_0 b_0 + (a_0 b_1 + a_1 b_0) w^m + a_1 b_1 w^{2m} \\
&= p_0 \;+\; p_1 \; w^m \;+\; p_2 \; w^{2m}
\end{aligned}
$$

The Karatsuba algorithm

Input: two n-digit integers a, b in base w.

Output: One integer c = a·b.

Divide:

$m = n/2.$

$a = a_0 + a_1 w^m.$

$b = b_0 + b_1 w^m.$

$$a \cdot b = a_0 b_0 + (a_0 b_1 + a_1 b_0) w^m + a_1 b_1 w^{2m}$$
$$= p_0 \;\; + \qquad p_1 \qquad w^m \; + \; p_2 \; w^{2m}$$

Conquer:

$q_0 = a_0 \times b_0.$

$q_1 = (a_0 + a_1) \times (b_1 + b_0).$

$q_2 = a_1 \times b_1$

Each x is a recursive call

The Karatsuba algorithm

Input: two n-digit integers a, b in base w.

Output: One integer c = a·b.

Divide:

$m = n/2.$

$a = a_0 + a_1 w^m.$

$b = b_0 + b_1 w^m.$

$$a \cdot b = a_0 b_0 + (a_0 b_1 + a_1 b_0) w^m + a_1 b_1 w^{2m}$$
$$= p_0 \;+\; p_1 \; w^m \;+\; p_2 \; w^{2m}$$

Conquer:

$q_0 = a_0 \,\text{x}\, b_0.$

$q_1 = (a_0 + a_1) \,\text{x}\, (b_1 + b_0).$

$q_2 = a_1 \,\text{x}\, b_1$

Each x is a recursive call

Combine:

$p_0 = q_0.$

$p_1 = q_1 - q_0 - q_2.$

$p_2 = q_2.$

Analysis of running time

T(n) = number of operations.

$T(n) = 3\, T(n/2) + O(n)$

$= ?$

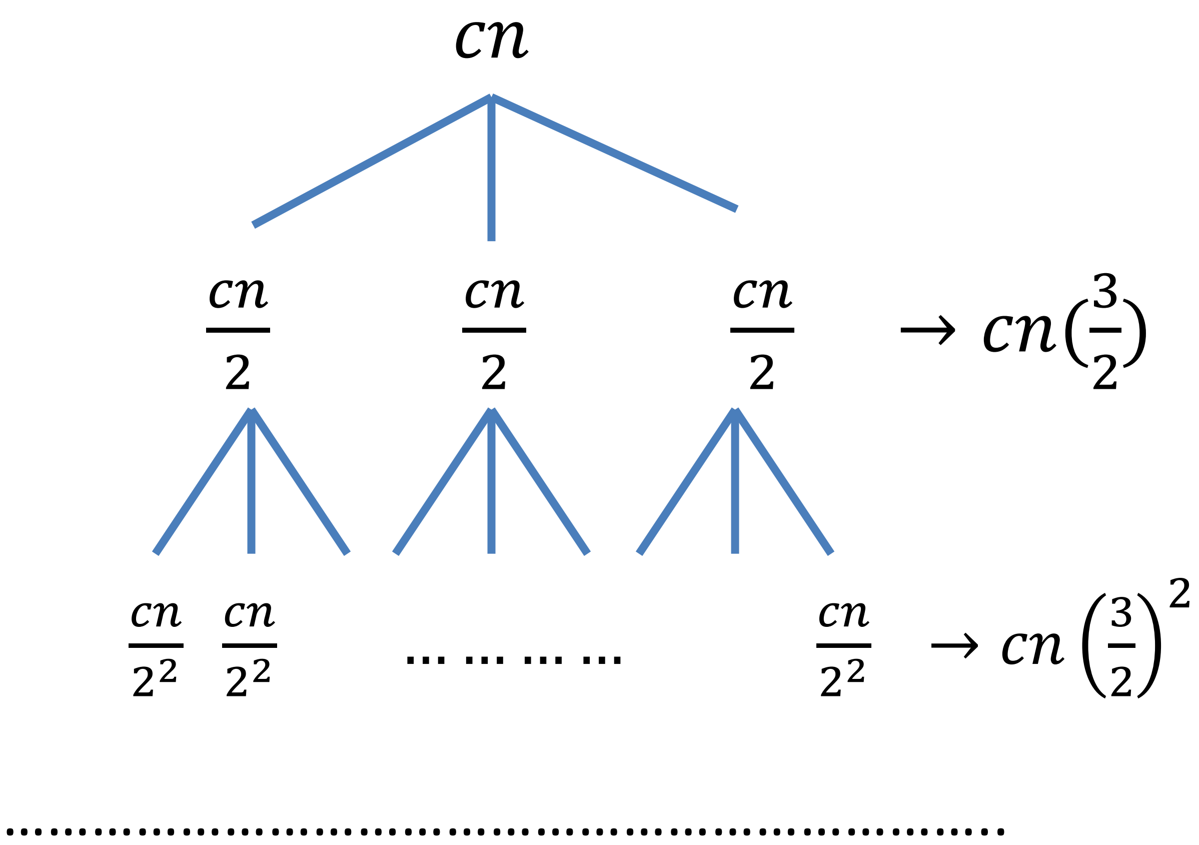
Analysis of running time

T(n) = number of operations.

T(n) = 3 T(n/2) + O(n)

= ?

Recursion tree



$$cn$$

$$\frac{cn}{2} \quad \frac{cn}{2} \quad \frac{cn}{2} \quad \rightarrow cn\left(\frac{3}{2}\right)$$

$$\frac{cn}{2^2} \quad \frac{cn}{2^2} \quad \dots\ \dots\ \dots\ \dots \quad \frac{cn}{2^2} \quad \rightarrow cn\left(\frac{3}{2}\right)^2$$

....

Cost at level $i = cn\left(\frac{3}{2}\right)^i$

Number of levels = $\log_2(n)$

Total cost = $\sum_{i=0}^{\log_2 n} cn\left(\frac{3}{2}\right)^i = O\left(n\left(\frac{3}{2}\right)^{\log_2 n}\right) = O(n^{\log_2 3})$

Analysis of running time

$T(n)$ = number of operations.

$T(n) = 3\ T(n/2) + O(n)$

$= \Theta(n^{\log 3})$ (log in base 2)

$= O(n^{1.59})$.

Karatsuba may be used in your computers to reduce, say, multiplication of 128-bit integers to 64-bit integers.

Are there faster algorithms for multiplication?

Algorithms taking essentially O(n log n) are known.

1971: Scho”nage-Strassen O(n log n log log n)

2007: Fu”rer O(n log n exp(log* n))

log*n = times you need to apply log to n to make it 1

They are all based on Fast Fourier Transform

Matrix Multiplication

n x n matrixes. Note input length is n^2

$$n=4 \left\{ \overset{A}{\begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}} \right. \bullet \overset{B}{\begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array}} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

Just to write down output need time $\Omega(n^2)$

The simple way to do matrix multiplication takes ?

Matrix Multiplication

n x n matrixes. Note input length is n^2

$$n=4 \left\{ \overset{A}{\begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}} \right. \bullet \overset{B}{\begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 \\ \hline \end{array}} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

Just to write down output need time $\Omega(n^2)$

The simple way to do matrix multiplication takes $O(n^3)$.

Strassen's Matrix Multiplication

Input: two n×n matrices A, B.

Output: One n×n matix C=A·B.

Strassen's Matrix Multiplication

Divide:

Divide each of the input matrices A and B into 4 matrices of size n/2xn/2, a follow:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$A.B = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Strassen's Matrix Multiplication

Conquer:

Compute the following 7 products:

$M_1 = (A_{11} + A_{22})(B_{11} + B_{22})$.

$M_2 = (A_{21} + A_{22})B_{11}$.

$M_3 = A_{11}(B_{12} - B_{22})$.

$M_4 = A_{22}(B_{21} - B_{11})$.

$M_5 = (A_{11} + A_{12})B_{22}$.

$M_6 = (A_{21} - A_{11})(B_{11} - B_{12})$.

$M_7 = (A_{12} - A_{22})(B_{21} - B_{22})$.

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

Strassen's Matrix Multiplication

Combine:

$C_{11} = M_1 + M_4 – M_5 + M_7.$

$C_{12} = M_3 + M_5 .$

$C_{21} = M_2 + M_4 .$

$C_{22} = M_1 – M_2 + M_3 + M_6 .$

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

Analysis of running time

T(n) = number of operations

T(n) = 7 T(n/2) + 18 {Time to do matrix addition}

$= 7\, T(n/2) + \Theta(n^2)$

= ?

Analysis of running time

T(n) = number of operations

T(n) = 7 T(n/2) + 18 {Time to do matrix addition}

$= 7\ T(n/2) + \Theta(n^2)$

$= \Theta(n^{\log 7})$

$= O(n^{2.81})$.

Definition: ω is the smallest number such that multiplication of n x n matrices can be computed in time $n^{\omega+\varepsilon}$ for every $\varepsilon > 0$

Meaning: time n^{ω} up to lower-order factors

$\omega \geq 2$ because you need to write the output

$\omega < 2.81$ Strassen, just seen

$\omega < 2.38$ state of the art

Determining ω is a prominent problem

Fast Fourier Transform (FFT)

We start with the most basic case

Walsh-Hadamard transform

Hadamard 2^i x 2^i matrix H_i :

$$H_0 = [1]$$

$$H_{i+1} = \begin{pmatrix} H_i & H_i \\ H_i & -H_i \end{pmatrix}$$

Problem: Given vector x of length $n = 2^k$, compute H_k x

Trivial: $O(n^2)$

Next: O(n log n)

Walsh-Hadamard transform

Write $x = [y\ z]^T$, and note that $H_{k+1}\, x =$

$$\begin{bmatrix} H_k\, y + H_k\, z \\ \\ H_k\, y - H_k\, z \end{bmatrix}$$

This gives T(n) = ?

Walsh-Hadamard transform

Write $x = [y\ z]^T$, and note that $H_{k+1}\ x =$

$$\begin{bmatrix} H_k\, y + H_k\, z \\ \\ H_k\, y - H_k\, z \end{bmatrix}$$

This gives $T(n) = 2\ T(n/2) + O(n) = O(n \log n)$

Polynomials and Fast Fourier Transform (FFT)

Polynomials

$A(x) = \sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

Evaluate at a point x = b with how many multiplications?

2n trivial

Polynomials

$A(x) = \sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

Evaluate at a point x = b with Horner's rule:

Compute a_{n-1} ,

$a_{n-2} + a_{n-1}x$,

$a_{n-3} + a_{n-2}\, x + a_{n-1}\, x^2$

…

Each step: multiply by x, and add a coefficient

There are ≤ n steps ⍰ n multiplications

Summing Polynomials

$\sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{n-1} b_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{n-1} c_i x^i$ the sum polynomial of degree n-1

$c_i = a_i + b_i$

Time O(n)

How to multiply polynomials?

$\sum_{i=0}^{n-1} a_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{n-1} b_i x^i$ a polynomial of degree n-1

$\sum_{i=0}^{2n-2} c_i x^i$ the product polynomial of degree n-1

$c_i = \sum_{j \leq i} a_j b_{i-j}$

Trivial algorithm: time $O(n^2)$

FFT gives time $O(n \log n)$

Polynomial representations

Coefficient: $(a_0, a_1, a_2, \ldots a_{n-1})$

Point-value: have points $x_0, x_1, \ldots x_{n-1}$ in mind

Represent polynomials A(X) by pairs

$\{ (x_0, y_0), (x_1, y_1), \ldots \}$ $A(x_i) = y_i$

To multiply in point-value, just need O(n) operations.

Approach to polynomial multiplication:

A, B given as coefficient representation

1) Convert A, B to point-value representation

2) Multiply C = AB in point-value representation

3) Convert C back to coefficient representation

2) done esily in time O(n)

FFT allows to do 1) and 3) in time O(n log n).
Note: For C we need 2n-1 points; we'll just think “n”

From coefficient to point-value:

$$\begin{matrix} y_0 \\ y_1 \\ \dots \\ \dots \\ \dots \\ y_{n-1} \end{matrix} = \begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-1} \end{pmatrix} \begin{matrix} a_0 \\ a_1 \\ \dots \\ \dots \\ \dots \\ a_{n-1} \end{matrix}$$

From point-value representation, note above matrix is invertible (if points distinct)

Alternatively, Lagrange's formula

We need to evaluate A at points $x_1 \ldots x_n$ in time O(n log n)

Idea: divide and conquer:

$A(x) = A^0 (x^2) + x A^1 (x^2)$

where A^0 has the even-degree terms, A^1 the odd

Example: $A = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$$A^0 (x^2) = a_0 + a_2 x^2 + a_4 x^4$$

$$A^1 (x^2) = a_1 + a_3 x^2 + a_5 x^4$$

How is this useful?

We need to evaluate A at points $x_1 \dots x_n$ in time O(n log n)

Idea: divide and conquer:

$A(x) = A^0 (x^2) + x A^1 (x^2)$

where A^0 has the even-degree terms, A^1 the odd

If my points are $x_1 , x_2 , x_{n/2} , -x_1 , -x_2 , -x_{n/2}$

I just need the evaluations of A^0 , A^1 at $x_1^2 , x_2^2, \dots x_{n/2}^2$

$T(n) \leq 2\, T(n/2) + O(n)$, with solution O(n log n). Are we done?

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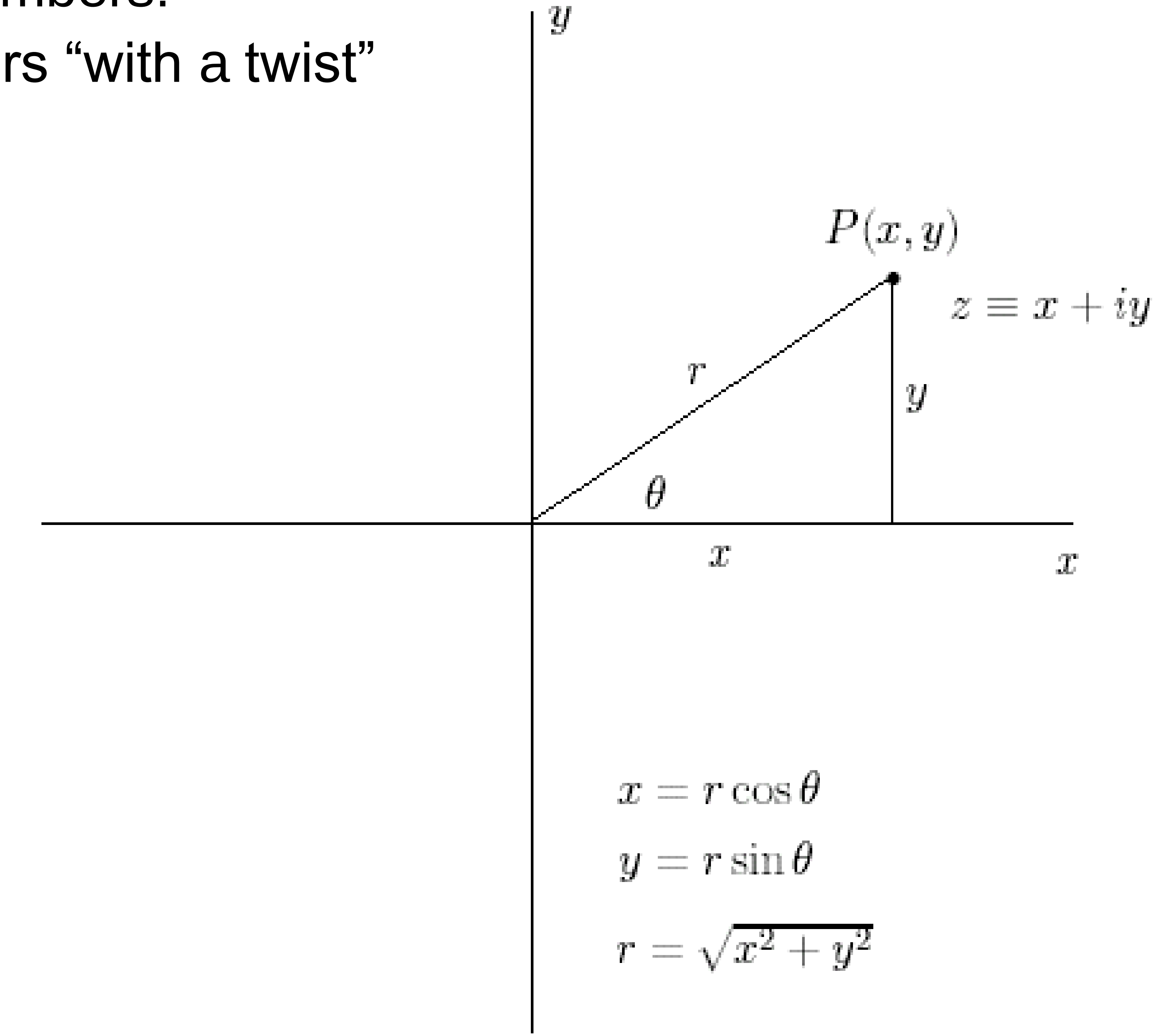
I just need the evaluations of A^0 , A^1 at x_1^2 , x_2^2, ... $x_{n/2}^2$

$T(n) \leq 2\, T(n/2) + O(n)$, with solution O(n log n). Are we done?

Need points which can be iteratively decomposed in + and -

Complex numbers:

Real numbers “with a twist”



$$x = r\cos\theta$$

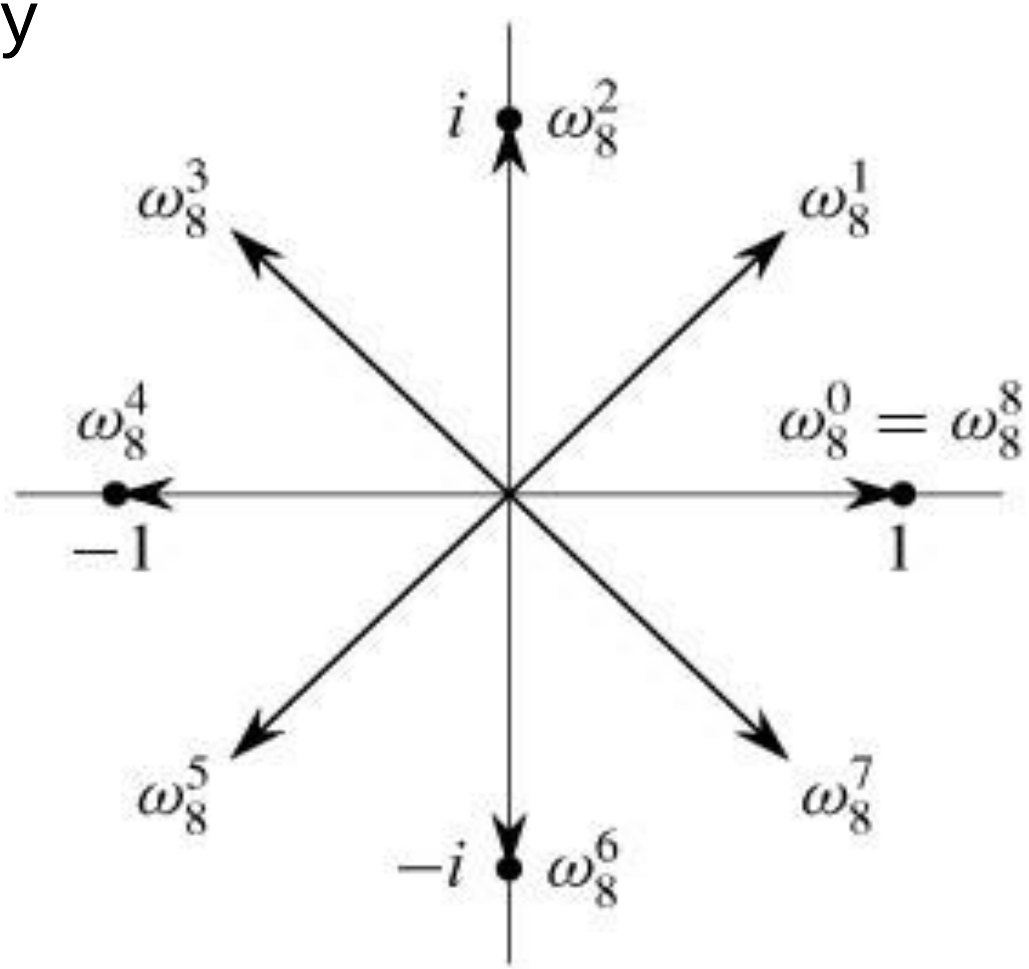
$$y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

ω_n = n-th primitive root of unity

$\omega_n^0, \ldots, \omega_n^{n-1}$

n-th roots of unity



We evaluate polynomial A

of degree n-1

at roots of unity

$\omega_n^0, \ldots, \omega_n^{n-1}$

Fact: The n squares of the n-th roots of unity are:

first the n/2 n/2-th roots of unity,

then again the n/2 n/2-th roots of unity.

from coefficient to point-value in O(n log n) (complex) steps

Summary: Evaluate A at n-th roots of unity ω_n^0 , … ,ω_n^{n-1}

Divide: $A(x) = A^0 (x^2) + x A^1 (x^2)$

where A^0 has the even-degree terms, A^1 the odd

Conquer: Evaluate A^0 , A^1 at n/2-th roots $\omega_{n/2}^0$,… ,$\omega_{n/2}^{n/2-1}$

This yields evaluation vectors y^0 , y^1

```
Combine: $z := 1 = \omega_n^0$
for (k = 0, k < n, k++) {
  $y[k] = y^0[k \text{ modulo } n/2] + z\, y^1[k \text{ modulo } n/2]$;  $z = z \bullet \omega_n$   }
```

$T(n) \leq 2\, T(n/2) + O(n)$, with solution $O(n \log n)$.

It only remains to go from point-value to coefficient represent.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

F

We need to invert F

It only remains to go from point-value to coefficient represent.

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

F

Fact: $(F^{-1})_{j,k} = \omega_n^{-jk} \;/\; n$ Note $j,k \in \{0,1,\ldots, n-1\}$

To compute inverse, use FFT with ω^{-1} instead of ω,
then divide by n.