Algorithms Slides

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2009 – present

Also, let me know if you use them.

Index

The slides are under construction.

The latest version is at http://www.ccs.neu.edu/home/viola/

Success stories of algorithms:

Shortest path (Google maps)

Pattern matching (Text editors, genome)

Fast-fourier transform (Audio/video processing)

http://cstheory.stackexchange.com/questions/19759/core-algorithms-deployed

This class:

General techniques:

- Divide-and-conquer,
- dynamic programming,
- data structures
- amortized analysis

Various topics:

- Sorting
- Matrixes
- Graphs
- Polynomials

What is an algorithm?

- Informally,

  an algorithm for a function $f : A \rightarrow B$ (the problem) is a simple, step-by-step, procedure

  that computes $f(x)$ on every input x

- Example: $A = N x N$ $B = N$, $f(x,y) = x+y$

- Algorithm: Kindergarten addition

$$\begin{array}{r} 112 \\ +\ 999 \\ \hline 1111 \end{array}$$

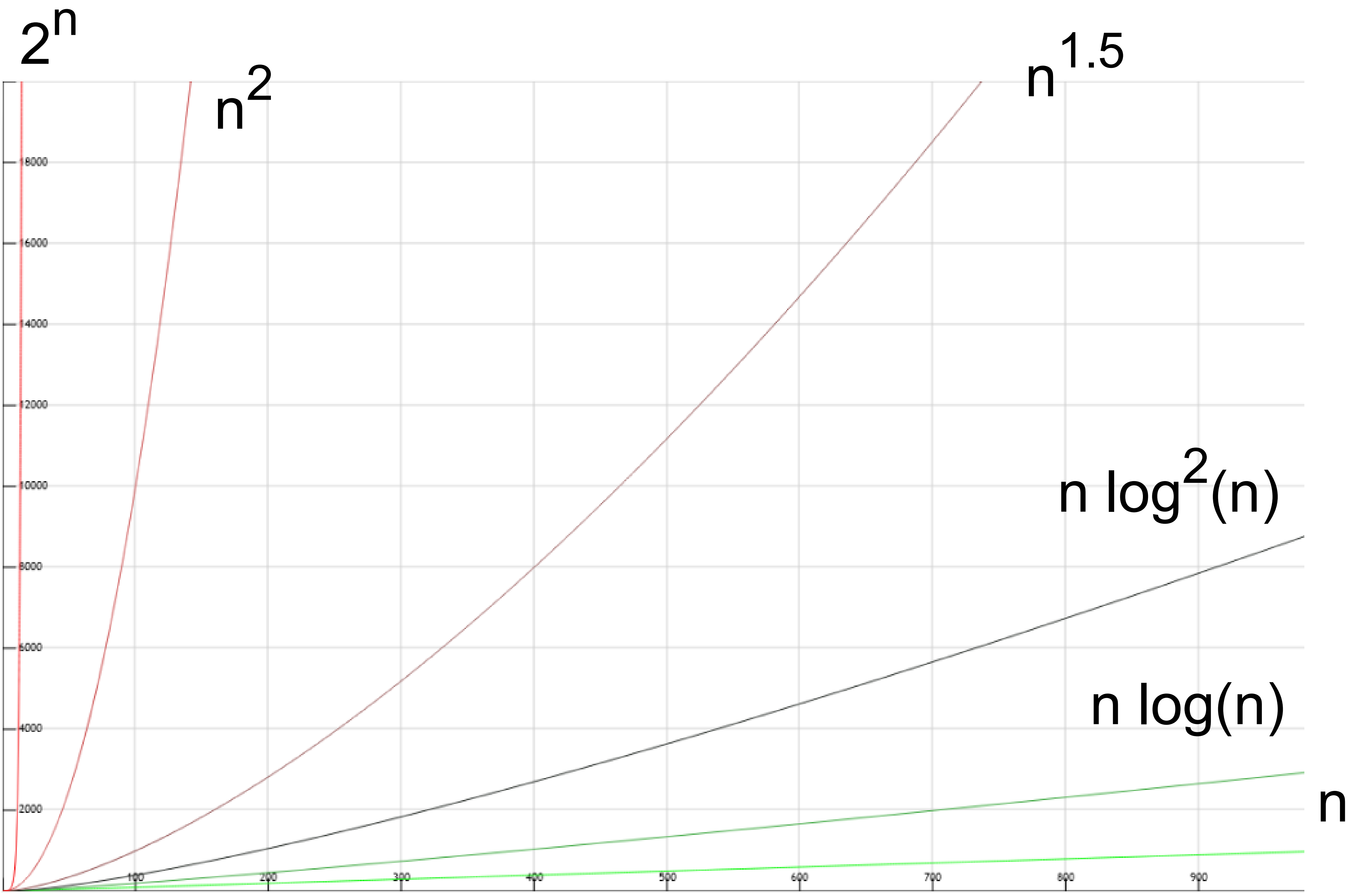
What operations are simple?

- If, for, while, etc.
- Direct addressing: A[n], the n-entry of array A
- Basic arithmetic and logic on variables
  - x * y, x + y, x AND y, etc.
  - Simple in practice only if the variables are “small”.

    For example, 64 bits on current PC
  - Sometimes we get cleaner analysis if we consider them simple regardless of size of variables.

Measuring performance

- We bound the running time, or the memory (space) used.
- These are measured as a function of the input length.
- Makes sense: need to at least read the input!
- The input length is usually denoted n
- We are interested in which functions of n grow faster


2^n
n^2
$n^{1.5}$
$n \log^2(n)$
$n \log(n)$
n
18000
16000
14000
12000
10000
8000
6000
4000
2000
100
200
300
400
500
600
700
800
900

Asymptotic analysis

- The exact time depends on the actual machine

- We ignore constant factors, to have more robust theory that applies to most computer

- Example:

  on my computer it takes 67 n + 15 operations,

  on yours 58 n – 15, but that's about the same

- We now give definitions that make this precise

Big-Oh

Definition:

$f(n) = O(g(n))$ if there are ($\exists$) constants c, n_0 such that

$f(n) \leq c \cdot g(n)$, for every ($\forall$) $n \geq n_0$.

Meaning: f grows no faster than g, up to constant factors

Big-Oh

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Example 1:

$5n + 2n^2 + \log(n) = O(n^2)$?

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Example 1:

$5n + 2n^2 + \log(n) = O(n^2)$ True

Pick $c = ?$

Big-Oh

Definition:

f(n) = O(g(n)) if there are ($\exists$) constants c, n_0 such that

f(n) $\leq$ c·g(n), for every ($\forall$) n $\geq$ n_0.

Example 1:

$5n + 2n^2 + \log(n) = O(n^2)$ True

Pick c = 3. For large enough n, $5n + \log(n) \leq n^2$.

Any c > 2 would work.

Example 2:

$100n^2 = O(2^n)$?

Example 2:

$100n^2 = O(2^n)$ True

Pick c = ?

Example 2:

$100n^2 = O(2^n)$ True

Pick c = 1.

Any c > 0 would work, for large enough n.

Example 3:

$n^2 \log n = O(n^2)$?

Example 3:

$n^2 \log n \neq O(n^2)$

$\forall c, n_0 \; \exists \; n \geq n_0$ such that $n^2 \log n > c \, n^2$.

$n > 2^c \Rightarrow n^2 \log n > n^2 c$

Example 4:

$2^n = O(2^{n/2})$?

Example 4:

$2^n \neq O(2^{n/2})$.

$\forall c, n_0 \ \exists \ n \geq n_0$ such that $2^n > c \cdot 2^{n/2}$.

Pick any $n > 2 \log c$

$2^n = 2^{n/2} \ 2^{n/2} > c \cdot 2^{n/2}$.

- $n \log n = O(n^2)$?
- $n^2 = O(n^{1.5} \log 10n)$?
- $2^n = O(n^{1000000})$?
- $(\sqrt{2})^{\log n} = O(n^{1/3})$?
- $n^{\log \log n} = O((\log n)^{\log n})$?
- $2^n = O(4^{\log n})$?
- $n! = O(2^n)$?
- $n! = O(n^n)$?
- $n2^n = O(2^{n \log n})$?

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- $n \log n = O(n^2)$.
- $n^2 \neq O(n^{1.5} \log 10n)$.
- $2^n \neq O(n^{1000000})$.
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- $n^{\log \log n} = O((\log n)^{\log n})$?
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- $n! = O(n^n)$?
- $n2^n = O(2^{n \log n})$?

$n^{\log \log n} =$

$2^{\log n . \log \log n} =$

$(\log n)^{\log n}$.

- $n \log n = O(n^2)$.
- $n^2 \neq O(n^{1.5} \log 10n)$.
- $2^n \neq O(n^{1000000})$.
- $(\sqrt{2})^{\log n} \neq O(n^{1/3})$.
- $n^{\log \log n} = O((\log n)^{\log n})$.
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- $(\sqrt{2})^{\log n} \neq O(n^{1/3})$.
- $n^{\log \log n} = O((\log n)^{\log n})$.
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- $n! = O(2^n)$?
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- $n^2 \neq O(n^{1.5} \log 10n)$.
- $2^n \neq O(n^{1000000})$.
- $(\sqrt{2})^{\log n} \neq O(n^{1/3})$.
- $n^{\log \log n} = O((\log n)^{\log n})$.
- $2^n \neq O(4^{\log n})$.
- $n! \neq O(2^n)$. $2.5 \sqrt{n} (n/e)^n \leq n! \leq 2.8 \sqrt{n} (n/e)^n$
- $n! = O(n^n)$?
- $n2^n = O(2^{n \log n})$?

- $n \log n = O(n^2)$.
- $n^2 \neq O(n^{1.5} \log 10n)$.
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- $n! \neq O(2^n)$.
- $n! = O(n^n)$.
- $n2^n = O(2^{n \log n})$?

- $n \log n = O(n^2)$.
- $n^2 \neq O(n^{1.5} \log 10n)$.
- $2^n \neq O(n^{1000000})$.
- $(\sqrt{2})^{\log n} \neq O(n^{1/3})$.
- $n^{\log \log n} = O((\log n)^{\log n})$.
- $2^n \neq O(4^{\log n})$.
- $n! \neq O(2^n)$.
- $n! = O(n^n)$.
- $n2^n = O(2^{n \log n})$? $n2^n = 2^{\log n+n}$.

- $n \log n = O(n^2)$.
- $n^2 \neq O(n^{1.5} \log 10n)$.
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Big-omega

Definition:

$f(n) = \Omega\ (g(n))$ means

$\exists\ c,\ n_0 > 0 \quad \forall\ n \geq n_0, \quad f(n) \geq c \cdot g(n).$

Meaning: f grows no slower than g, up to constant factors

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$\exists\ c,\ n_0 > 0 \quad \forall\ n \geq n_0, \quad f(n) \geq c \cdot g(n).$

Example 1:

$0.01\ n = \Omega\ (\log n)$?

Big-omega

Definition:

f(n) = Ω (g(n)) means

$\exists\, c, n_0 > 0 \quad \forall\, n \geq n_0, \quad f(n) \geq c \cdot g(n).$

Example 1:

0.01 n = Ω (log n) True

Pick c = 1. Any c > 0 would work

Example 2:

$n^2/100 = \Omega(n \log n)$?

Example 2:

$n^2/100 = \Omega(n \log n)$.

$c = 1/100$ Again, any c would work.

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$n^2/100 = \Omega(n \log n)$.

$c = 1/100$ Again, any c would work.

Example 3:

$\sqrt{n} = \Omega(n/100)$?

Example 2:

$n^2/100 = \Omega(n \log n)$.

c = 1/100 Again, any c would work.

Example 3:

$\sqrt{n} \neq \Omega(n/100)$

$\forall c, n_0\ \exists\ n \geq n_0$ such that , $\sqrt{n} < c \cdot n/100$.

Example 4:

$2^{n/2} = \Omega(2^n)$?

Example 4:

$2^{n/2} \neq \Omega(2^n)$

$\forall c, n_0\ \exists\ n \geq n_0$ such that $2^{n/2} < c \cdot 2^n$.

Big-omega, Big-Oh

Note: $f(n) = \Omega(g(n)) \Leftrightarrow g(n) = O(f(n))$

$f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$.

Example:

$10 \log n = O(n)$, and $n = \Omega(10 \log n)$.

$5n = O(n)$, and $n = \Omega(5n)$

Theta

Definition:

$f(n) = \Theta\ (g(n))$ means

$\exists\ n_0, c_1, c_2 > 0 \quad \forall\ n \geq n_0,$

$f(n) \leq c_1 \cdot g(n)$ and $g(n) \leq c_2 \cdot f(n)$.

Meaning: f grows like g, up to constant factors

Theta

Definition:

$f(n) = \Theta\ (g(n))$ means

$\exists\ n_0, c_1, c_2 > 0 \quad \forall\ n \geq n_0,$

$f(n) \leq c_1 \cdot g(n)$ and $g(n) \leq c_2 \cdot f(n)$.

Example:

$n = \Theta\ (n + \log n)$?

Theta

Definition:

$f(n) = \Theta\ (g(n))$ means

$\exists\ n_0, c_1, c_2 > 0 \quad \forall\ n \geq n_0,$

$f(n) \leq c_1 \cdot g(n)$ and $g(n) \leq c_2 \cdot f(n)$.

Example:

$n = \Theta\ (n + \log n)$ True

$c_1 = ?$, $c_2 = ?$ $n_0 = ?$ such that $\forall n \geq n_0,$

$n \leq c_1(n + \log n)$ and $n + \log n \leq c_2 n$.

Theta

Definition:

$f(n) = \Theta(g(n))$ means

$\exists\, n_0, c_1, c_2 > 0 \quad \forall\, n \geq n_0,$

$f(n) \leq c_1 \cdot g(n)$ and $g(n) \leq c_2 \cdot f(n)$.

Example:

$n = \Theta(n + \log n)$ True

$c_1 = 1$, $c_2 = 2$ $n_0 = 2$ such that $\forall n \geq 2$,

$n \leq 1\,(n + \log n)$ and $n + \log n \leq 2\,n$.

Theta

Definition:

$f(n) = \Theta\ (g(n))$ means

$\exists\ n_0,\ c_1,\ c_2 > 0 \quad \forall\ n \geq n_0,$

$f(n) \leq c_1 \cdot g(n)$ and $g(n) \leq c_2 \cdot f(n)$.

Note:

$f(n) = \Theta\ (g(n)) \Leftrightarrow f(n) = \Omega\ (g(n))$ and $f(n) = O(g(n))$

$f(n) = \Theta\ (g(n)) \Leftrightarrow g(n) = \Theta\ (f(n))$.

Mixing things up

- $n + O(\log n) = O(n)$

  Means $\forall\ c\ \ \exists\ c', n_0 : \forall\ n > n_0\ \ n + c \log n < c' n$

- $n^3 \log (n) = n^{O(1)}$

  Means $\exists\ c, n_0 : \forall\ n > n_0\ \ n^3 \log (n) = n^c$

- $2^n + n^{O(1)} = \Theta(2^n)$

  Means $\forall\ c\ \exists\ c_1, c_2, n_0 : \forall\ n > n_0$

$$c_2\, 2^n \leq 2^n + n^c \leq c_1\, 2^n$$

Sorting

Sorting problem:

- Input:

  A sequence (or array) of n numbers (a[1], a[2], …, a[n]).

- Desired output:

  A sequence (b[1], b[2], …, b[n]) of sorted numbers (in increasing order).

Example:

Input = (5, 17, -9, 76, 87, -57, 0).

Output = ?

Sorting problem:

- Input:

  A sequence (or array) of n numbers (a[1], a[2], …, a[n]).
- Desired output:

  A sequence (b[1], b[2], …, b[n]) of sorted numbers (in increasing order).

Example:

Input = (5, 17, -9, 76, 87, -57, 0).

Output = (-57, -9, 0, 5, 17, 76, 87).

Sorting problem:

- Input:

  A sequence (or array) of n numbers (a[1], a[2], …, a[n]).

- Desired output:

  A sequence (b[1], b[2], …, b[n]) of sorted numbers (in increasing order).

Who cares about sorting?

- Sorting is a basic operation that shows up in countless other algorithms
- Often when you look at data you want it sorted
- It is also used in the theory of NP-hardness!

Bubblesort:

```
Input (a[1], a[2], …, a[n]).
 for (i=n; i > 1; i - -)
   for (j=1; j < i; j++)
     if (a[j] > a[j+1])
       swap a[j] and a[j+1];
```

Bubblesort:

```
Input (a[1], a[2], …, a[n]).
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Claim: Bubblesort sorts correctly

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Proof: Fix i. Let a'[1], …, a'[n] be array at start of inner loop.

Note at the end of the loop: a'[i] = ?

Bubblesort:

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Proof: Fix i. Let a'[1], …, a'[n] be array at start of inner loop.

Note at the end of the loop: $a'[i] = \max_{k \leq i} a'[k]$

and the positions k > i are

Bubblesort:

Input (a[1], a[2], …, a[n]).

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for (i=n; i > 1; i - -)
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      swap a[j] and a[j+1];
```

Claim: Bubblesort sorts correctly

Proof: Fix i. Let a'[1], …, a'[n] be array at start of inner loop.

Note at the end of the loop: $a'[i] = \max_{k \leq i} a'[k]$

and the positions $k > i$ are not touched.

Since the outer loop is from n down to 1, the array is sorted. ■

Analysis of running time

T(n) = number of comparisons

```
Bubble sort:
Input (a[1], a[2], …, a[n]).
 for (i=n; i > 1; i--)
    for (j=1; j < i; j++)
      if (a[j] > a[j+1])
        swap a[j] and a[j+1];
```

i = n-1 ⇨ n -1 comparisons.

i = n-2 ⇨ n -2 comparisons.

…

i = 1 ⇨ 1 comparison.

$T(n) = (n-1) + (n-2) + \ldots + 1 < n^2$

Is this tight? Is also $T(n) = \Omega(n^2)$?

Analysis of running time

T(n) = number of comparisons

Bubble sort:

```
Input (a[1], a[2], …, a[n]).
 for (i=n; i > 1; i--)
    for (j=1; j < i; j++)
      if (a[j] > a[j+1])
        swap a[j] and a[j+1];
```

i = n-1 ⇨ n -1 comparisons.

i = n-2 ⇨ n -2 comparisons.

…

i = 1 ⇨ 1 comparison.

$$T(n) = (n-1) + (n-2) + \ldots + 1 = n(n-1)/2 = \Theta(n^2)$$

Space (also known as Memory)

We need to keep track of i, j

We need an extra element
to swap values of input array a.

Space = O(1)

```
Bubble sort:
Input (a[1], a[2], …, a[n]).
 for (i=n; i > 1; i--)
    for (j=1; j < i; j++)
      if (a[j] > a[j+1])
        swap a[j] and a[j+1];
```

Bubble sort takes quadratic time

Can we sort faster?

We now see two methods that can sort in linear time,

under some assumptions

Countingsort:

Assumption: all elements of the input array are integers in the range 0 to k.

Idea: determine, for each A[i], the number of elements in the input array that are smaller than A[i].

This way we can put element A[i] directly into its position.

```
// Sorts A[1..n] into array B
Countingsort (A[1..n]) {
  // Initializes C to 0
  for (i=0; k ; i++)   C[i] = 0;

  // Set C[i] = number of elements = i.
  for (i=1; n ; i++)  C[ A[ i ] ] =C[ A[ i ] ]+1;

  // Set C[i] = number of elements ≤ i.
  for (i=1; k ; i++)  C[i] = C[i]+C[i-1] ;

  for (i=n; 1 ; i - -) {
    B[ C[ A[ i ] ] ] = A[ i ] ;  //Place A[i] at right location
    C[ A[ i ] ] = C[ A[ i ] ]-1; //Decrease for equal elements
  }
```

Analysis of running time

T(n) = number of operations

$= O(k) + O(n) + O(k) + O(n)$

$= \Theta(n + k).$

If $k = O(n)$ then $T(n) = \Theta(n)$

```
Countingsort (A[1..n])
  for (i =0; i<k ; i++)
      C[i] = 0;
  for (i =1; i<n ; i++)
      C[A[i]] =C[A[i]] +1;
 for (i =1; i<k ; i++)
      C[i] = C[i] +C[i-1] ;
 for (i =n; i>1 ; i--) {
    B[ C[ A[ i ] ] ] = A[ i ] ;
    C[ A[ i ] ] = C[ A[ i ] ] -1;
 }
```

Space

O(k) for C

Recall numbers in 0..k.

O(n) for B, where output is

Total space: O(n + k)

If k = O(n) then Θ(n)

```
Countingsort (A[1..n])
  for (i =0; i<k ; i++)
      C[i] = 0;
  for (i =1; i<n ; i++)
      C[A[i]] =C[A[i]] +1;
  for (i =1; i<k ; i++)
      C[i] = C[i] +C[i-1] ;
  for (i =n; i>1 ; i--) {
     B[ C[ A[ i ] ] ] = A[ i ] ;
     C[ A[ i ] ] = C[ A[ i ] ] -1;
  }
```

Radix sort

Assumption: all elements of the input array are d-digit integers.

Idea: first sort by least significant digit,
    then according to the next digit,
    …,
    and finally according to the most significant digit.

It is essential to use a digit sorting algorithm that is stable: elements with the same digit appear in the output array in the same order as in the input array.

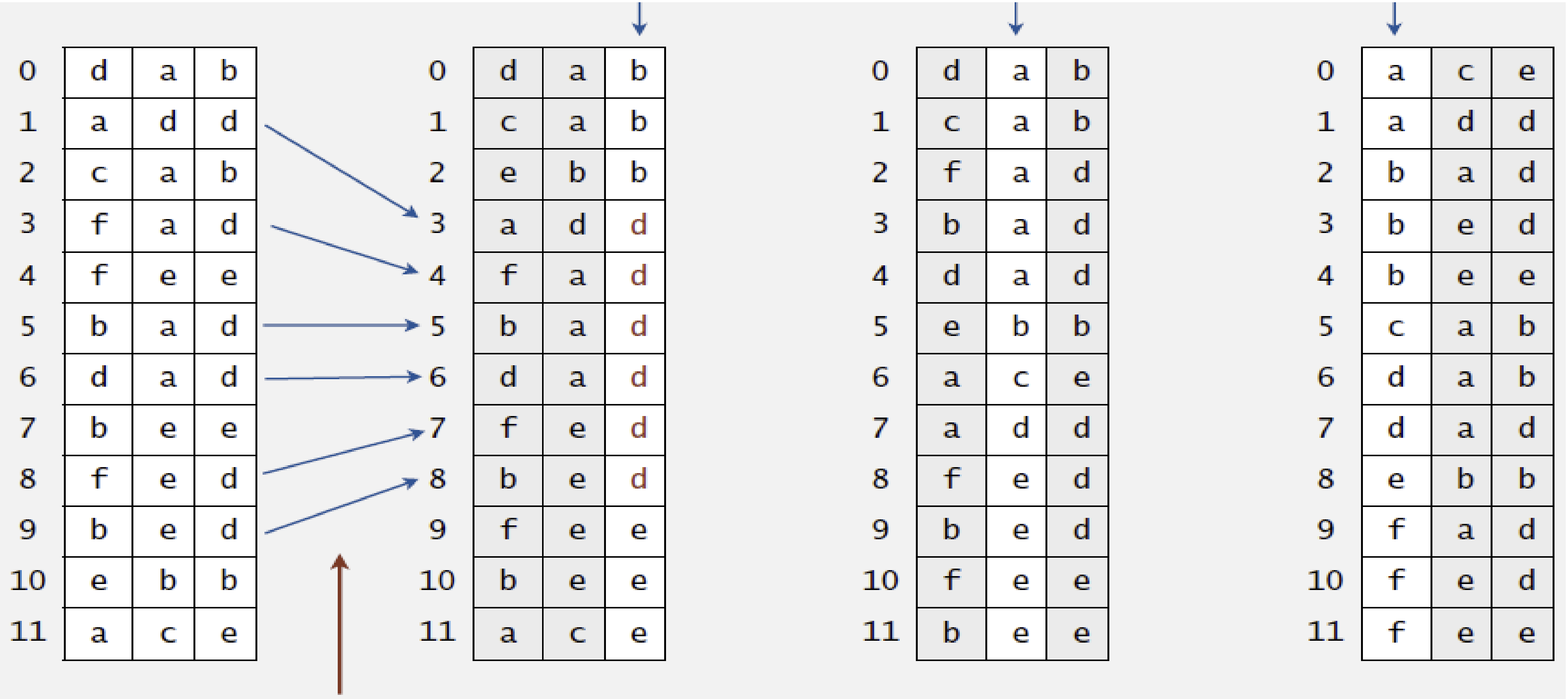
- Fact: Counting sort is stable.

```
Radixsort(A[1..n]) {
  for i that goes from least significant digit to most {
    use counting sort algorithm to sort array A on digit i
  }
}
```

Example:

Sort in ascending order (3,2,1,0) (two binary digits).

```
Radixsort(A[1..n]) {
  for i that goes from least significant digit to most {
    use counting sort algorithm to sort array A on digit i
  }
}
```



0	d	a	b	0	d	a	b	0	d	a	b	0	a	c	e
1	a	d	d	1	c	a	b	1	c	a	b	1	a	d	d
2	c	a	b	2	e	b	b	2	f	a	d	2	b	a	d
3	f	a	d	3	a	d	d	3	b	a	d	3	b	e	d
4	f	e	e	4	f	a	d	4	d	a	d	4	b	e	e
5	b	a	d	5	b	a	d	5	e	b	b	5	c	a	b
6	d	a	d	6	d	a	d	6	a	c	e	6	d	a	b
7	b	e	e	7	f	e	d	7	a	d	d	7	d	a	d
8	f	e	d	8	b	e	d	8	f	e	d	8	e	b	b
9	b	e	d	9	f	e	e	9	b	e	d	9	f	a	d
10	e	b	b	10	b	e	e	10	f	e	e	10	f	e	d
11	a	c	e	11	a	c	e	11	b	e	e	11	f	e	e

sort must be stable
(arrows do not cross)

Image source: http://www.programering.com/a/MTOyYjNwATM.html

Analysis of running time

```
Radixsort(A[1..n]) {
  for i from least significant
        digit to most {
    use counting sort to
    sort array A on digit i
  }
}
```

T(n) = number of operations

$T(n) = d\cdot(\text{running time of Counting sort on n elements})$

$= \Theta(d\cdot(n+k))$

Example: To sort numbers in range 0.. n^{10}

T(n) = ?

(hint: think numbers in base n)

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$$T(n) = \Theta(10\ n) = \Theta(n)$$

While counting sort would take $T(n) = ?$

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$T(n) = \Theta(10\ n) = \Theta(n)$

While counting sort would take $T(n) = \Theta(n^{10})$

Space

We need as much space as we did for Counting sort on each digit

Space = O(d • (n+k))

Can you improve this?

```
Radixsort(A[1..n])  {
  for i from least significant
       digit to most {
   use counting sort to
   sort array A on digit i
    }
}
```

Can we sort faster than n^2 without extra assumptions?

Next we show how to sort with O(n log n) comparisons

We introduce a new general paradigm

Deleted scenes

- 3SAT problem: Given a 3CNF formula such as

  $\varphi := (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z)$

  can we set variables True/False to make φ True?

  Such φ is called satisfiable.

- Theorem [3SAT is NP-complete]

  Let $M : \{0,1\}^n \to \{0,1\}$ be an algorithm running in time T

  Given $x \in \{0,1\}^n$ we can efficiently compute 3CNF φ :

  $$M(x) = 1 \Leftrightarrow \varphi \text{ satisfiable}$$

- How efficient?

- **Theorem [3SAT is NP-complete]**

  Let $M : \{0,1\}^n \to \{0,1\}$ be an algorithm running in time T

  Given $x \in \{0,1\}^n$ we can efficiently compute 3CNF φ :

  $$M(x) = 1 \iff \varphi \text{ satisfiable}$$

- Standard proof: φ has $\Theta(T^2)$ variables (and size), $x_{i,\,j}$

$$\boxed{\begin{array}{cccc} x_{1,\,1} & x_{1,\,2} & \cdots & x_{1,\,T} \\ & \ldots & & \\ x_{i,\,1} & x_{i,\,2} & \cdots & x_{i,\,T} \end{array}}$$

row i = memory, state at time i=1..T

φ ensures that memory and state evolve according to M

- Theorem [3SAT is NP-complete]

  Let M : $\{0,1\}^n \rightarrow \{0,1\}$ be an algorithm running in time T

  Given x $\in \{0,1\}^n$ we can efficiently compute 3CNF φ :

  $$M(x) = 1 \Leftrightarrow \varphi \text{ satisfiable}$$

- Better proof: φ has $O(T \log^{O(1)} T)$ variables (and size),

  $C_i := x_{i,1}\, x_{i,2} \dots\ x_{i, \log T}$ = state and what algorithm reads, writes at time $i = 1..T$

  Note only 1 memory location is represented per time step.

  How do you check C_i correct? What does φ do?

- Theorem [3SAT is NP-complete]

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  φ : Check C_{i+1} follows from C_i assuming read correct

  Compute $C'_i := C_i$ sorted on memory location accessed

  Check C'_{i+1} follows from C'_i assuming state correct

- Theorem [3SAT is NP-complete]

  Let M : $\{0,1\}^n \to \{0,1\}$ be an algorithm running in time T

  Given x $\in \{0,1\}^n$ we can efficiently compute 3CNF φ .

  $M(x) = 1 \Leftrightarrow \varphi$ satisfiable

THAT'S WHY SORTING MATTERS!

- Better proof: φ has $O(T \log^{O(1)} T)$ variables (and size),

  $C_i := x_{i,1}, x_{i,2}, \ldots, x_{i,\log T}$ = state and what algorithm reads, writes at time $i = 1..T$

  φ : Check C_{i+1} follows from C_i assuming read correct

  Let C' be C_i sorted on memory location accessed

  Check C'_{i+1} follows from C'_i assuming state