Big picture

- All languages
- Decidable

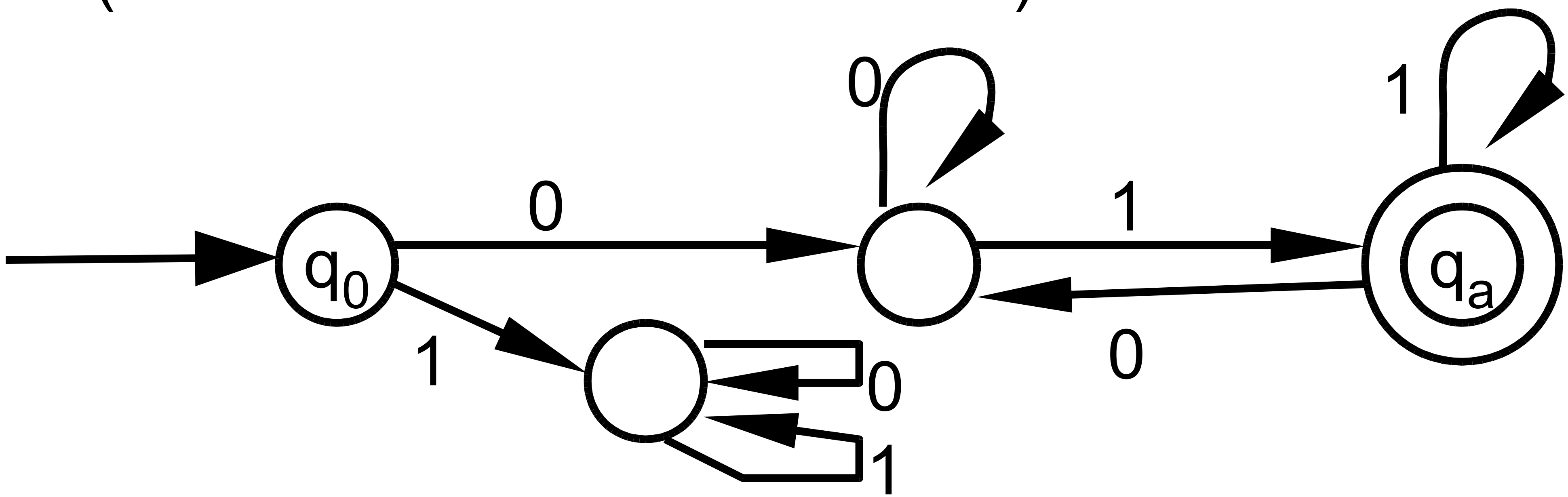
  Turing machines
- NP
- P
- Context-free

  Context-free grammars, push-down automata
- Regular

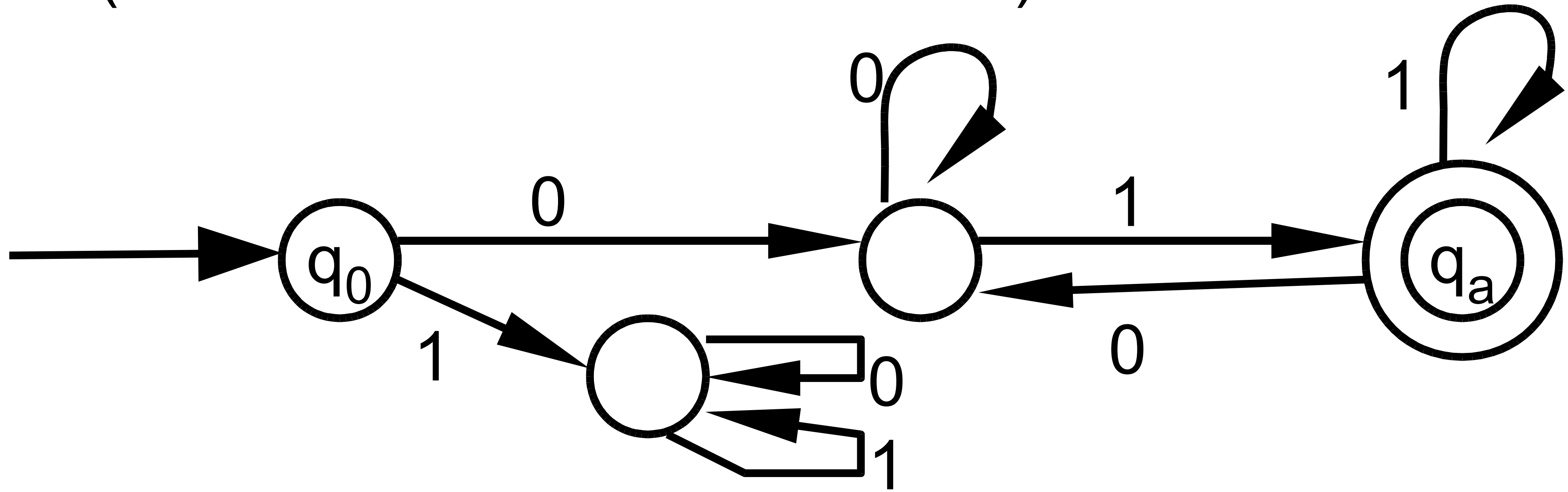
  Automata, non-deterministic automata,

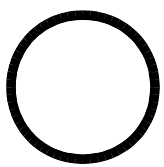
  regular expressions

DFA (Deterministic Finite Automata)



DFA (Deterministic Finite Automata)

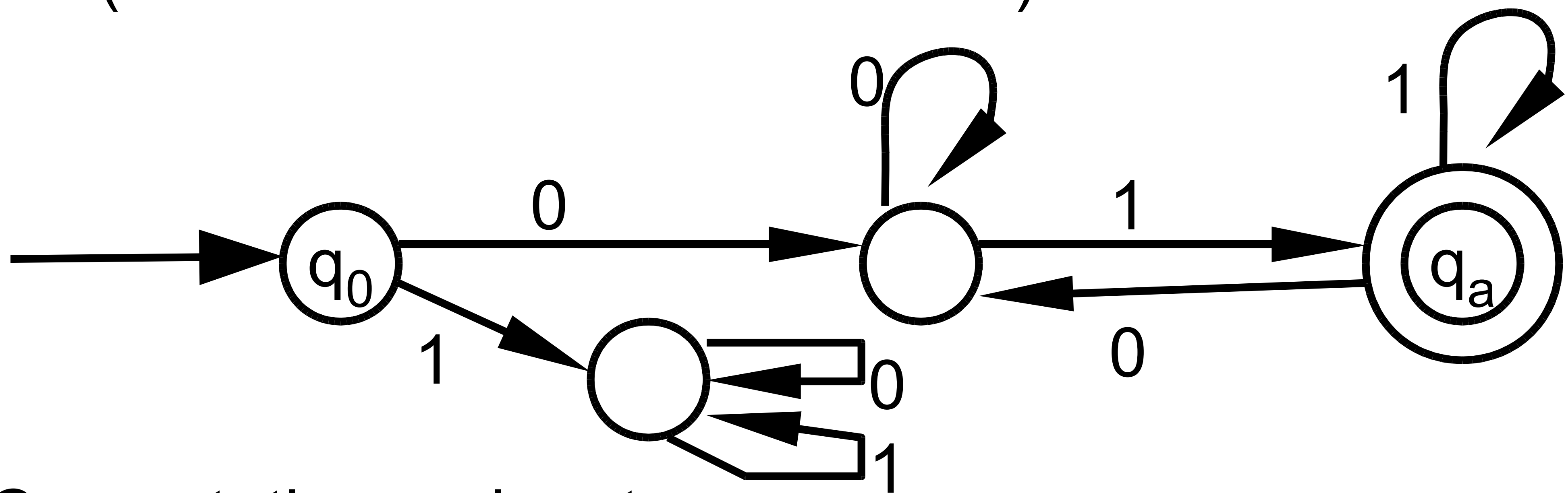


- States  , this DFA has 4 states

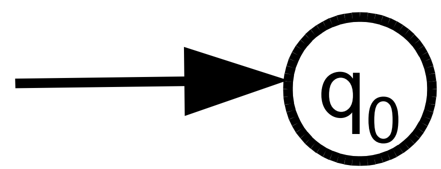
- Transitions 

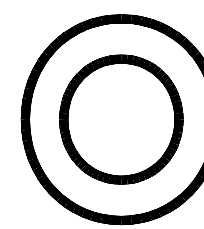
  labelled with elements of the alphabet $\Sigma = \{0,1\}$

DFA (Deterministic Finite Automata)



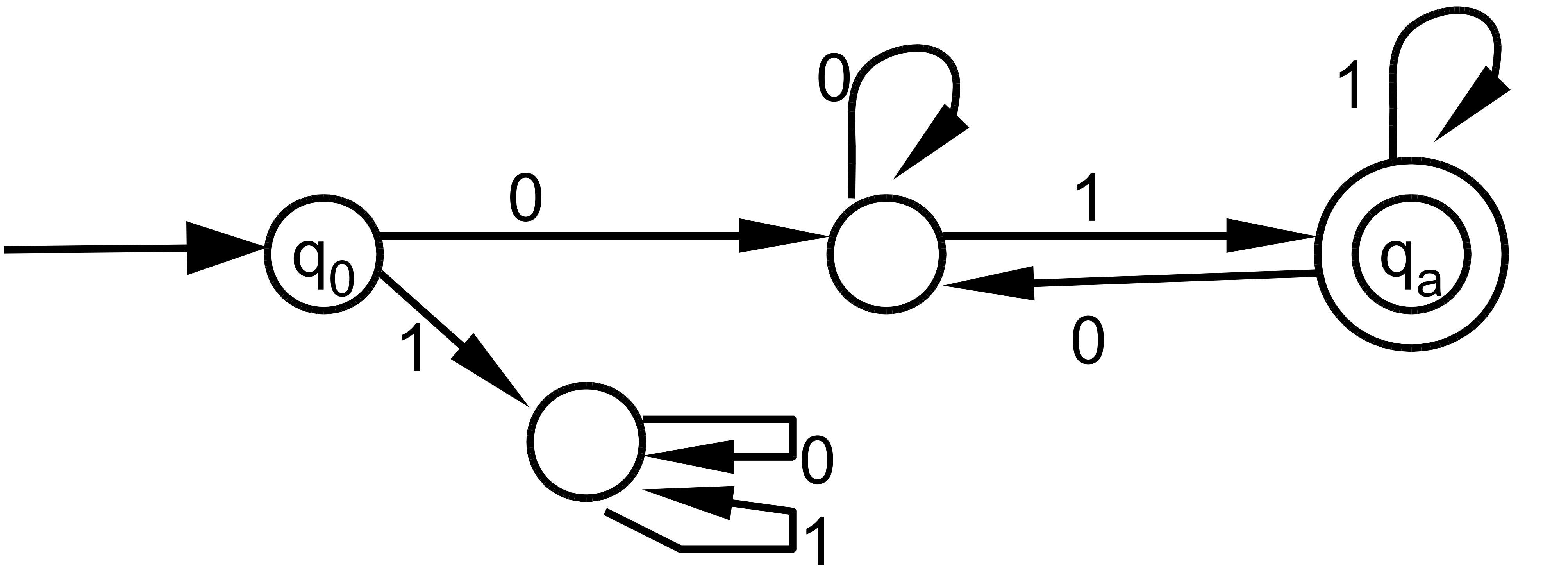
Computation on input w:

- Begin in start state 

- Read input string in a one-way fashion
- Follow the arrows matching input symbols
- When input ends: ACCEPT if in accept state 

  REJECT if not

DFA (Deterministic Finite Automata)

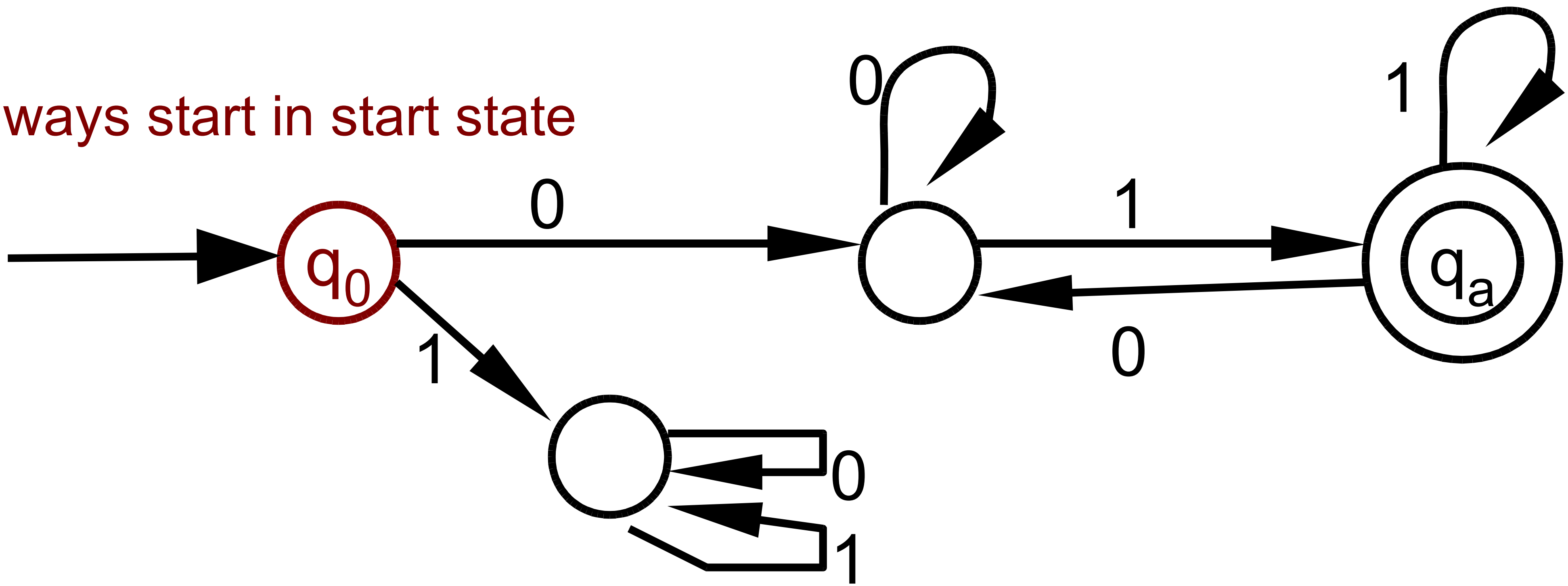


Example: Input string

w = 0011

DFA (Deterministic Finite Automata)

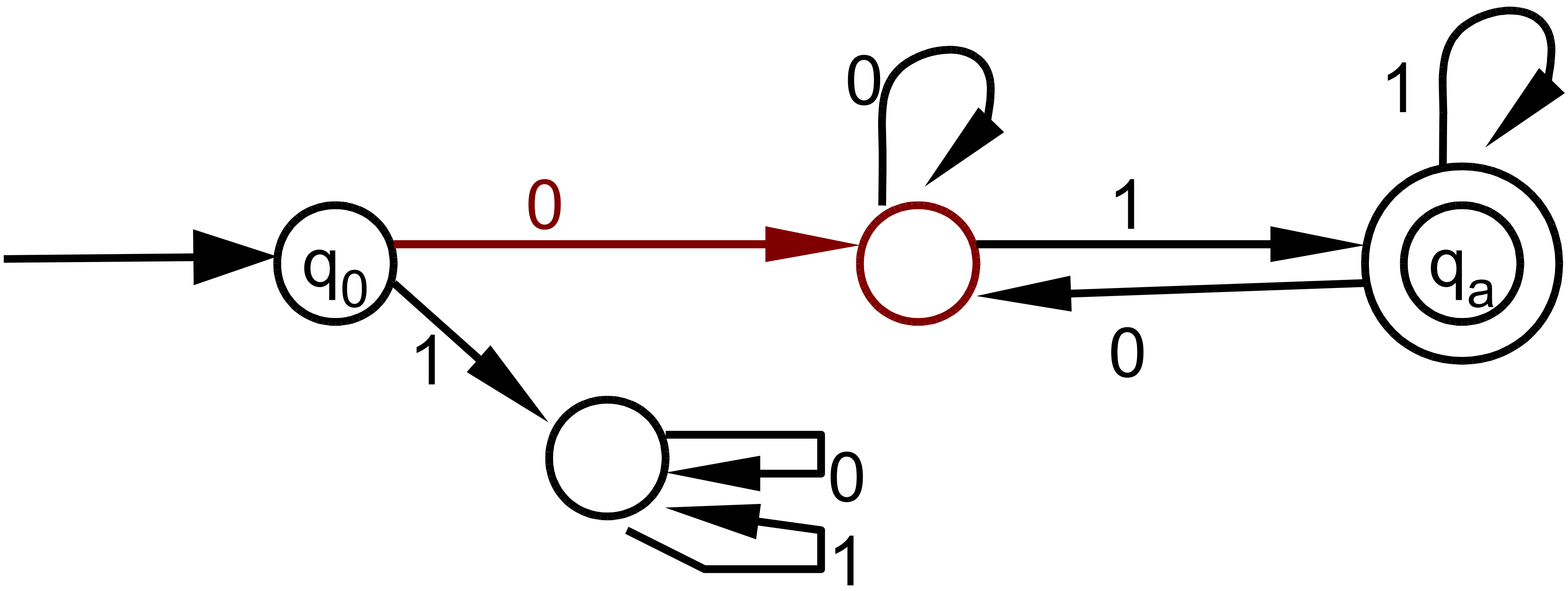
always start in start state



Example: Input string

w = 0011

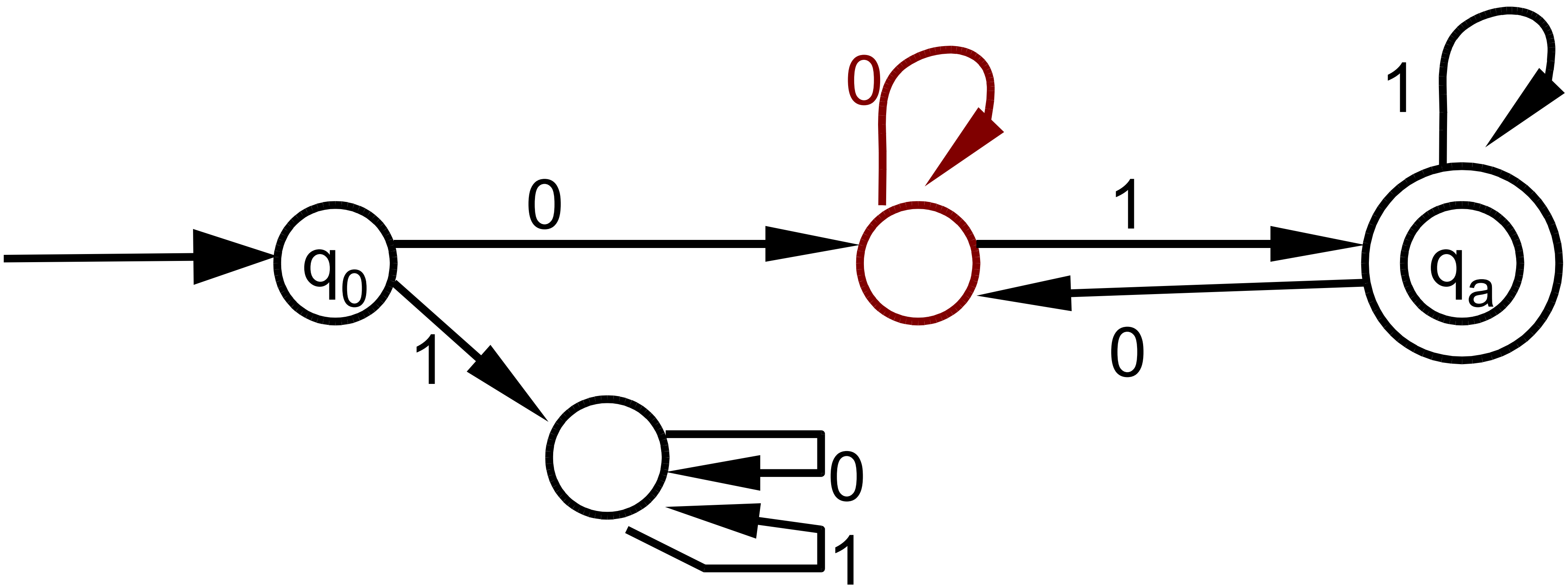
DFA (Deterministic Finite Automata)



Example: Input string

w = **0**011

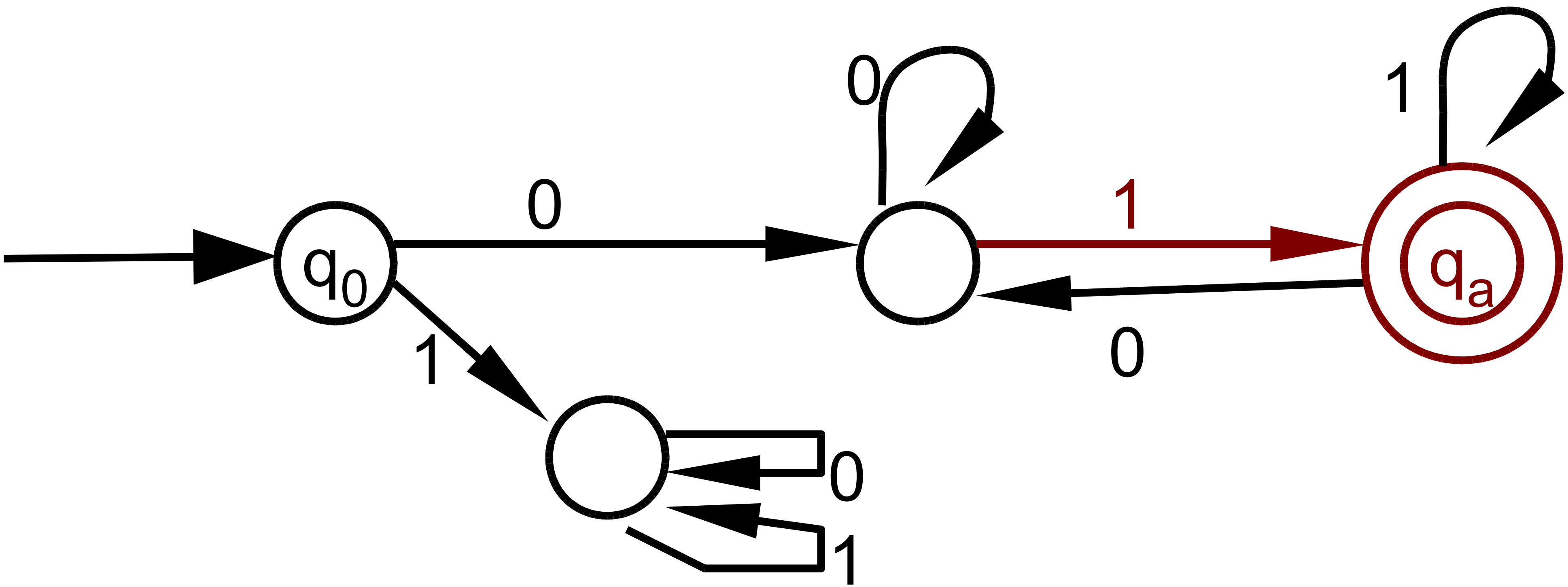
DFA (Deterministic Finite Automata)



Example: Input string

$w = 0\mathbf{0}11$

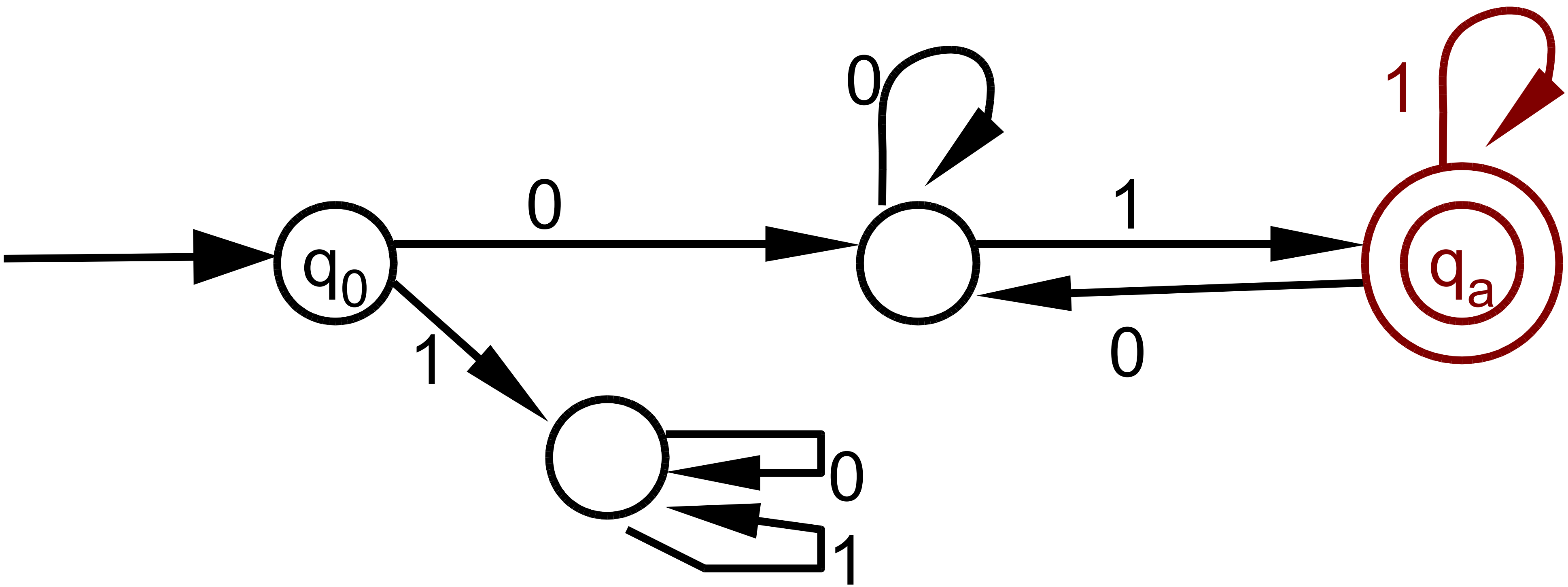
DFA (Deterministic Finite Automata)



Example: Input string

$w = 0011$

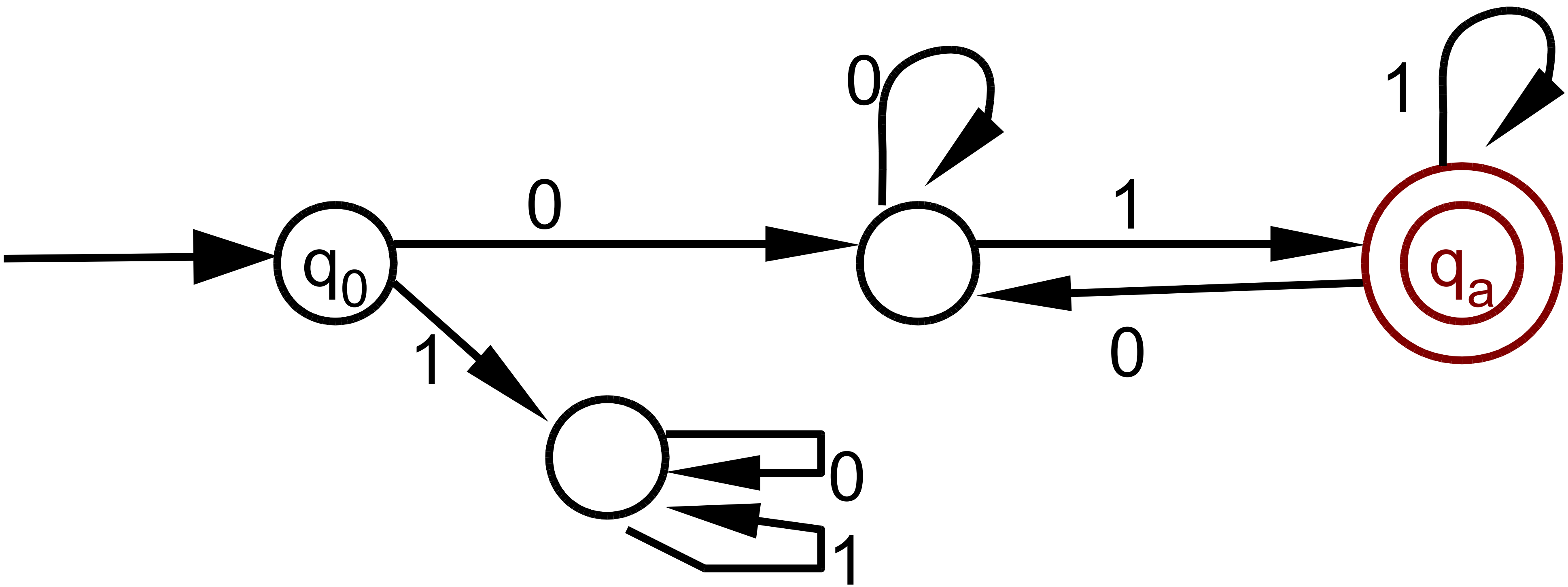
DFA (Deterministic Finite Automata)



Example: Input string

w = 001**1**

DFA (Deterministic Finite Automata)



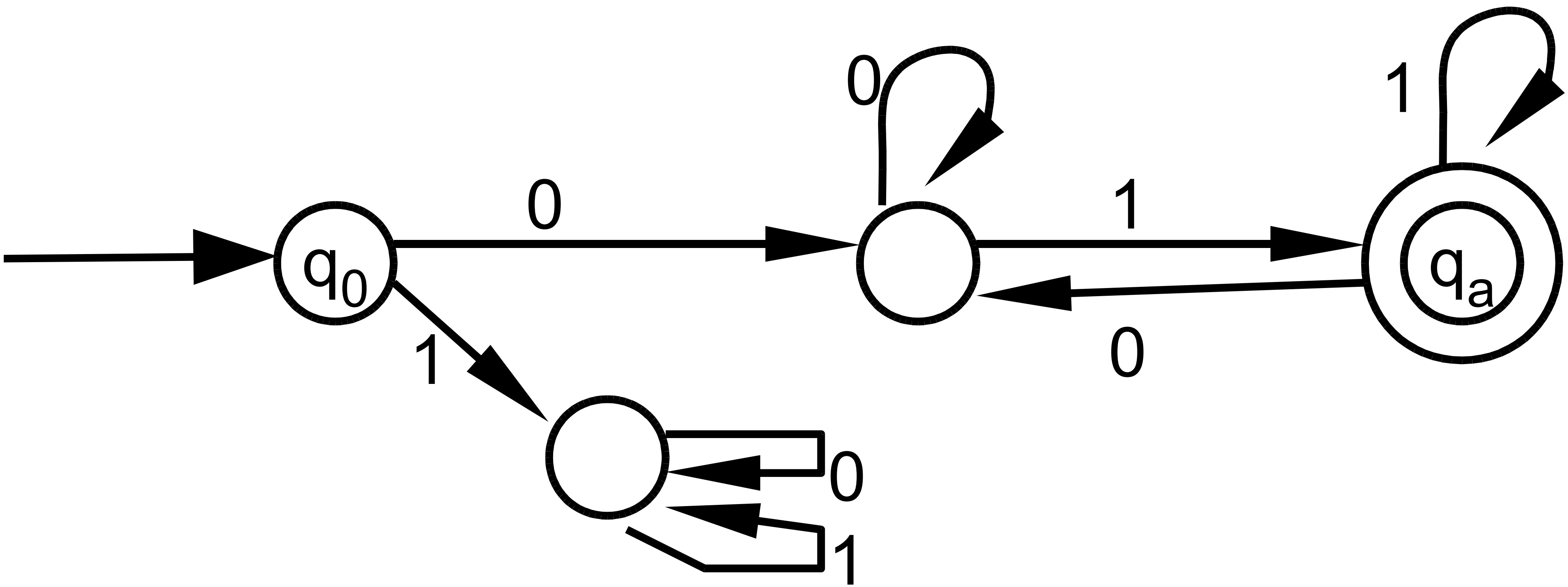
Example: Input string

w = 0011 **ACCEPT**

because end in

accept state

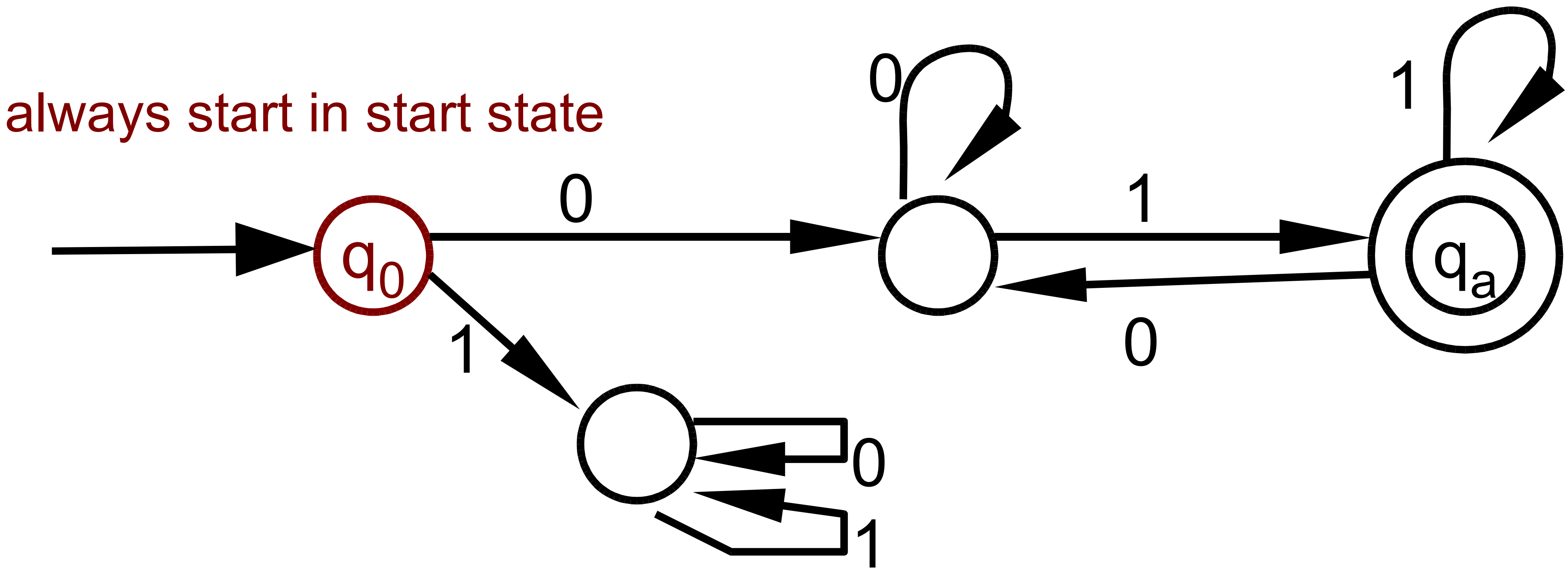
DFA (Deterministic Finite Automata)



Example: Input string

w = 010

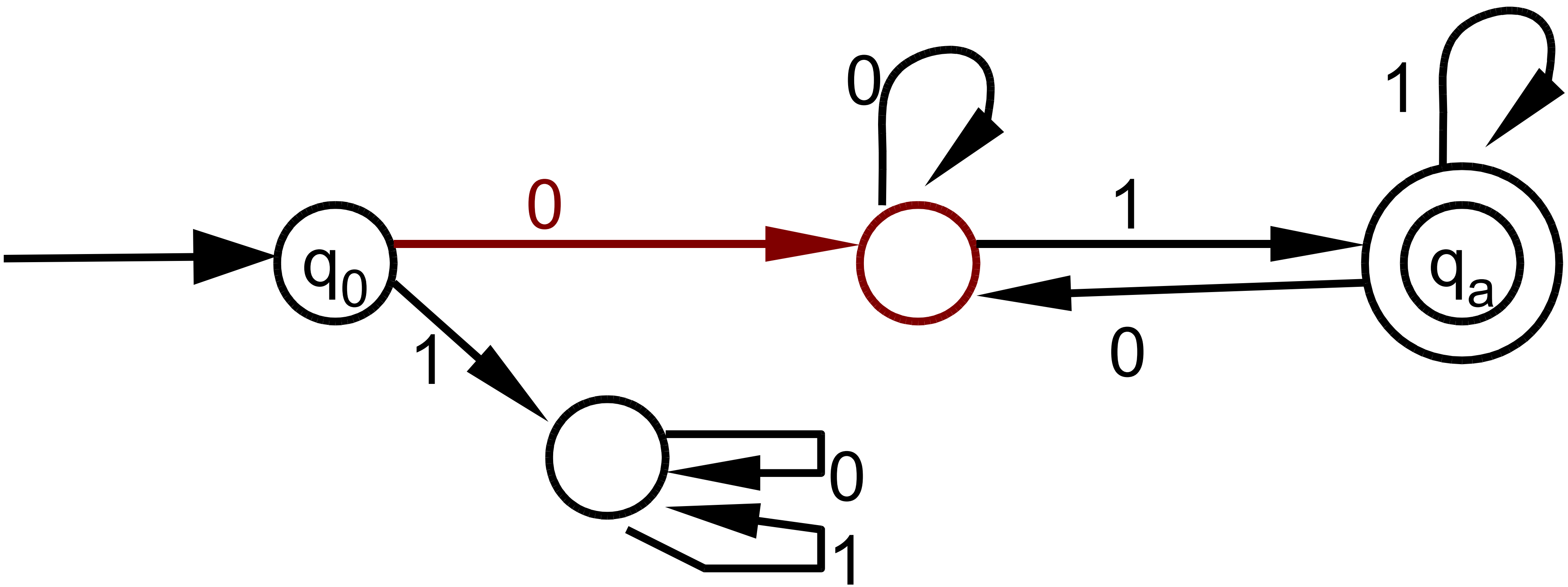
DFA (Deterministic Finite Automata)



Example: Input string

$w = 010$

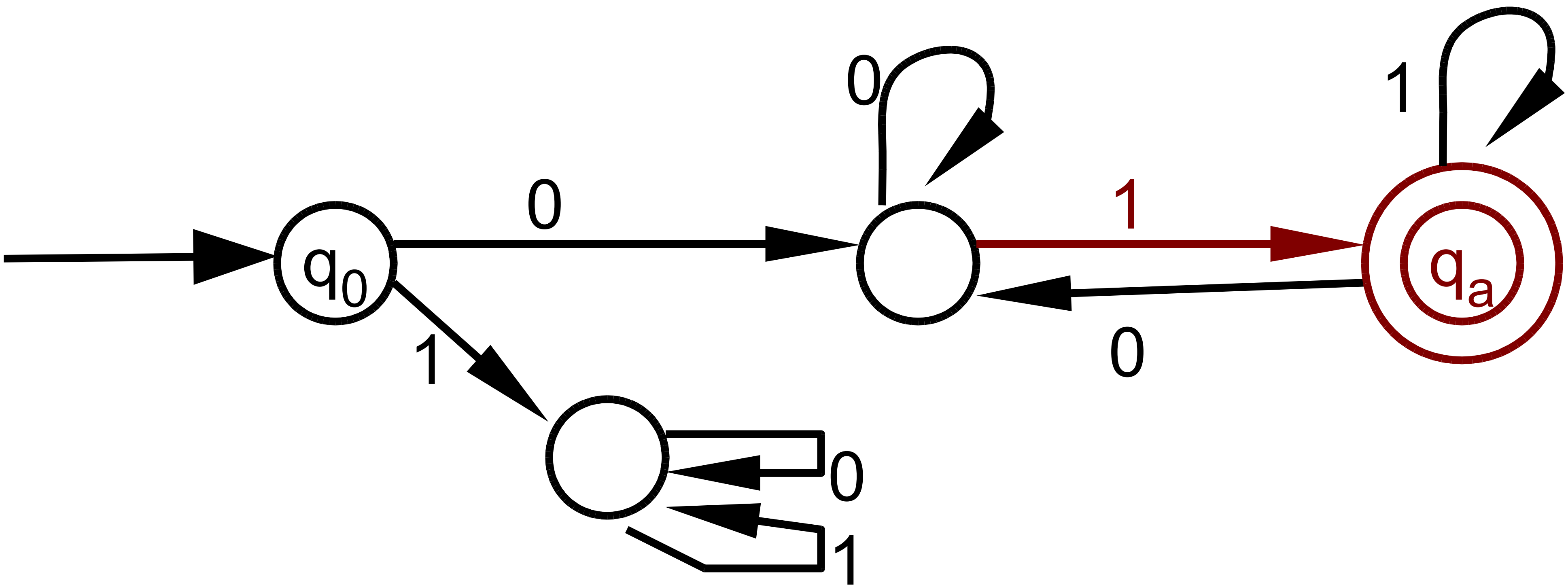
DFA (Deterministic Finite Automata)



Example: Input string

$w = \mathbf{0}10$

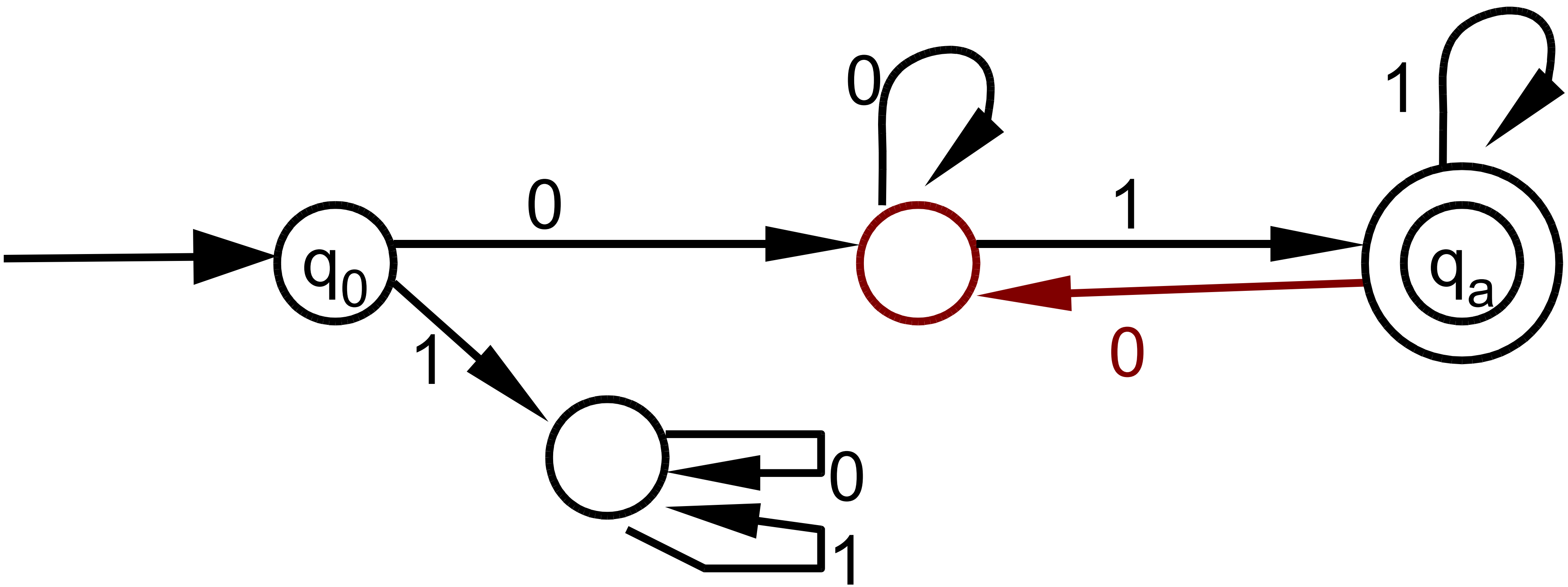
DFA (Deterministic Finite Automata)



Example: Input string

$w = 010$

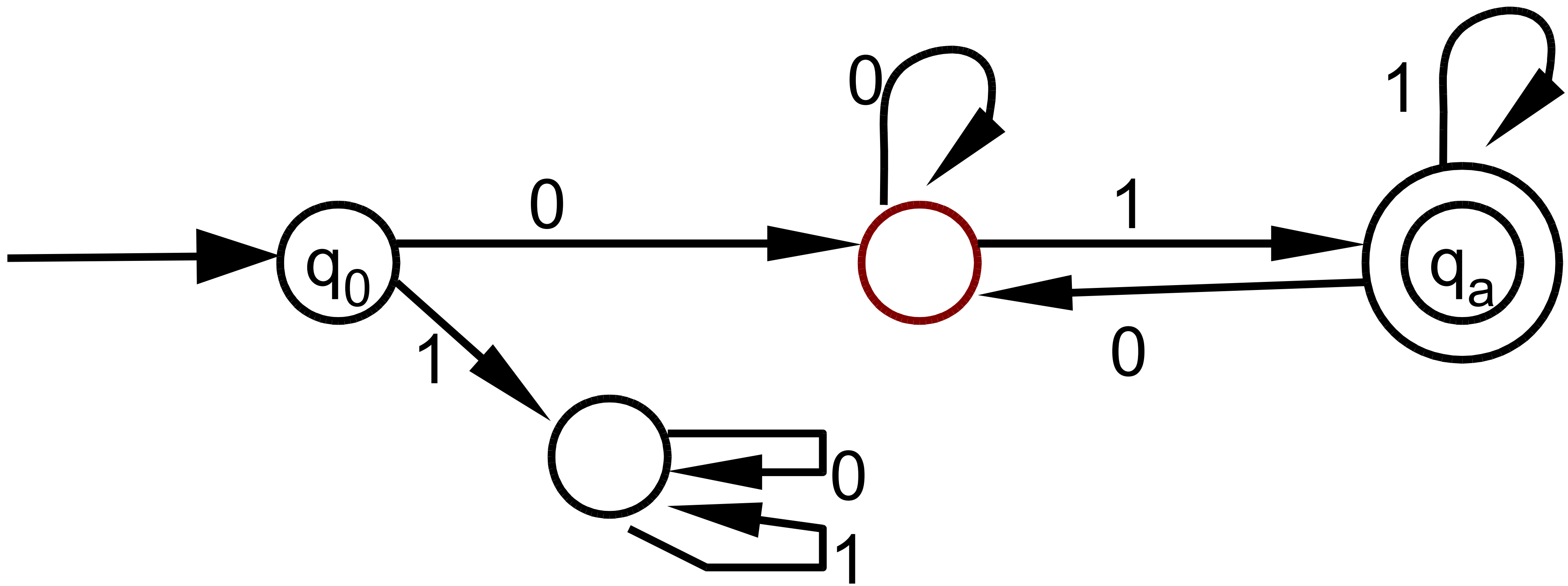
DFA (Deterministic Finite Automata)



Example: Input string

$w = 01\mathbf{0}$

DFA (Deterministic Finite Automata)

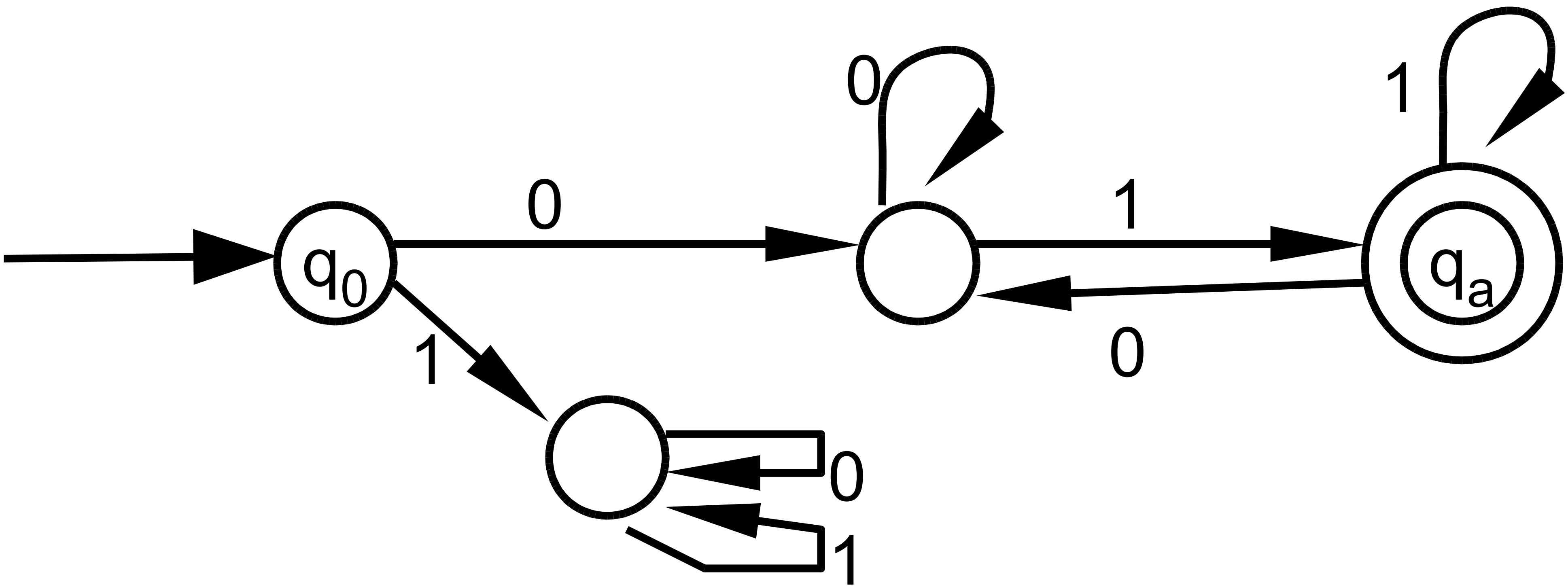


Example: Input string

$w = 010$ **REJECT**

because does not end in accept state

DFA (Deterministic Finite Automata)



Example: Input string w = 01 ACCEPT

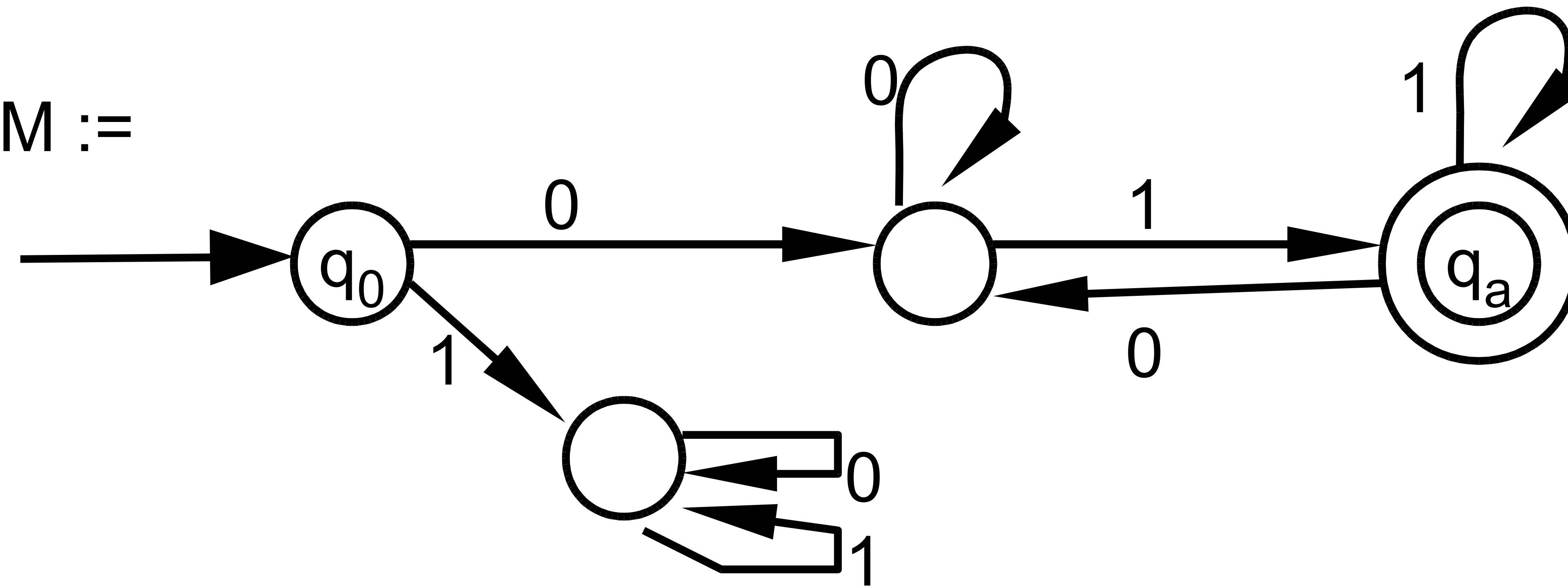
w = 010 REJECT

w = 0011 ACCEPT

w = 00110 REJECT

DFA (Deterministic Finite Automata)

M :=



M recognizes language

L(M) = { w : w starts with 0 and ends with 1 }

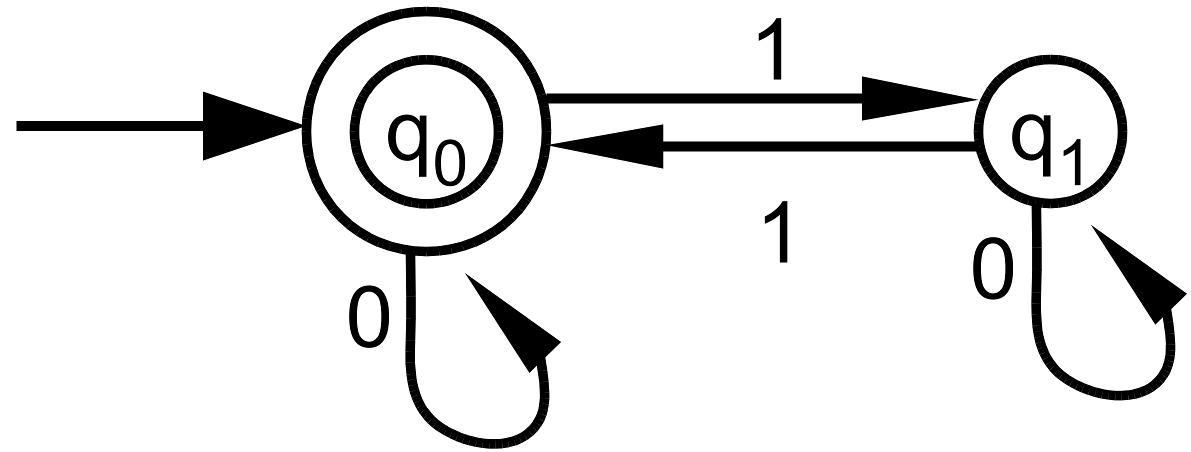
L(M) is the language of strings causing M to accept

Example: 0101 is an element of L(M), $0101 \in L(M)$

Example

$\Sigma = \{0,1\}$

M :=



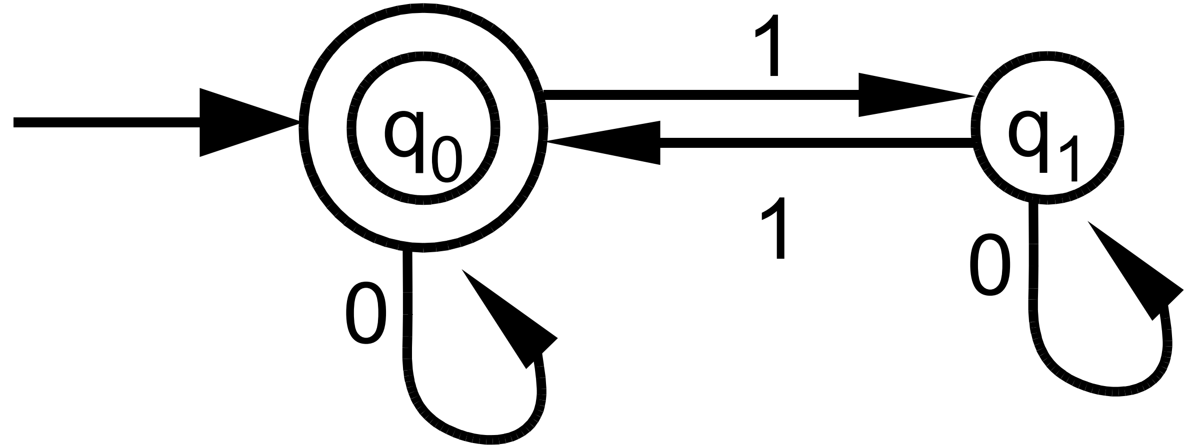
- 00 causes M to accept, so 00 is in L(M) $00 \in L(M)$
- 01 does not cause M to accept, so 01 not in L(M),

  $01 \notin L(M)$

- 0101 $\in L(M)$
- 01101100 $\in L(M)$
- 011010 $\notin L(M)$

Example

$M :=$



$\Sigma = \{0,1\}$

$L(M) = \{w : w$ has an even number of 1 $\}$

Note: If there is no 1, then there are zero 1,

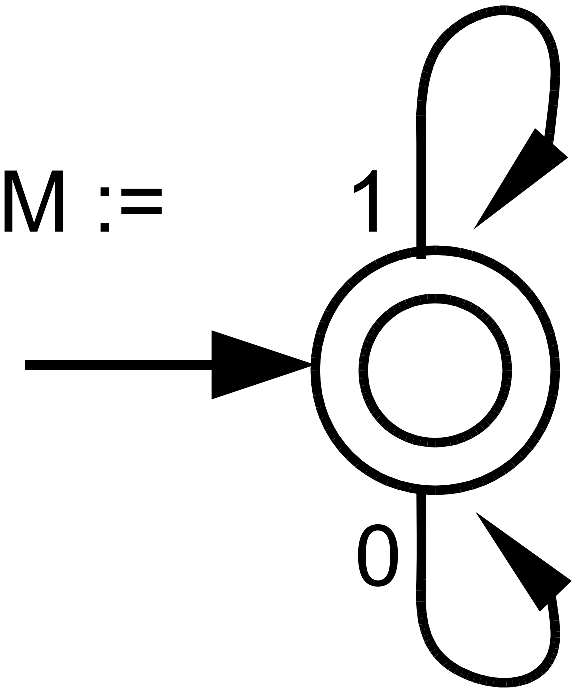
zero is an even number, so M should accept.

Indeed $0000000 \in L(M)$

Example

M :=

$\Sigma = \{0,1\}$

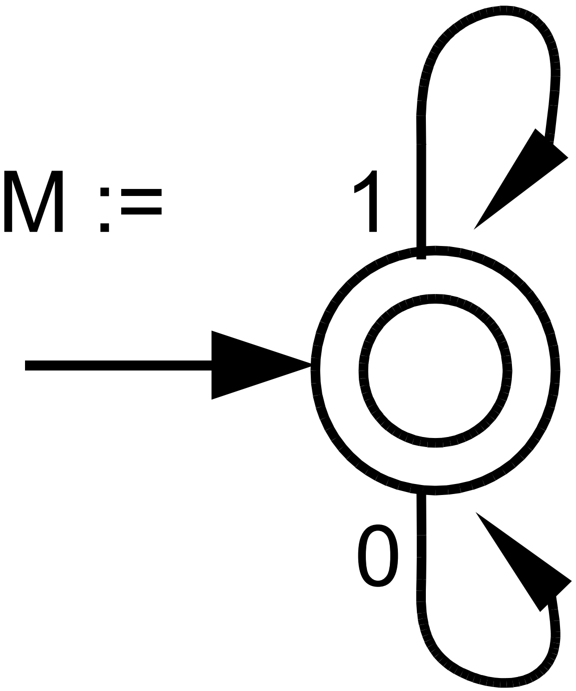


- L(M) = ?

Example

$\Sigma = \{0,1\}$

M :=



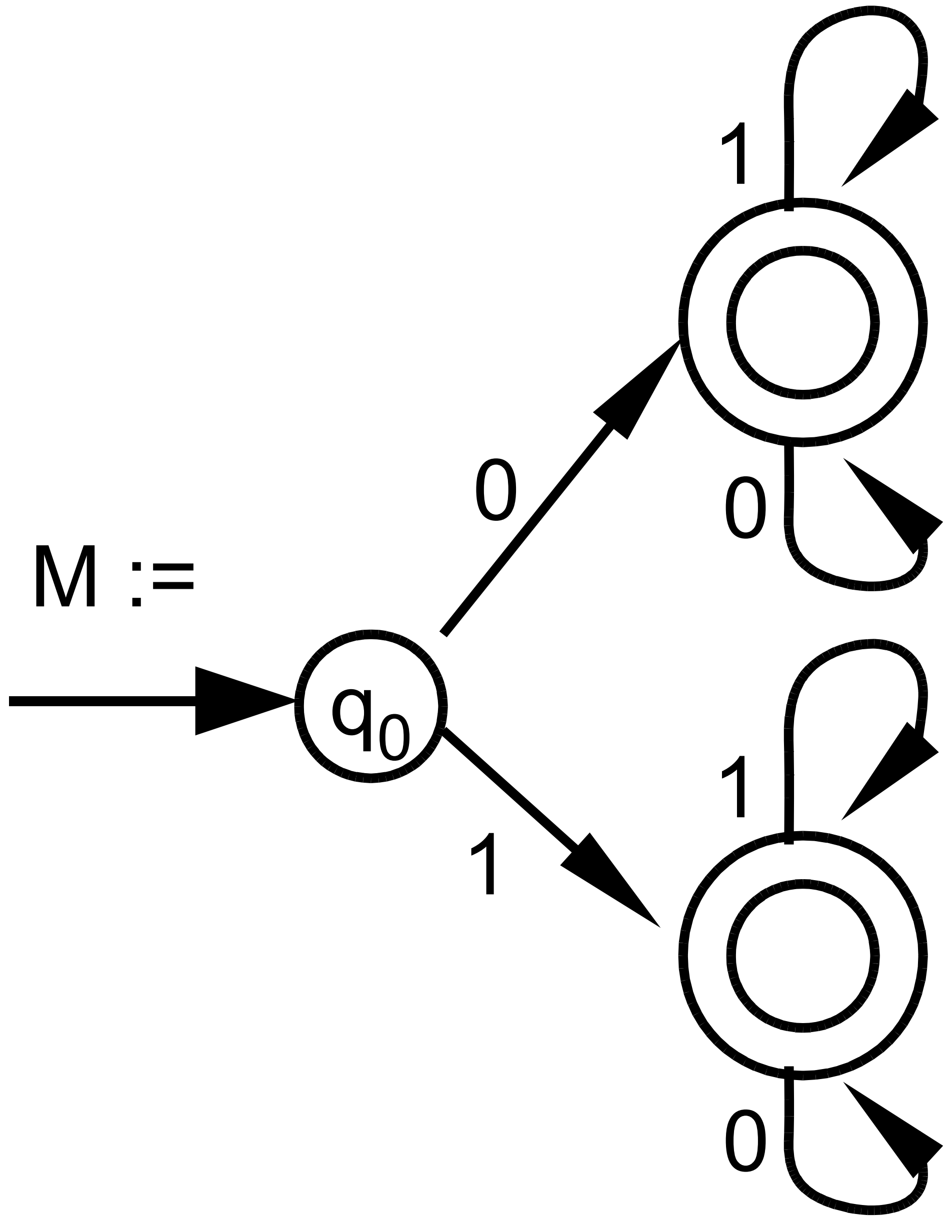
- L(M) = every possible string over {0,1}

  $= \{0,1\}^*$

Example

$\Sigma = \{0,1\}$

M :=

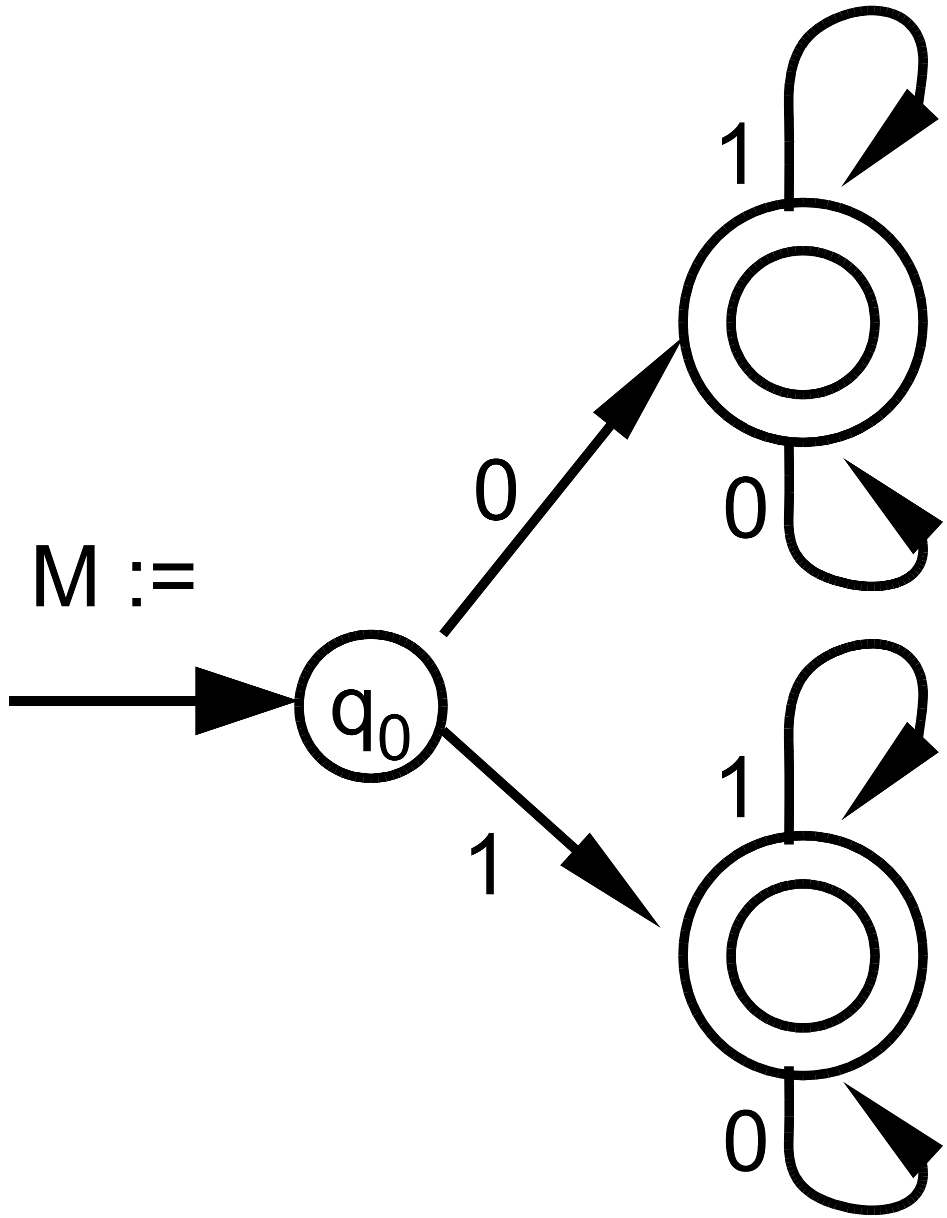


- L(M) = ?

Example

$\Sigma = \{0,1\}$

M :=

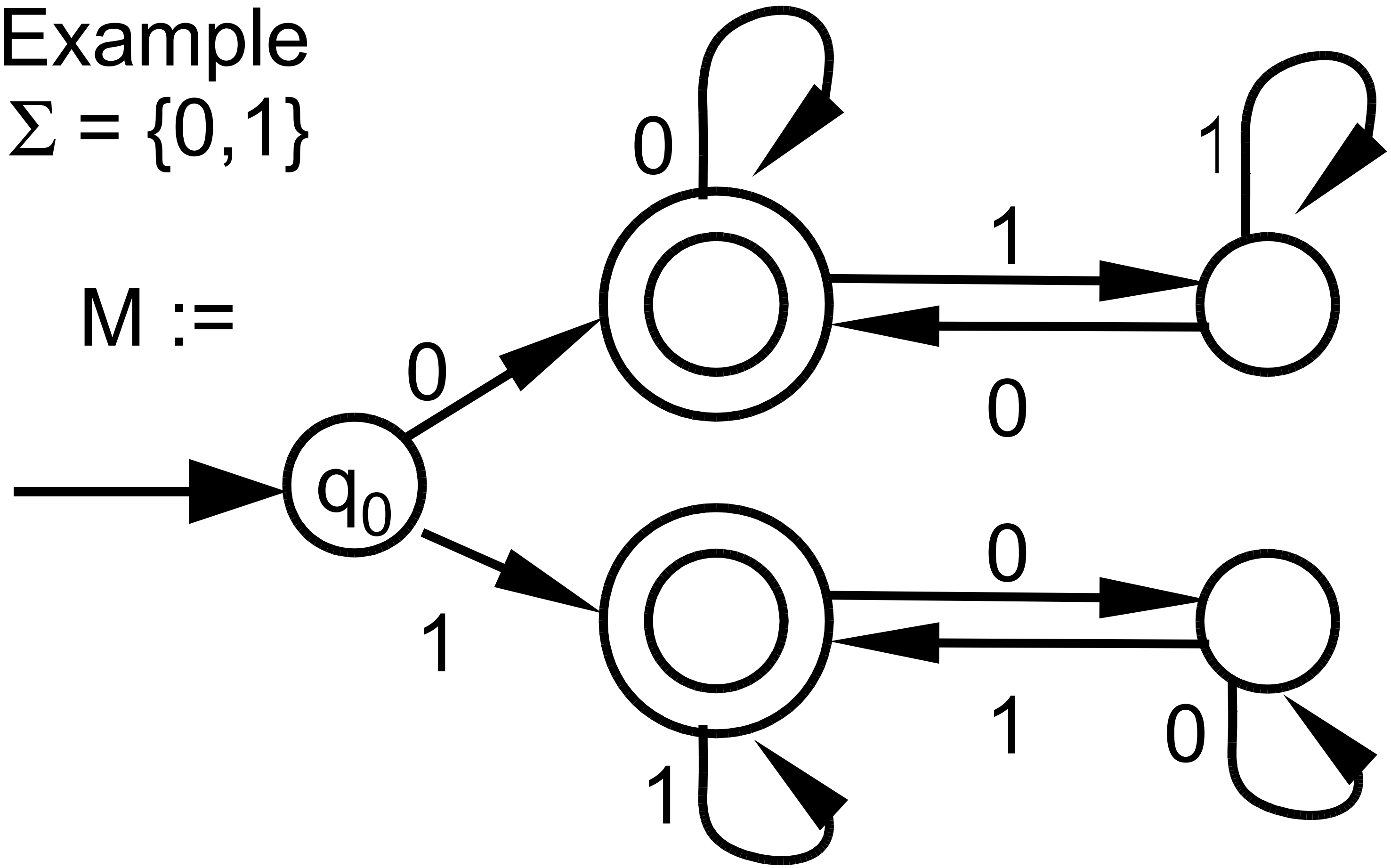


• L(M) = all strings over {0,1} except empty string ε

$= \{0,1\}^* - \{ \varepsilon \}$

Example
$\Sigma = \{0,1\}$

M :=

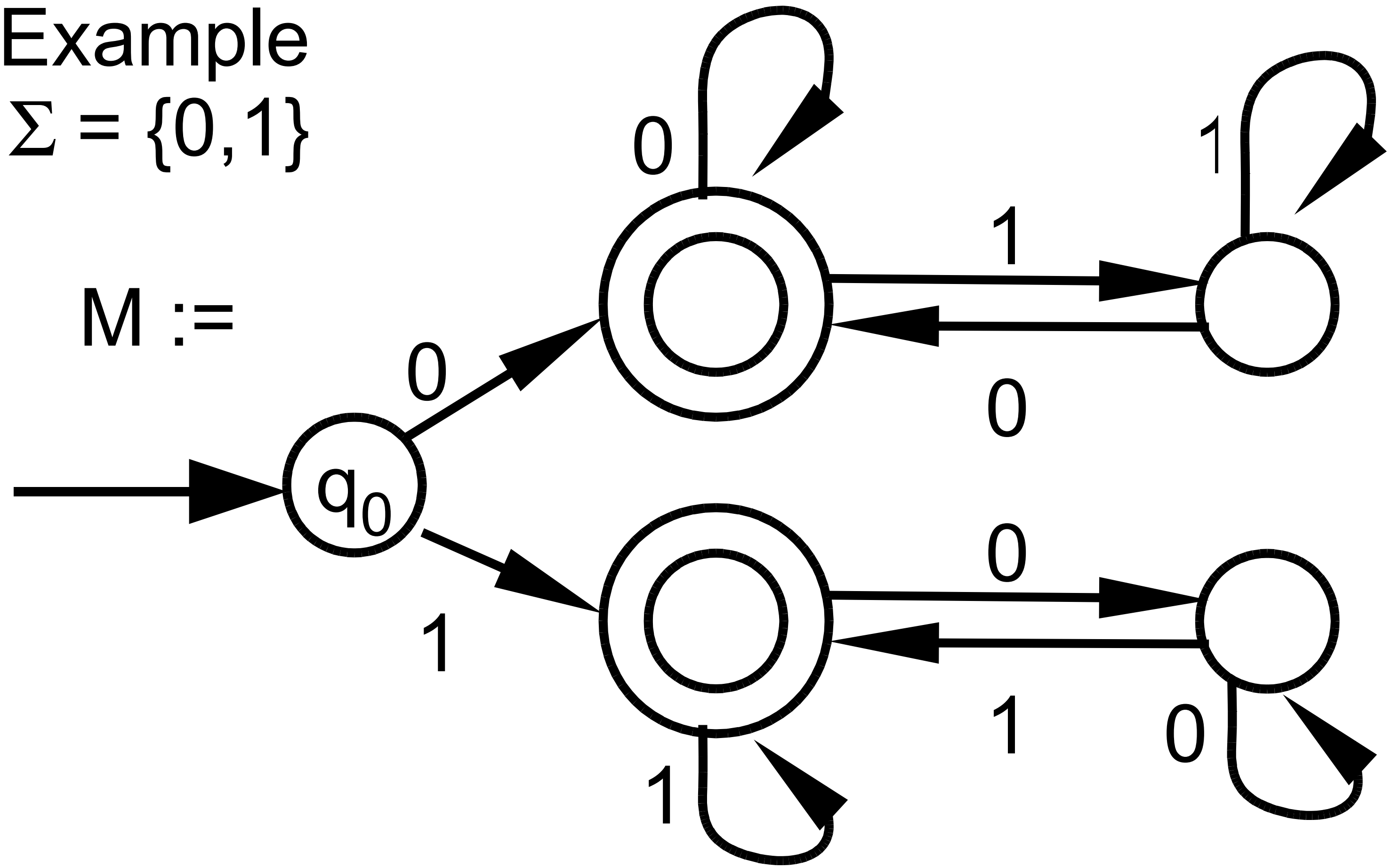


- L(M) = ?

Example

$\Sigma = \{0,1\}$

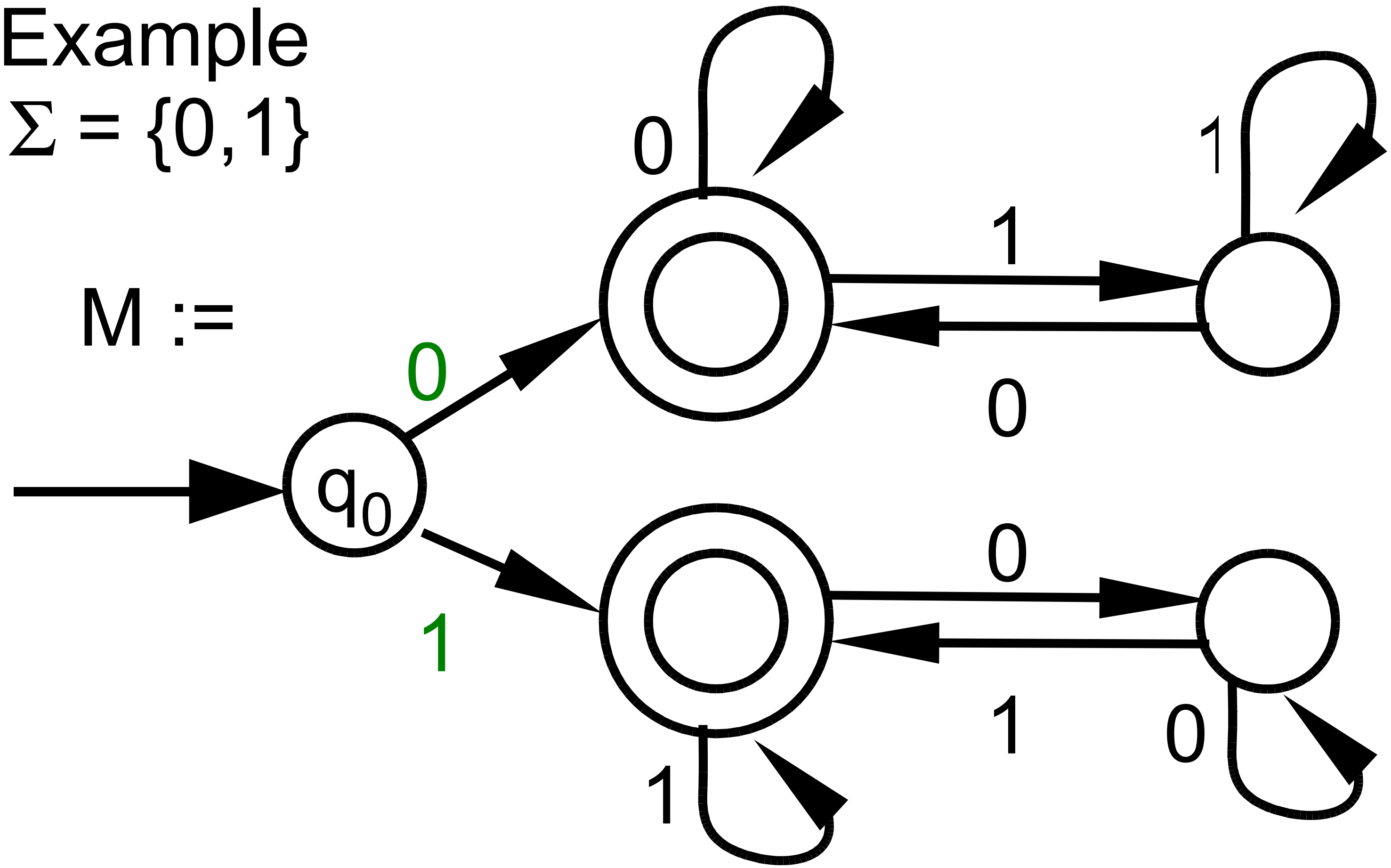
M :=



- L(M) = { w : w starts and ends with same symbol }
- Memory is encoded in … what ?

Example
$\Sigma = \{0,1\}$

M :=



Remember 0

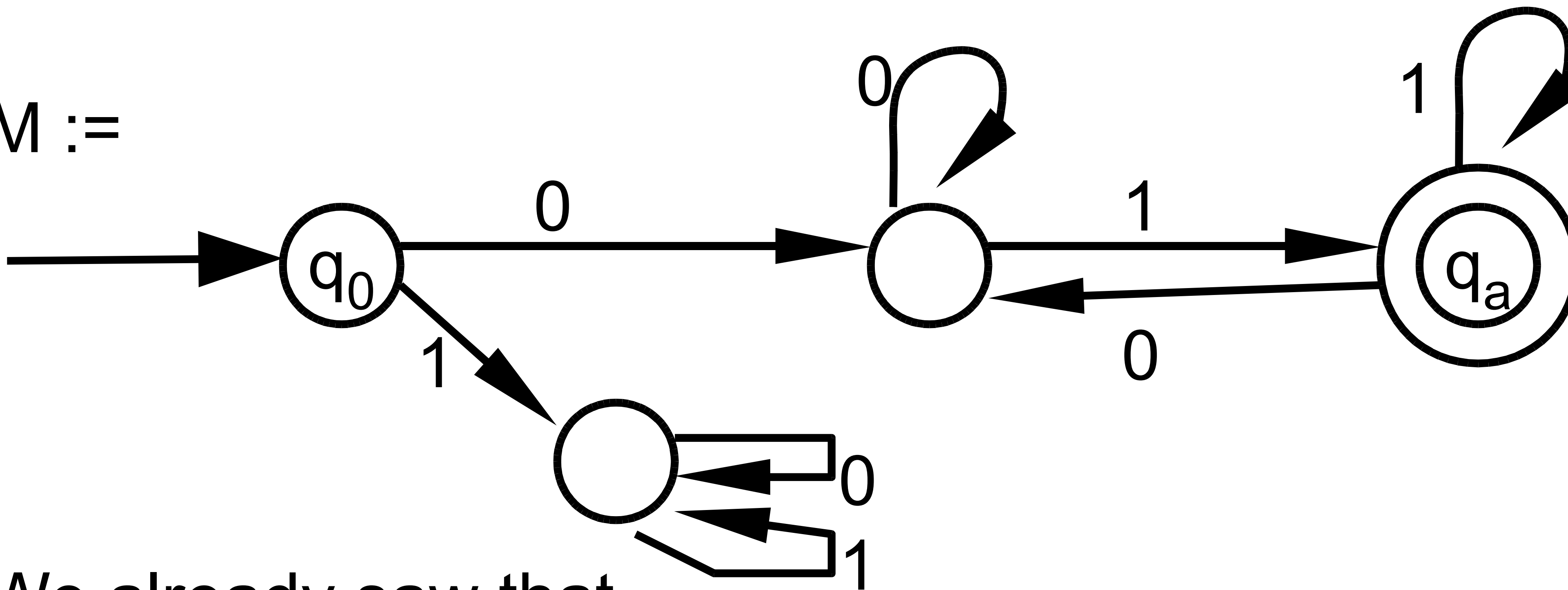
Remember 1

- L(M) = { w : w starts and ends with same symbol }
- Memory is encoded in states.

DFA have finite states, so finite memory

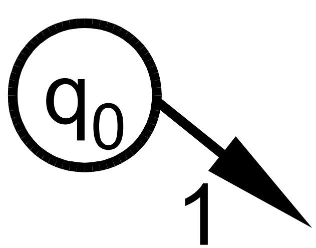
Convention:

M :=



We already saw that

L(M) = { w : w starts with 0 and ends with 1 }

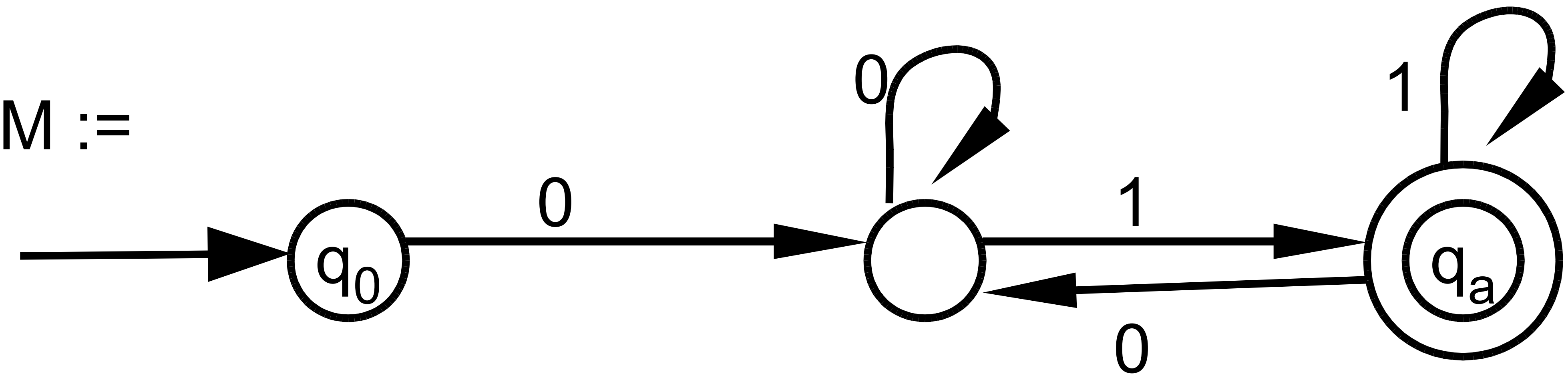
The arrow 

leads to a “sink” state.

If followed, M can never accept

Convention:

M :=



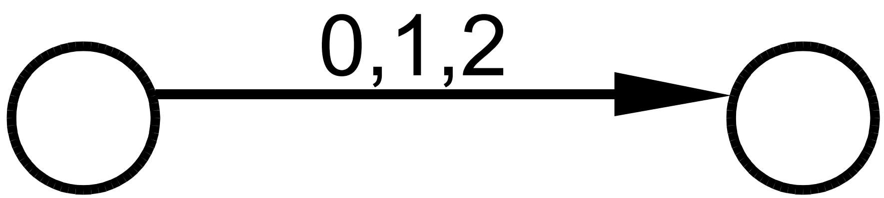
Don't need to write such arrows:

If, from some state, read symbol with no corresponding arrow, imagine M goes into “sink state” that is not shown, and REJECT.

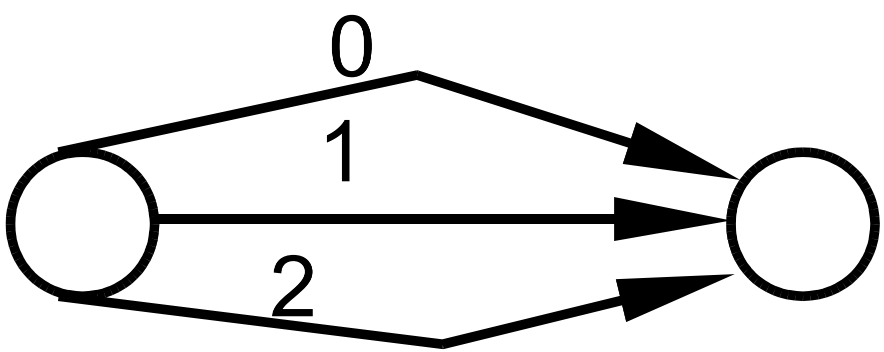
This makes pictures more compact.

Another convention:

List multiple transition on same arrow:



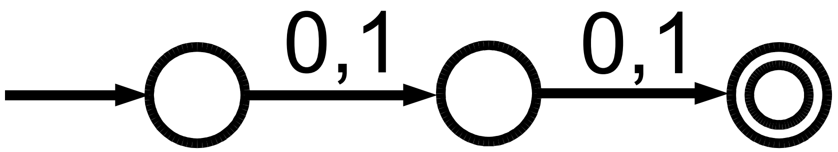
Means



This makes pictures more compact.

Example $\sum = \{0,1\}$

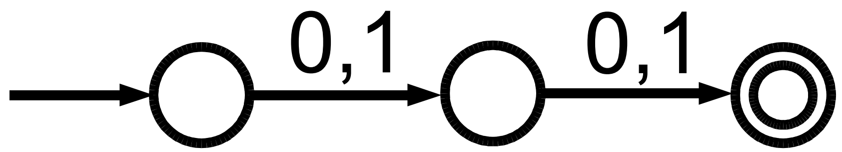
M =



L(M) = ?

Example $\sum = \{0,1\}$

M =

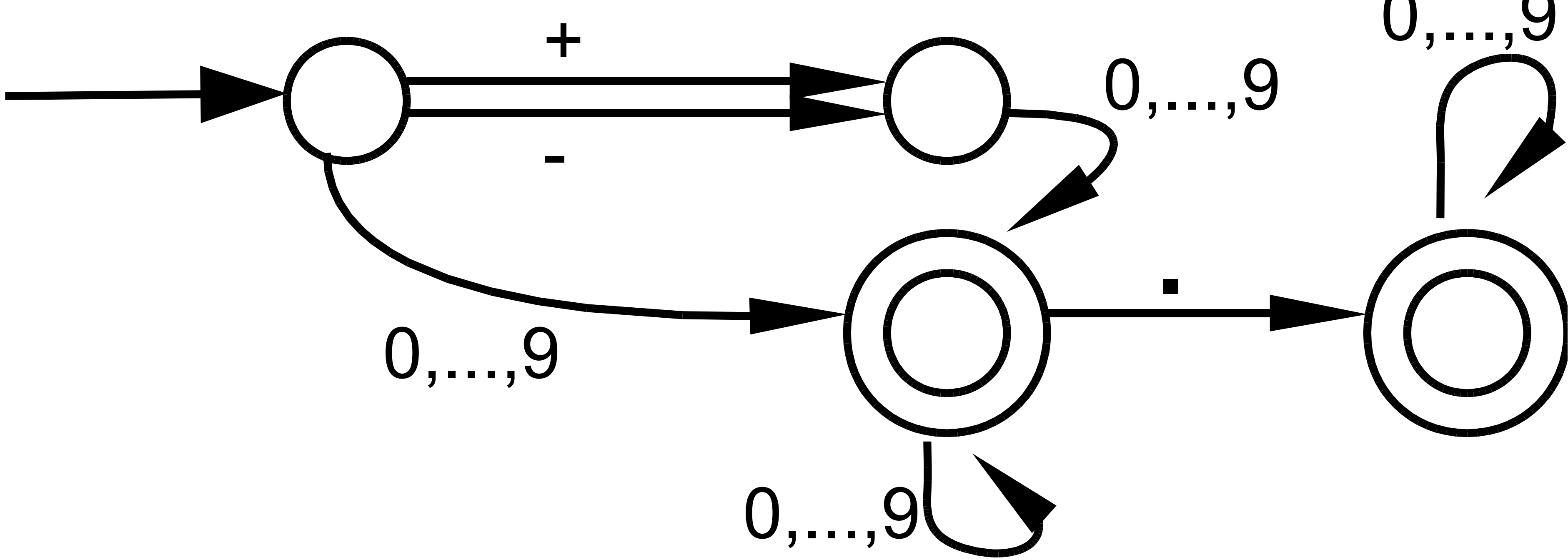


$L(M) = \sum^2 = \{00,01,10,11\}$

Example from programming languages:

Recognize strings representing numbers:

$\Sigma = \{0,1,2,3,4,5,6,7,8,9, +, -, .\}$

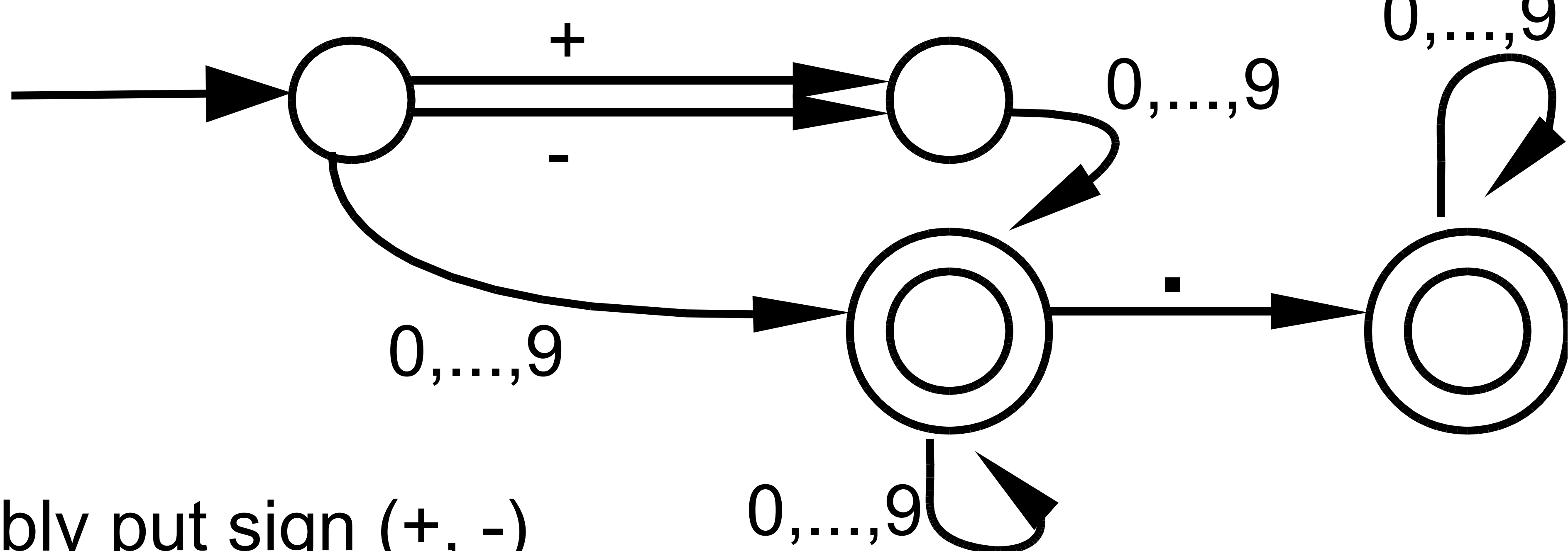


Note: 0,...,9 means 0,1,2,3,4,5,6,7,8,9: 10 transitions

Example from programming languages:

Recognize strings representing numbers:

$\Sigma = \{0,1,2,3,4,5,6,7,8,9, +, -, .\}$



Possibly put sign (+, -)

Follow with arbitrarily many digits, but at least one

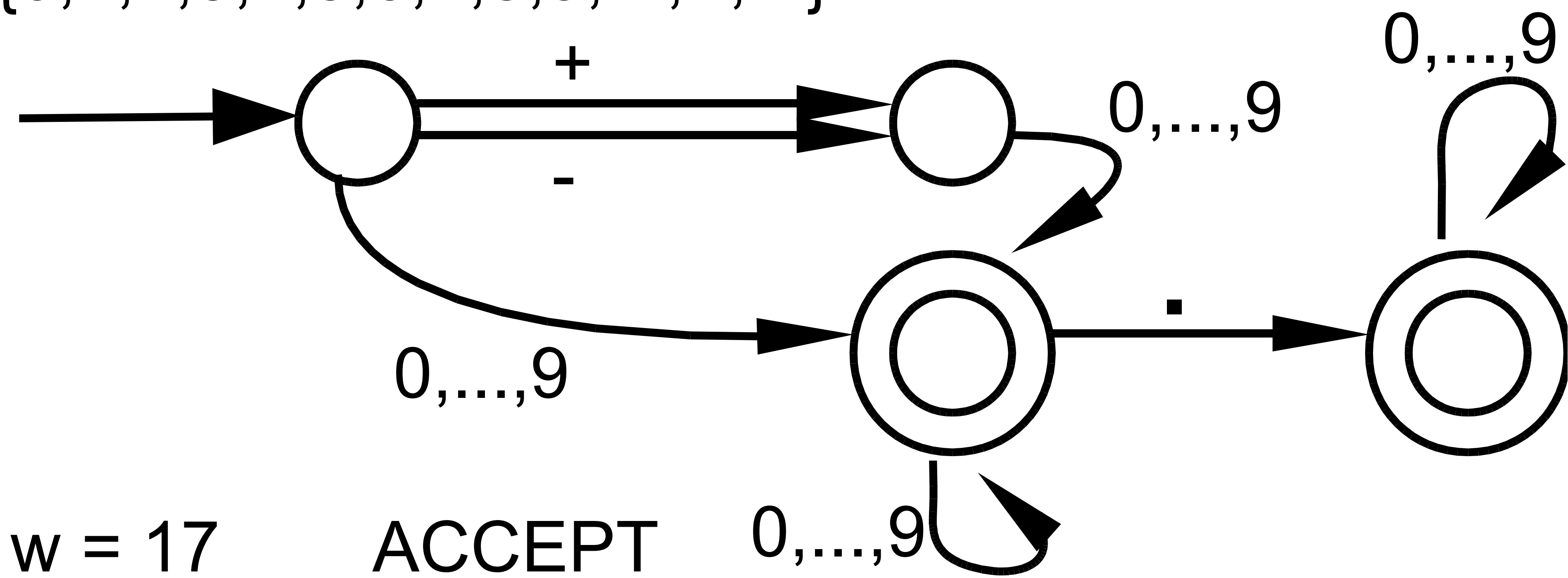
Possibly put decimal point

Follow with arbitrarily many digits, possibly none

Example from programming languages:

Recognize strings representing numbers:

$\Sigma = \{0,1,2,3,4,5,6,7,8,9, +, -, .\}$



Input w = 17 ACCEPT

Input w = + REJECT

Input w = -3.25 ACCEPT

Input w = +2.35-. REJECT

Example
$\Sigma = \{0,1\}$

- What about { w : w has same number of 0 and 1 }

- Can you design a DFA that recognizes that?

- It seems you need infinite memory

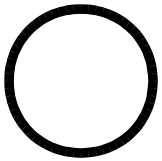
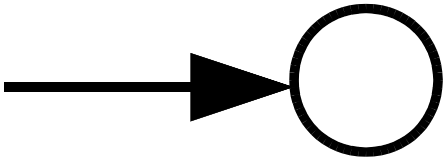
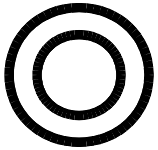
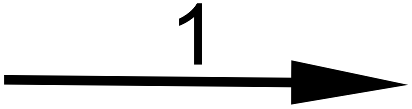
- We will prove later that

there is no DFA that recognizes that language !

Next: formal definition of DFA

- Useful to prove various properties of DFA
- Especially important to prove that things CANNOT be recognized by DFA.
- Useful to practice mathematical notation

State diagram of a DFA:

- One or more states 
- Exactly one start state 
- Some number of accept states 
- Labelled transitions exiting each state, 

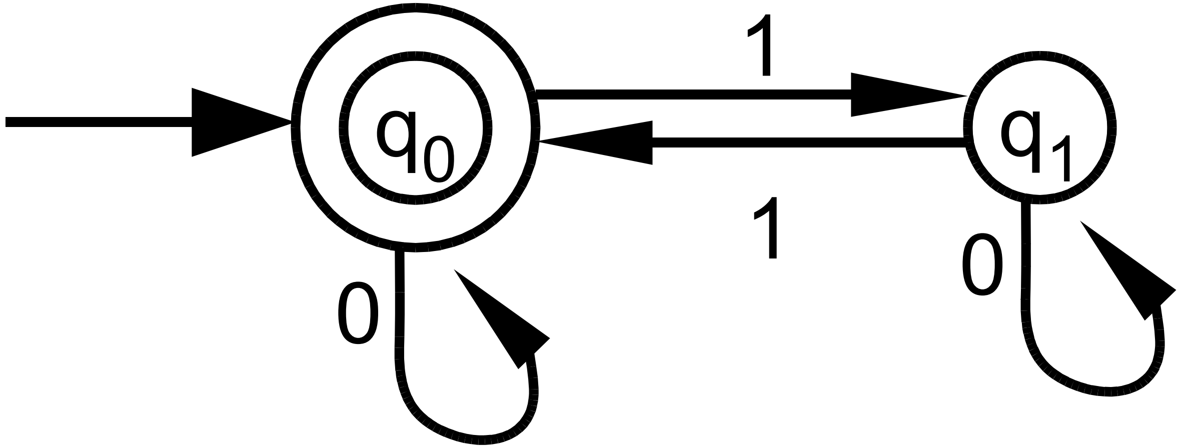
for every symbol in Σ

- Definition: A finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

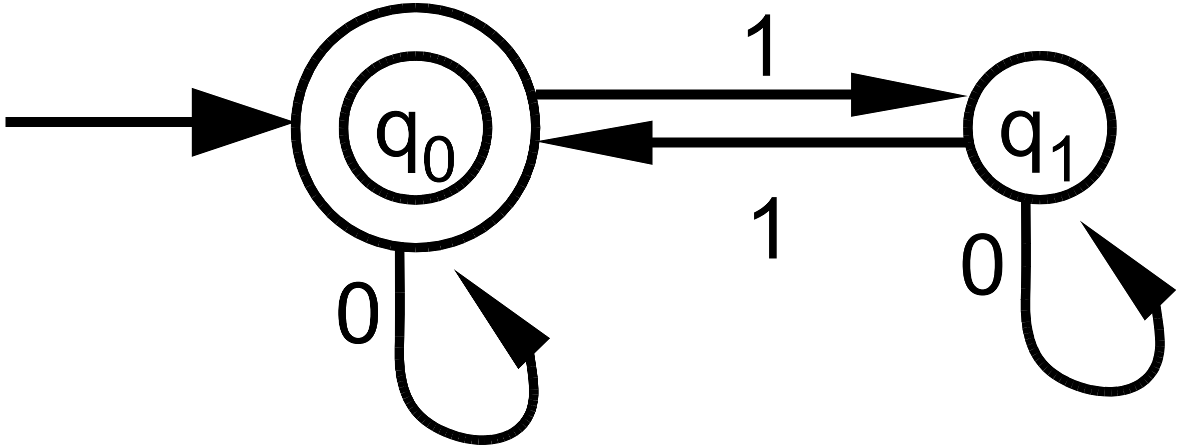
- Q is a finite set of states
- Σ is the input alphabet
- $\delta : Q \times \Sigma \rightarrow Q$ is the transition function
- q_0 in Q is the start state
- $F \subseteq Q$ is the set of accept states

$Q \times \Sigma$ is the set of ordered pairs $(a,b) : a \in Q, b \in \Sigma$

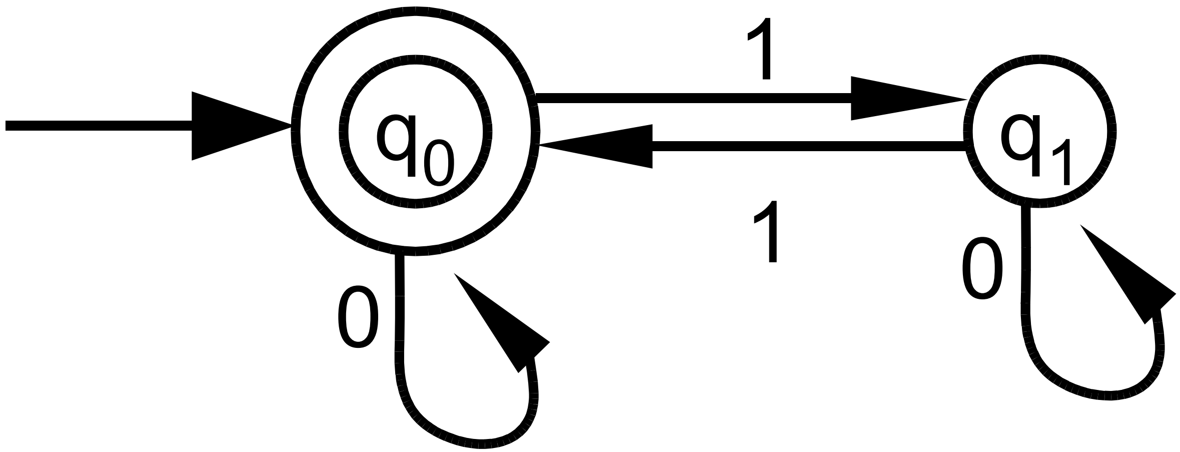
Example $\{q,r,s\} \times \{0,1\} = \{(q,0),(q,1),(r,0),(r,1),(s,0),(s,1)\}$



- Example: above DFA is 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
- $Q = \{ q_0, q_1\}$
- $\Sigma = \{0,1\}$
- $\delta(q_0 ,0) = ?$

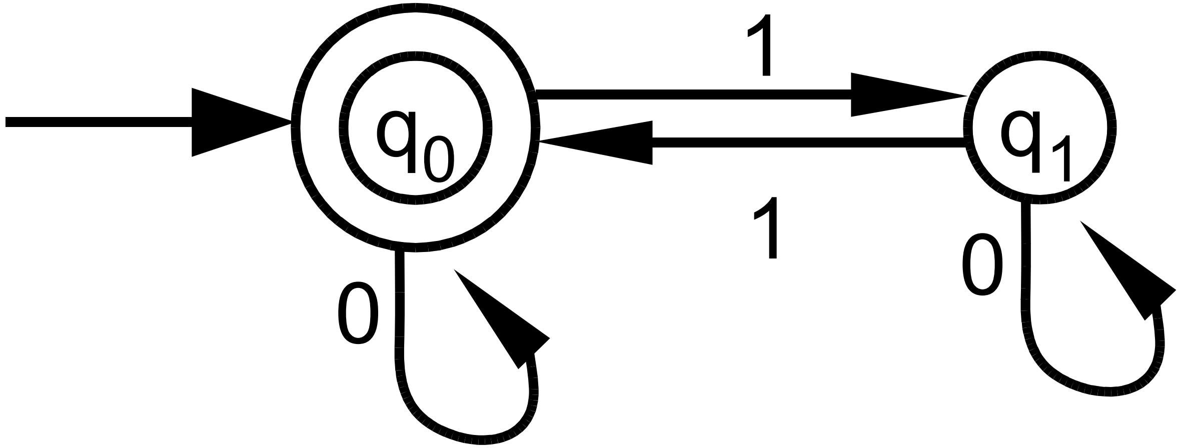


- Example: above DFA is 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
- $Q = \{ q_0, q_1\}$
- $\Sigma = \{0,1\}$
- $\delta(q_0, 0) = q_0$ $\delta(q_0, 1) = ?$



- Example: above DFA is 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
- $Q = \{ q_0, q_1\}$
- $\Sigma = \{0,1\}$
- $\delta(q_0, 0) = q_0$ $\delta(q_0, 1) = q_1$

  $\delta(q_1, 0) = q_1$ $\delta(q_1, 1) = q_0$
- q_0 in Q is the start state
- $F = ?$



- Example: above DFA is 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
- $Q = \{ q_0, q_1\}$
- $\Sigma = \{0,1\}$
- $\delta(q_0 ,0) = q_0 \quad \delta(q_0 ,1) = q_1$

  $\delta(q_1 ,0) = q_1 \quad \delta(q_1 ,1) = q_0$
- q_0 in Q is the start state
- $F = \{ q_0\} \subseteq Q$ is the set of accept states

- Definition: A DFA $(Q, \Sigma, \delta, q_0, F)$ accepts a string w if

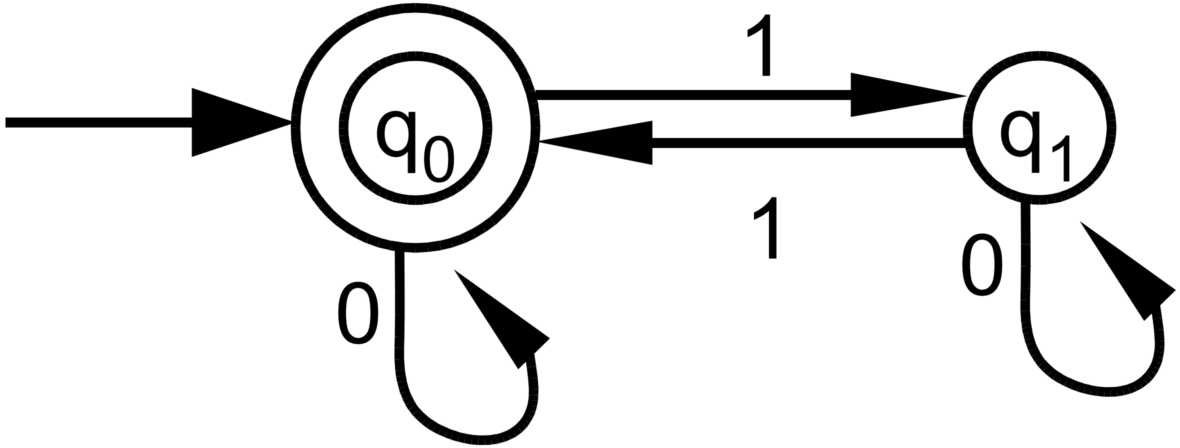
- $w = w_1 w_2 \ldots w_k$ where, $\forall\ 1 \le i \le k$, w_i is in Σ

  (the k symbols of w)

- $\exists$ sequence of k+1 states $r_0, r_1, .., r_k$ in Q such that:
- $r_0 = q_0$
- $r_{i+1} = \delta(r_i, w_{i+1})\ \forall\ 0 \le i < k$
- r_k is in F

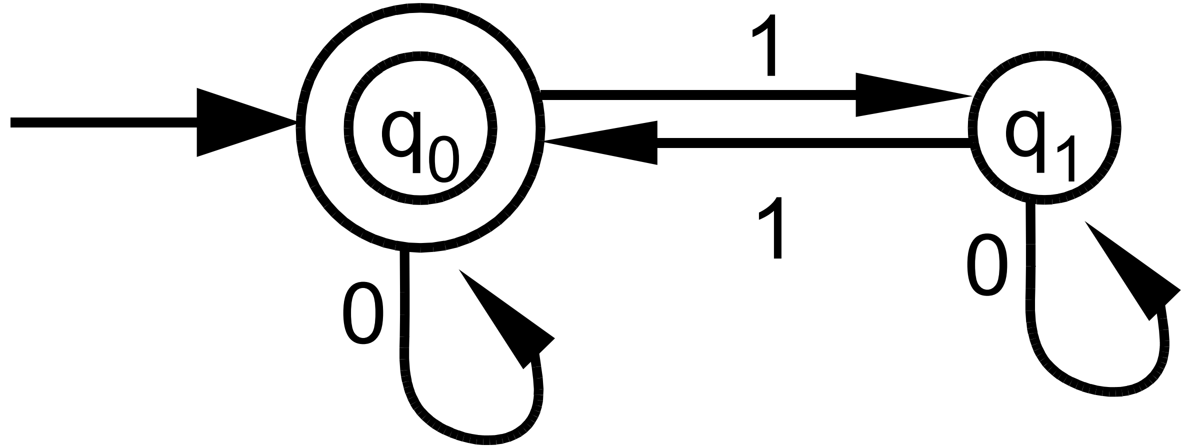
  (r_i = state DFA is in after reading i-th symbol in w)

Example



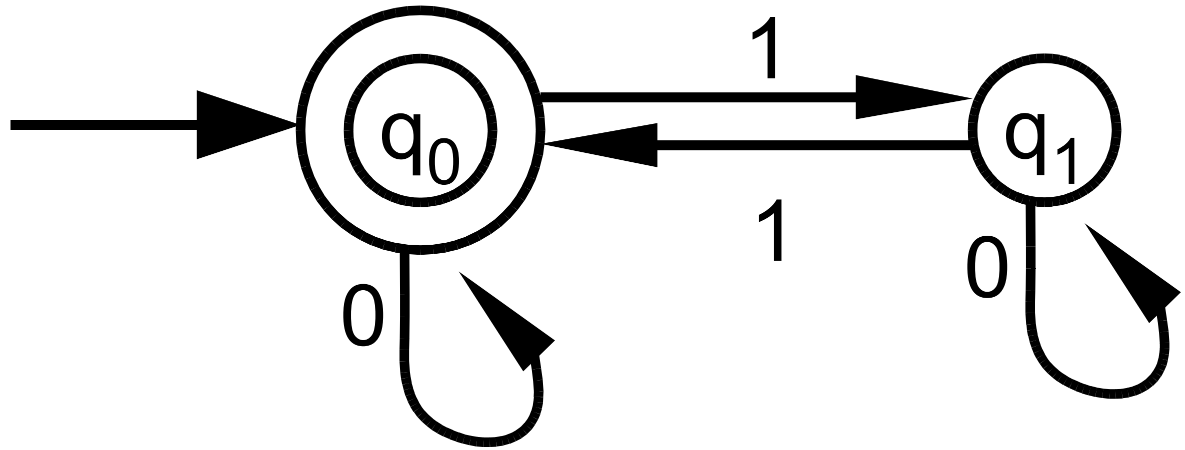
- Above DFA $(Q, \Sigma, \delta, q_0, F)$ accepts w = 011

Example



- Above DFA $(Q, \Sigma, \delta, q_0, F)$ accepts $w = 011$
- $w = 011 = w_1\ w_2\ w_3$ $\qquad w_1 = 0 \quad w_2 = 1 \quad w_3 = 1$

Example

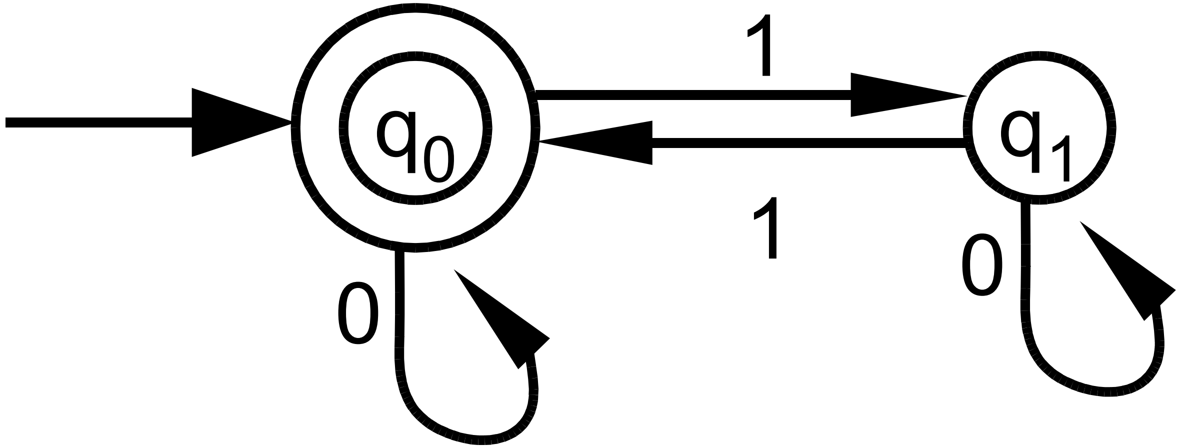


- Above DFA $(Q, \Sigma, \delta, q_0, F)$ accepts $w = 011$
- $w = 011 = w_1\, w_2\, w_3$ $\qquad w_1 = 0 \quad w_2 = 1 \quad w_3 = 1$

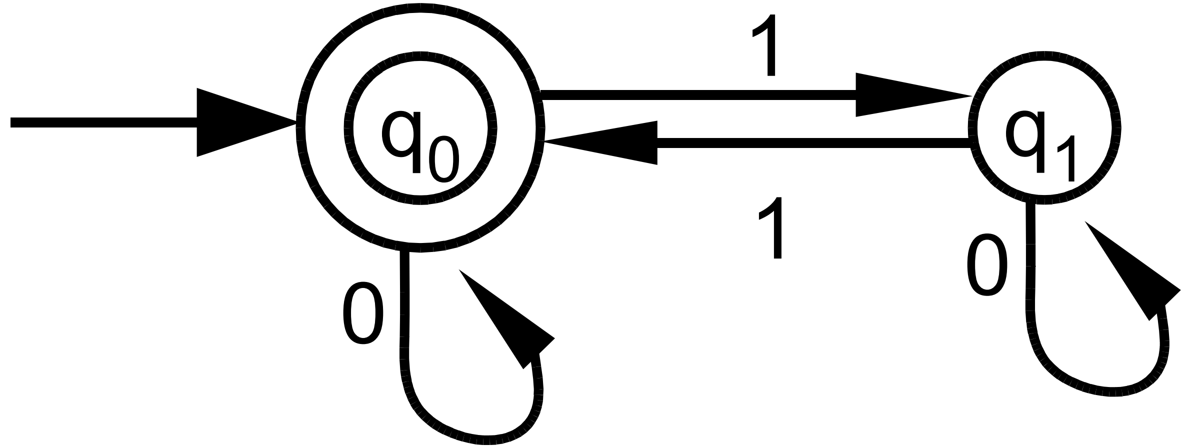
WE MUST SHOW THAT

- $\exists$ sequence of 3+1=4 states r_0, r_1, r_2, r_3 in Q that:
- $r_0 = q_0$
- $r_{i+1} = \delta(r_i, w_{i+1})\ \forall\ 0 \le i < 3$
- r_3 is in F

Example



- Above DFA $(Q, \Sigma, \delta, q_0, F)$ accepts $w = 011$
- $w = 011 = w_1\ w_2\ w_3$ $\qquad$ $w_1 = 0 \quad w_2 = 1 \quad w_3 = 1$
- consider 4 states $r_0 :=$?

Example 

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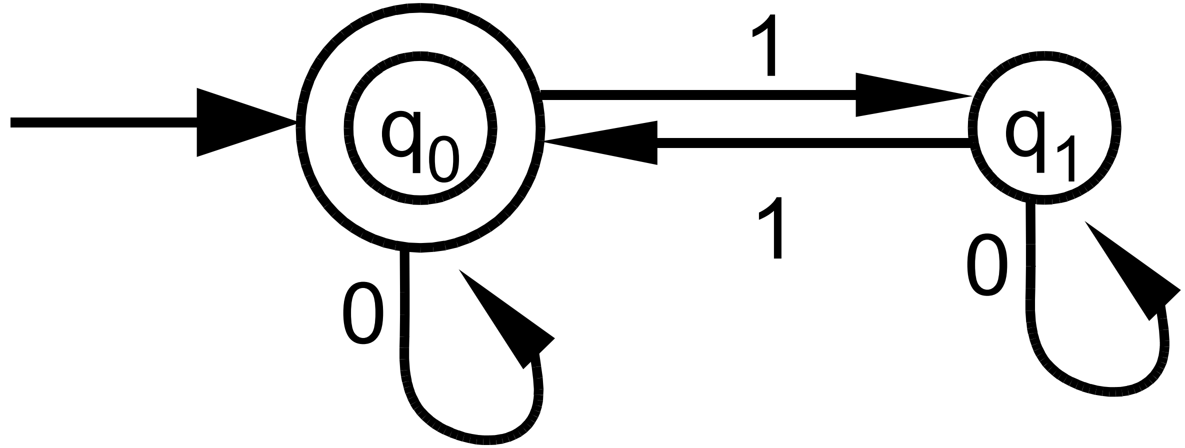
- $r_0 = q_0$ OK

Example 

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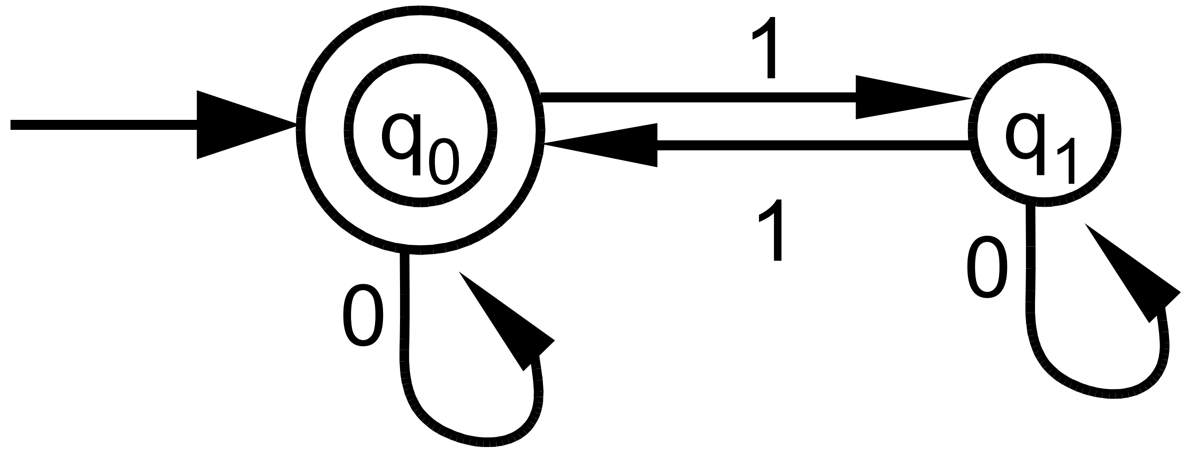
- $r_0 = q_0$ OK
- $r_1 = \delta(r_0, w_1) = \delta(q_0, 0) = q_0$ OK

Example



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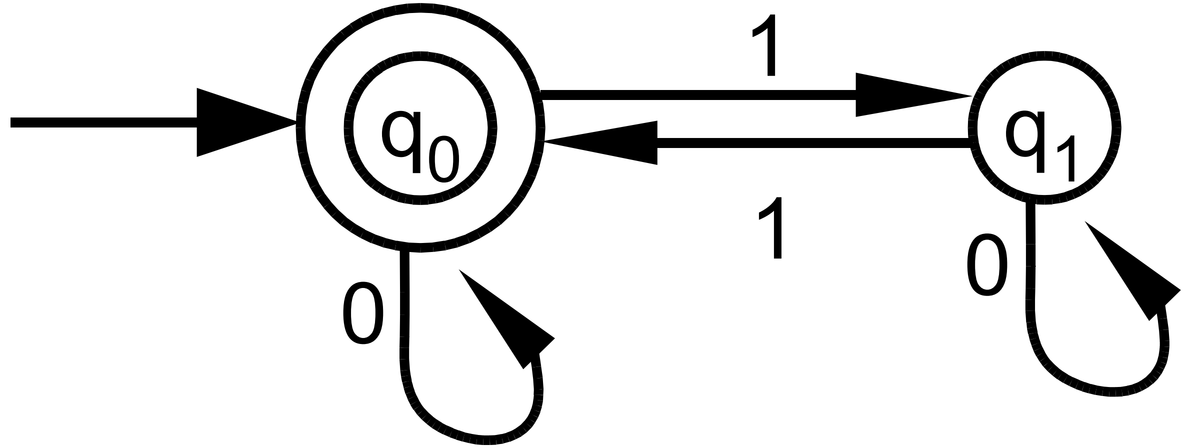
Example



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- $r_0 = q_0$ OK
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- $r_2 = \delta(r_1, w_2) = \delta(q_0, 1) = q_1$ OK
- $r_3 = \delta(r_2, w_3) = \delta(q_1, 1) = q_0$ OK
- $r_3 = q_0$ in F OK DONE!

- Definition: For a DFA M, we denote by L(M) the set of strings accepted by M:

$$L(M) := \{ w : M \text{ accepts } w\}$$

We say M accepts or recognizes the language L(M)

- Definition: A language L is regular

$$\text{if } \exists \text{ DFA } M : L(M) = L$$

In the next lectures we want to:

• Understand power of regular languages

• Develop alternate, compact notation to specify regular languages

Example: Unix command ***grep '\<c.*h\>'*** *file*
selects all words starting with c and ending with h in *file*

- Understand power of regular languages:

- Suppose A, B are regular languages, what about
- not A := { w : w is not in A }
- A U B := { w : w in A or w in B }
- A o B := { $w_1 w_2$: w_1 in A and w_2 in B }
- A* := { $w_1 w_2 \ldots w_k$: $k \geq 0$, w_i in A for every i }

- Are these languages regular?

- Understand power of regular languages:

- Suppose A, B are regular languages, what about
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- Terminology: Are regular languages closed under not, U, o, * ?

- Theorem:

  If A is a regular language, then so is (not A)

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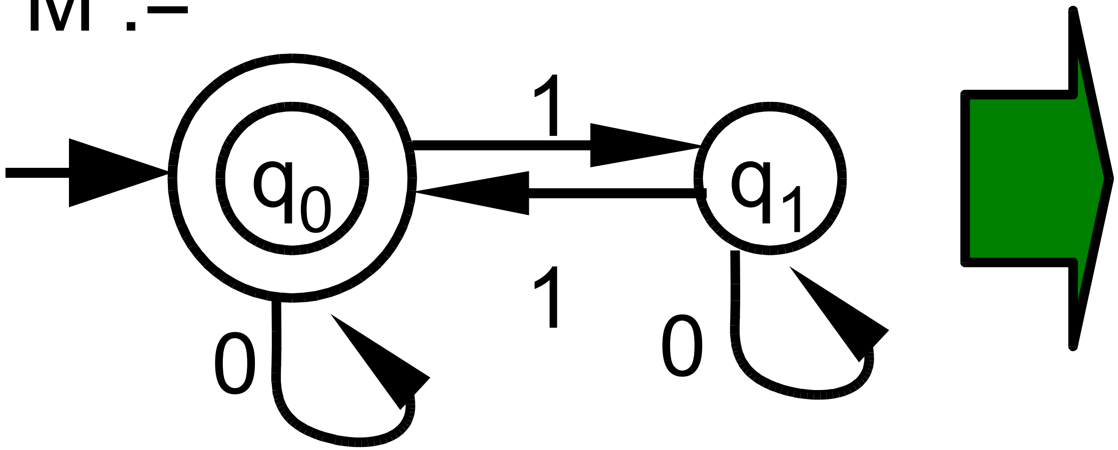
M :=



L(M) =

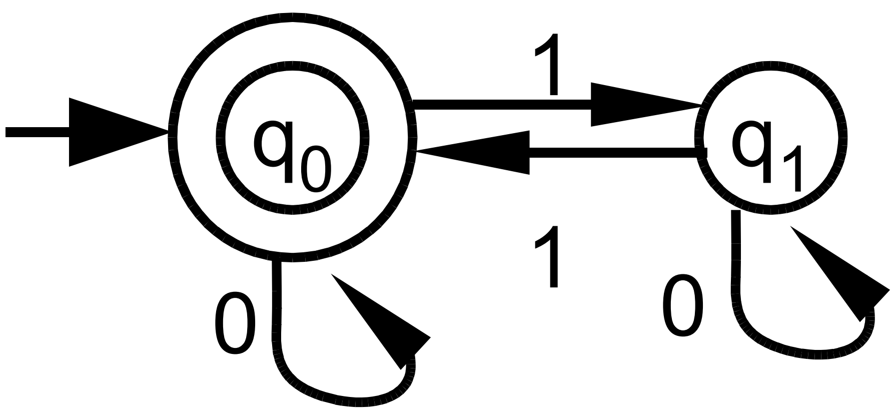
{ w : w has even number of 1}

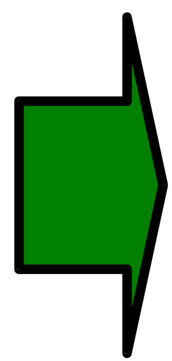
- Theorem:

    If A is a regular language, then so is (not A)

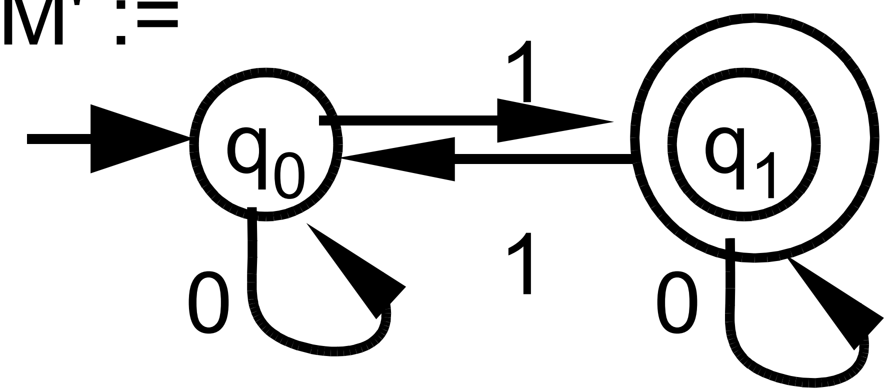
- Proof idea: Complement the set of accept states
- Example:

M :=





M' :=



L(M) =

{ w : w has even number of 1}

L(M') = not L(M) =

{ w : w has odd number of 1}

- Theorem:

If A is a regular language, then so is (not A)

- Formal construction:

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $L(M) = A$

Construct DFA $M' = (Q, \Sigma, \delta, q_0, F')$

$F' :=$ not F

- $L(M') =$ not A because

M' accepts w $\Leftrightarrow$ M does not accept w

- **Theorem**:

  If A is a regular language, then so is (not A)

- Formal constr

  Given DFA M … L(M) = A

  Construct DFA M … q_0, F')

  F' := not F

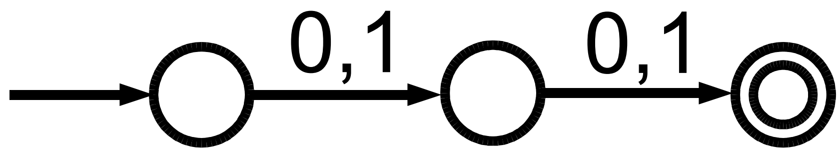
Use formal definition of accept to see this!

- L(M') = not A because

  M' accepts w ⇔ M does not accept w

Example $\Sigma = \{0,1\}$

M =



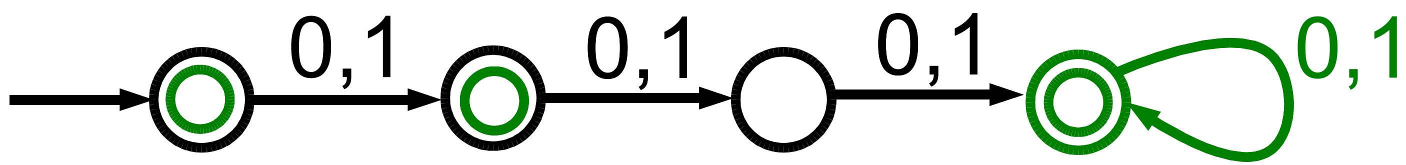
$L(M) = \Sigma^2 = \{00,01,10,11\}$

What is a DFA M' :

$L(M')$ = not Σ^2 = all strings except those of length 2 ?

Example $\sum = \{0,1\}$

M' =



$L(M') = \text{not } \sum^2 = \{0,1\}^* - \{00,01,10,11\}$

Do not forget the convention about the sink state!

- Suppose A, B are regular languages, what about
- not A := { w : w is not in A } REGULAR
- A U B := { w : w in A or w in B }
- A o B := { w_1 w_2 : w_1 in A and w_2 in B }
- A* := { w_1 w_2 … w_k : $k \geq 0$, w_i in A for every i }

- Other three are more complicated!

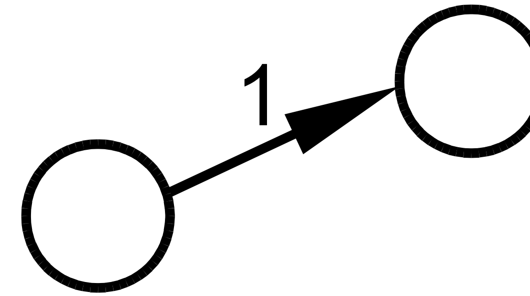
- Plan: we introduce NFA

  prove that NFA are equivalent to DFA

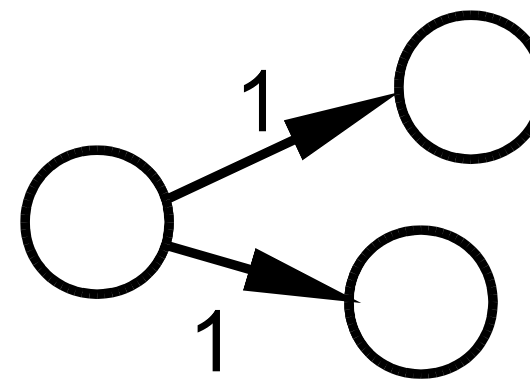
  prove A U B, A o B, A* regular, using NFA

Non deterministic finite automata (NFA)

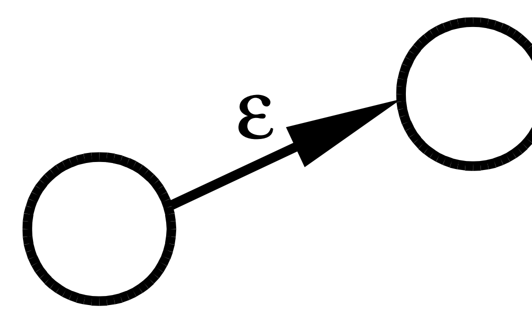
- DFA: given state and input symbol,
  unique choice for next state,
  deterministic:



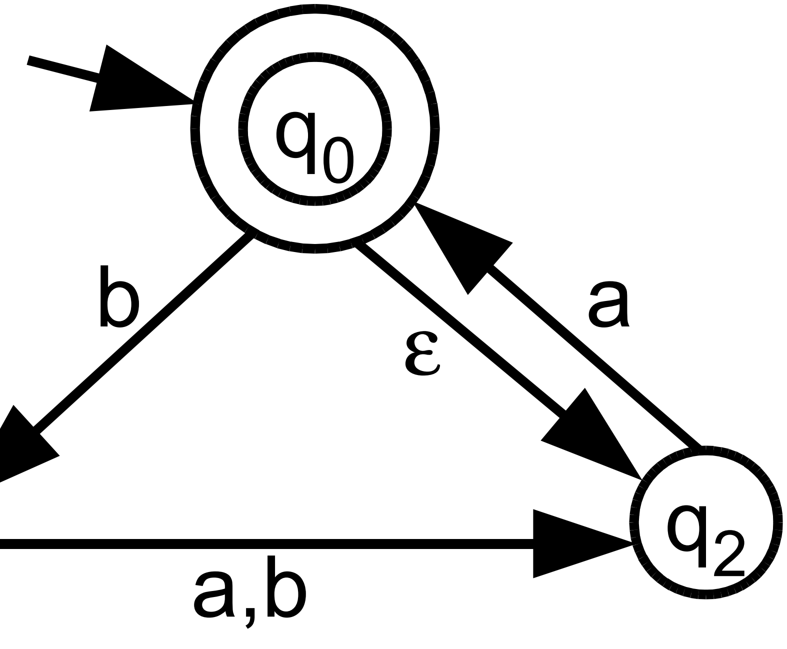
- Next we allow multiple choices,
  non-deterministic



- We also allow ε-transitions:
  can follow without reading anything



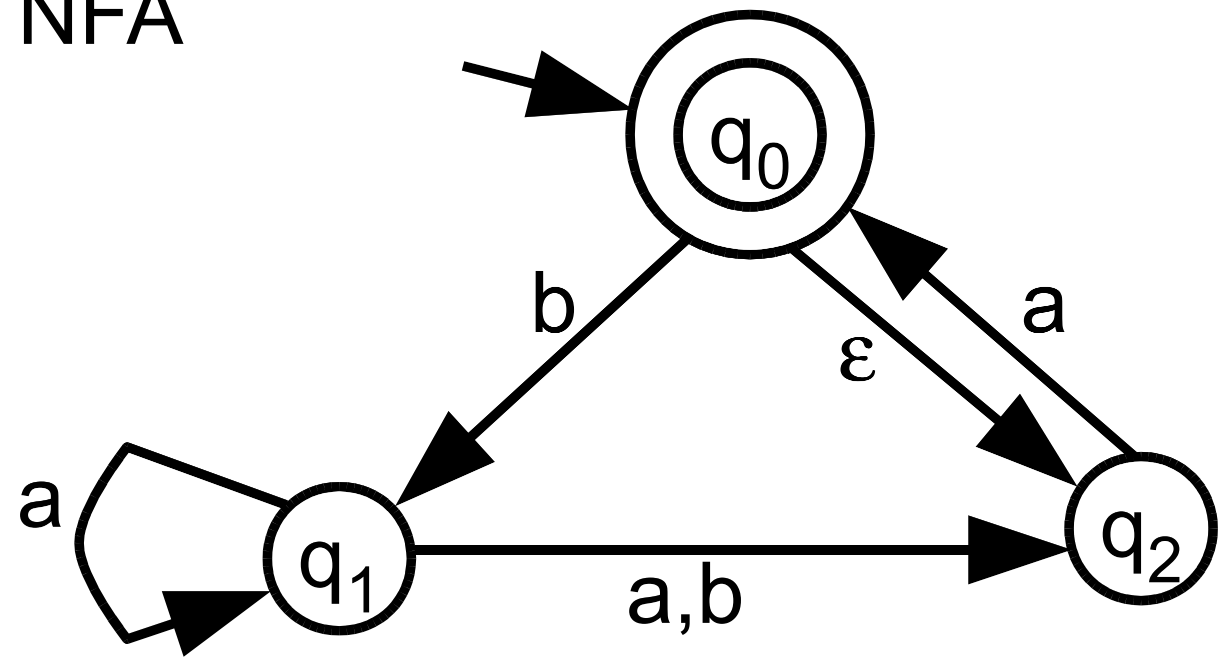
Example of NFA



Intuition of how it computes:

- Accept string w if there is a way to follow transitions that ends in accept state
- Transitions labelled with symbol in $\Sigma = \{a,b\}$ must be matched with input
- ε transitions can be followed without matching

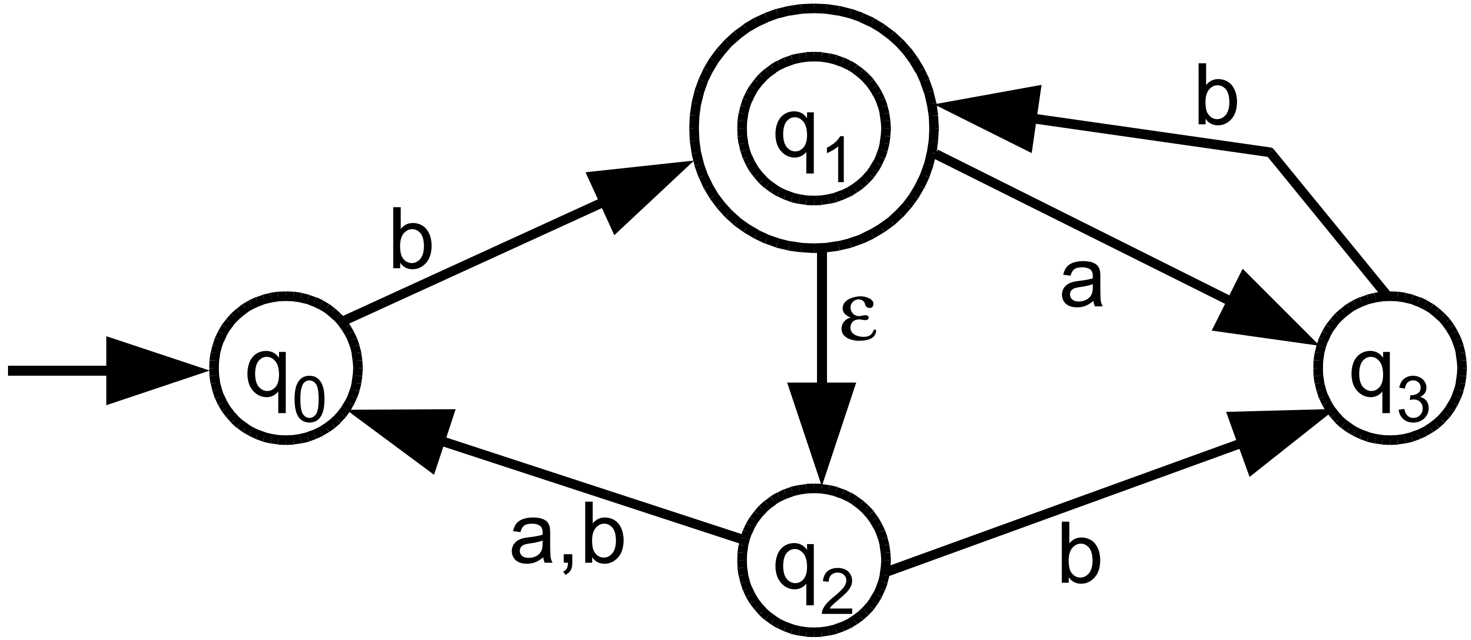
Example of NFA



Example:

- Accept a (first follow ε-transition)
- Accept baaa

ANOTHER Example of NFA



Example:

- Accept bab (two accepting paths, one uses the ε-transition)
- Reject ba (two possible paths, but neither has final state = q_1)

- Definition: A non-deterministic finite automaton (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states
- Σ is the input alphabet
- $\delta : Q \times (\Sigma \cup \{\varepsilon\}) \to \text{Powerset}(Q)$
- q_0 in Q is the start state
- $F \subseteq Q$ is the set of accept states

- Recall: Powerset(Q) = set of all subsets of Q

Example: Powerset({1,2}) = ?

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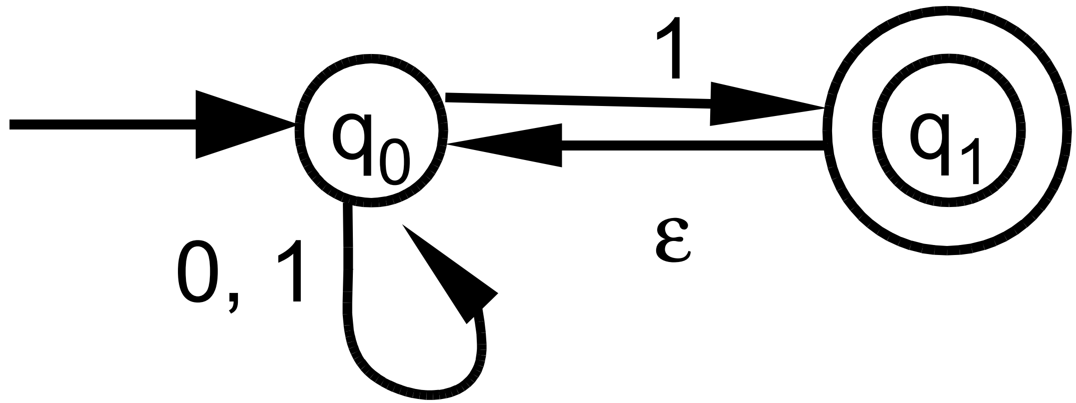
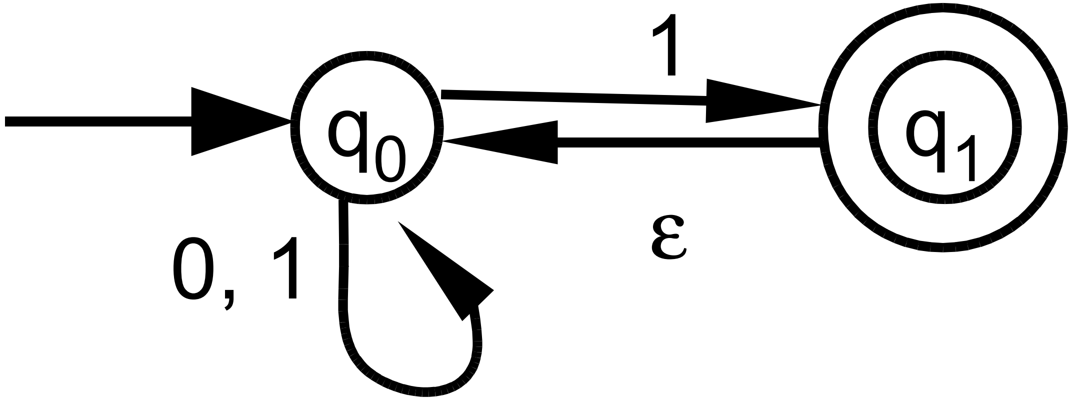
  Example: $\text{Powerset}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$



- Example: above NFA is 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
- $Q = \{ q_0, q_1\}$
- $\Sigma = \{0,1\}$
- $\delta(q_0, 0) =$?



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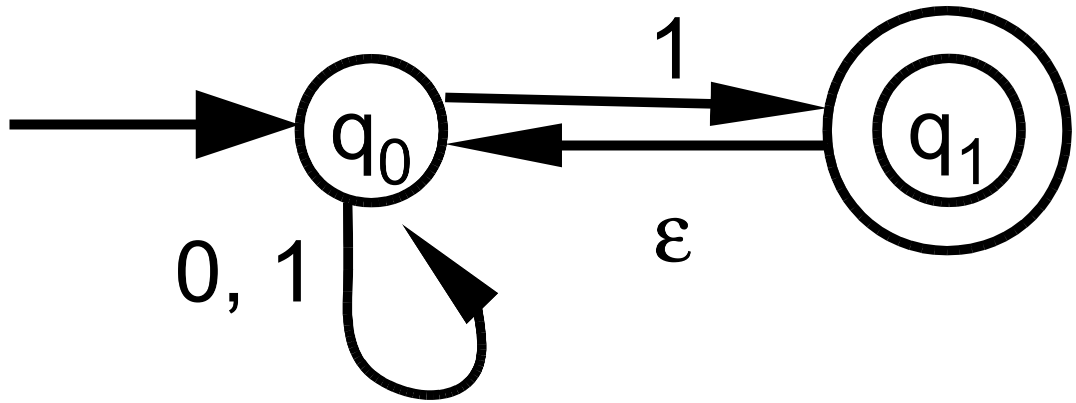

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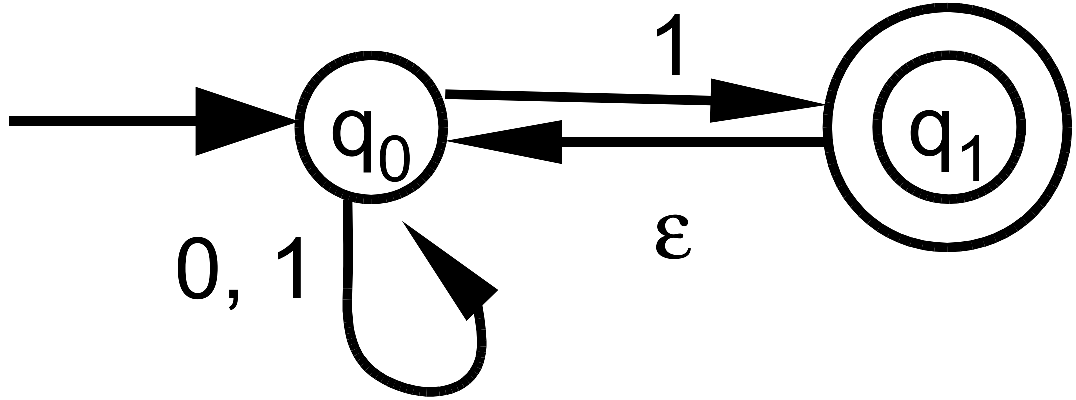

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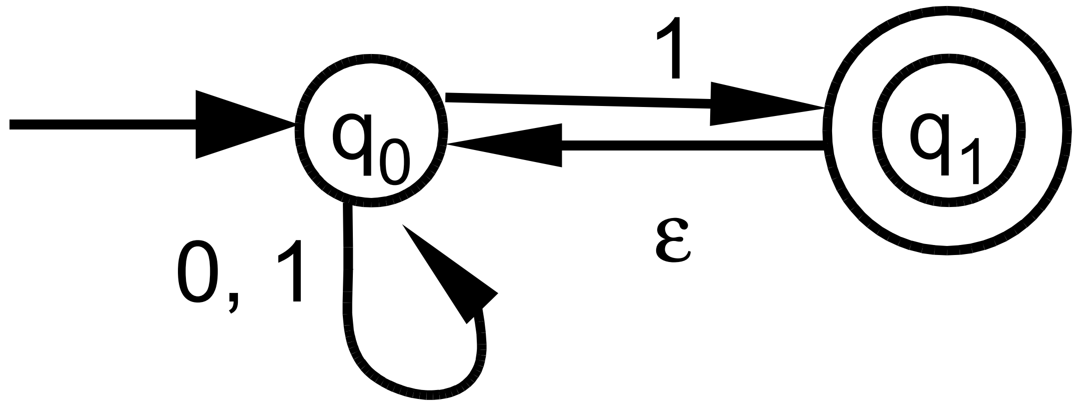

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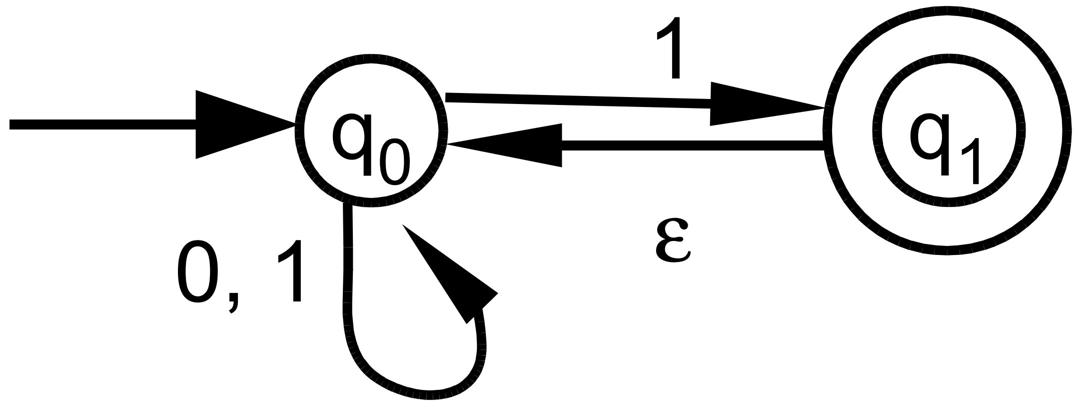

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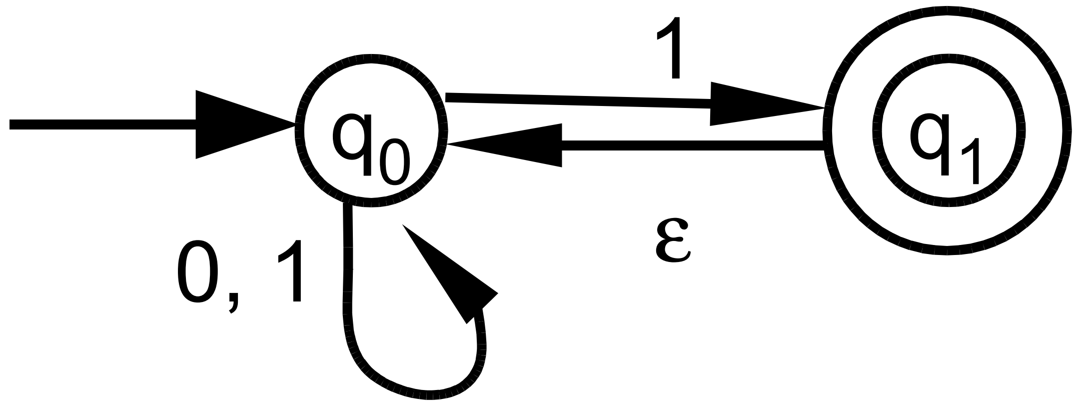

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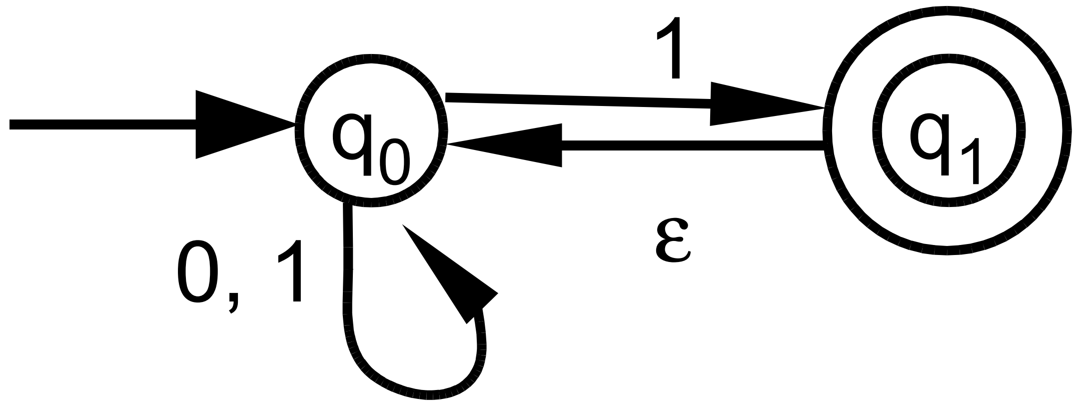
  $\delta(q_1 ,0) = \varnothing$ $\delta(q_1 ,1) = \varnothing$ $\delta(q_1 ,\varepsilon) = \{q_1\}$
- q_0 in Q is the start state
- F = ?



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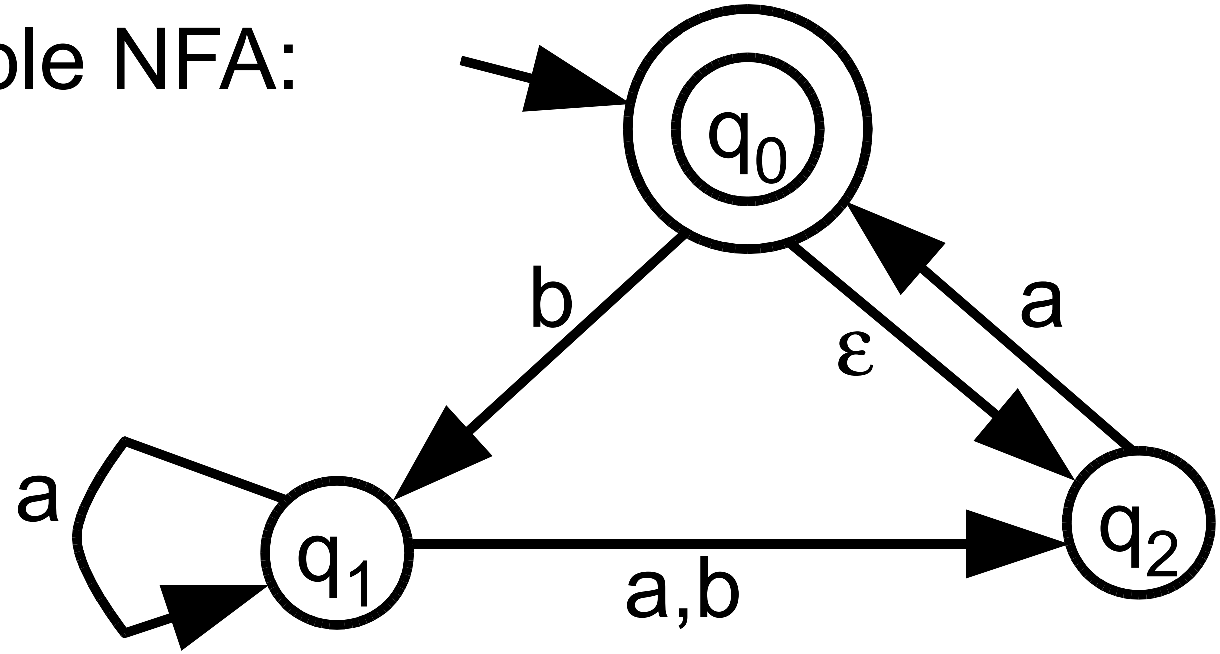
- $F = \{ q_1\} \subseteq Q$ is the set of accept states

- Definition: A NFA $(Q, \Sigma, \delta, q_0, F)$ accepts a string w if $\exists$ integer k, $\exists$ k strings $w_1, w_2, \ldots, w_k$ such that
- $w = w_1 w_2 \ldots w_k$ where $\forall\ 1 \leq i \leq k,\ w_i \in \Sigma \cup \{\varepsilon\}$ (the symbols of w, or ε)

- $\exists$ sequence of k+1 states $r_0, r_1, .., r_k$ in Q such that:
- $r_0 = q_0$
- $r_{i+1} \in \delta(r_i, w_{i+1})\ \forall\ 0 \leq i < k$
- r_k is in F

- Differences with DFA are in green

Back to first example NFA:



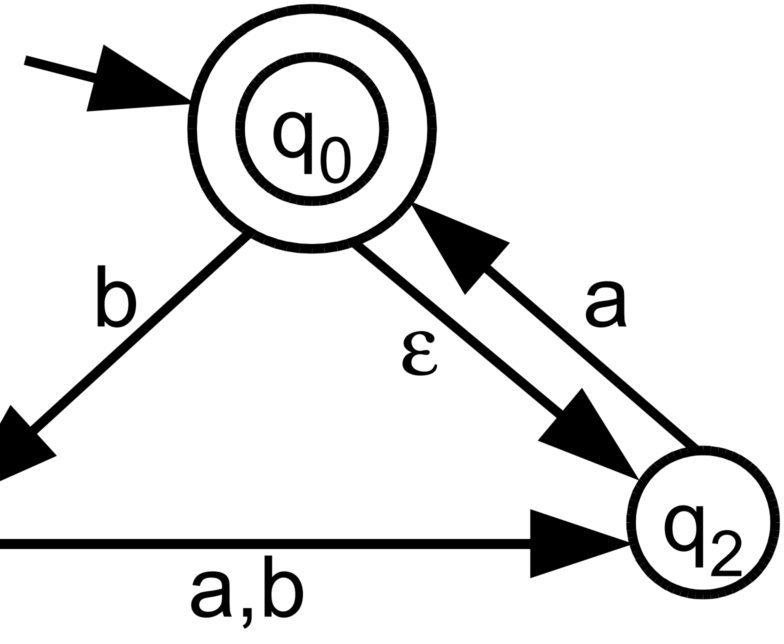
Accepts w = baaa

w_1 = b, w_2 = a, w_3 = a, w_4 = ε, w_5 = a

Accepting sequence of 5+1 = 6 states:

r_0 = ?

Back to first example NFA:



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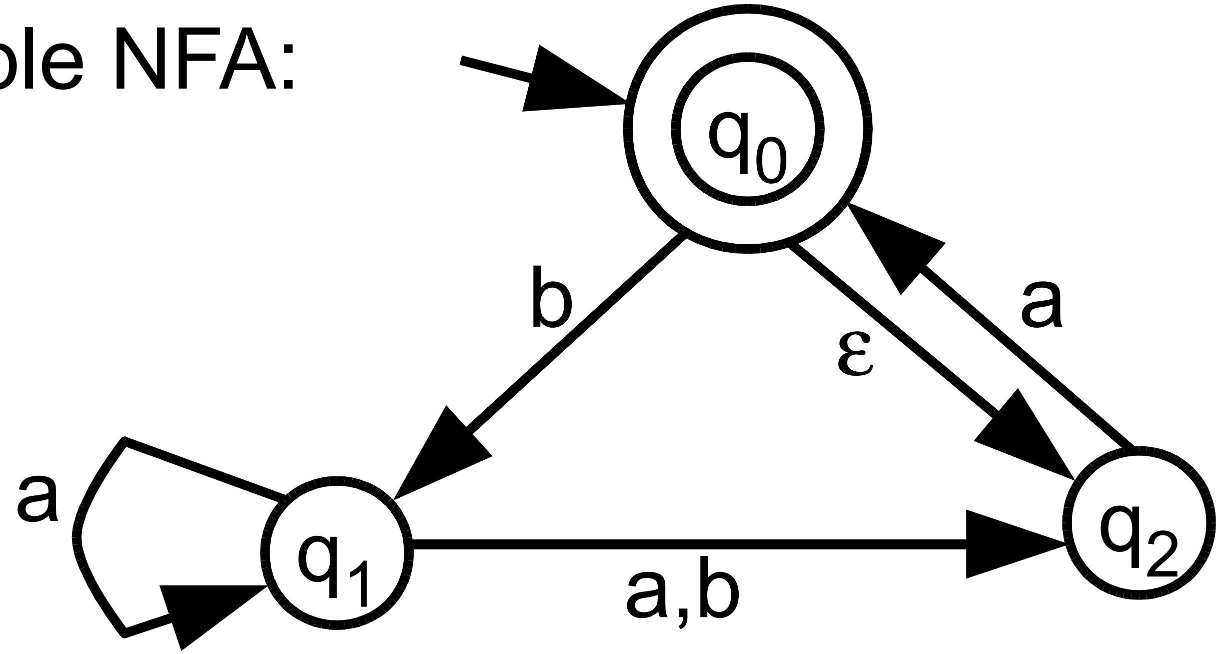
Accepting sequence of 5+1 = 6 states:

$r_0 = q_0$, $r_1 = ?$

Back to first example NFA:



Accepts w = baaa

$w_1 = b, \quad w_2 = a, \quad w_3 = a, \quad w_4 = \varepsilon, \quad w_5 = a$

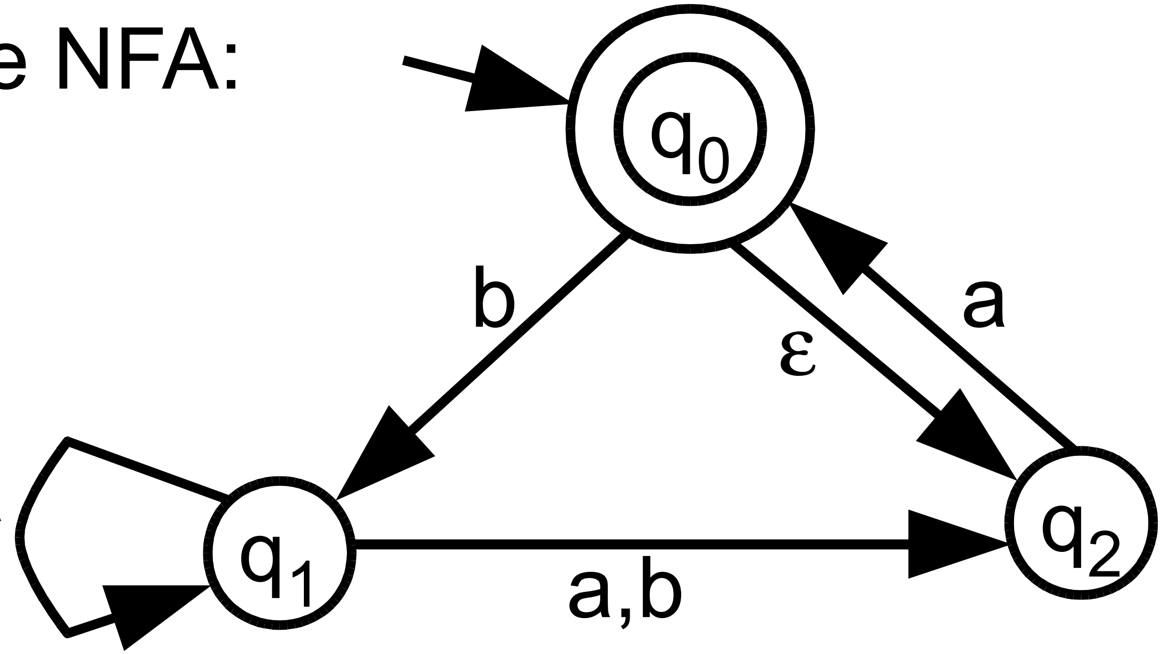
Accepting sequence of 5+1 = 6 states:

$r_0 = q_0, \quad r_1 = q_1, \quad r_2 =$?

Transitions:

$r_1 \in \delta(r_0, b) = \{q_1\}$

Back to first example NFA:



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$w_1 = b, \quad w_2 = a, \quad w_3 = a, \quad w_4 = \varepsilon, \quad w_5 = a$

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Transitions:

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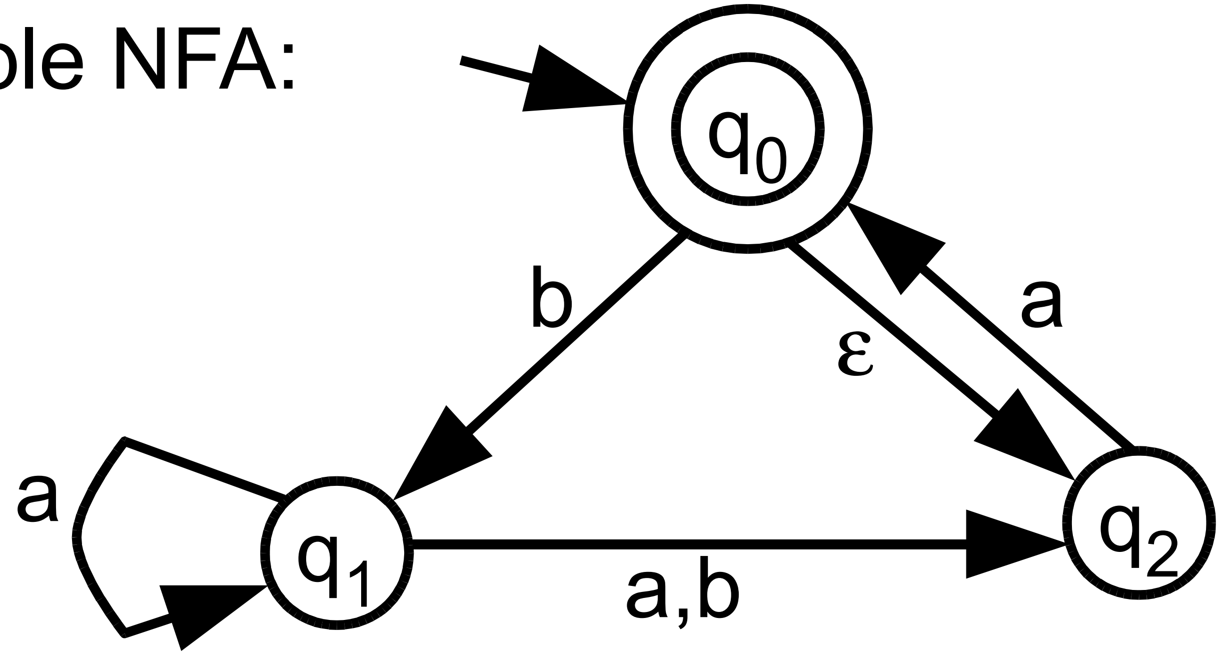
Back to first example NFA:



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$w_1 = b, \quad w_2 = a, \quad w_3 = a, \quad w_4 = \varepsilon, \quad w_5 = a$

Accepting sequence of 5+1 = 6 states:

$r_0 = q_0, \quad r_1 = q_1, \quad r_2 = q_2, \quad r_3 = q_0, \quad r_4 =$?

Transitions:

$r_1 \in \delta(r_0,b) = \{q_1\} \quad r_2 \in \delta(r_1,a) = \{q_1,q_2\}$

$r_3 \in \delta(r_2,a) = \{q_0\}$

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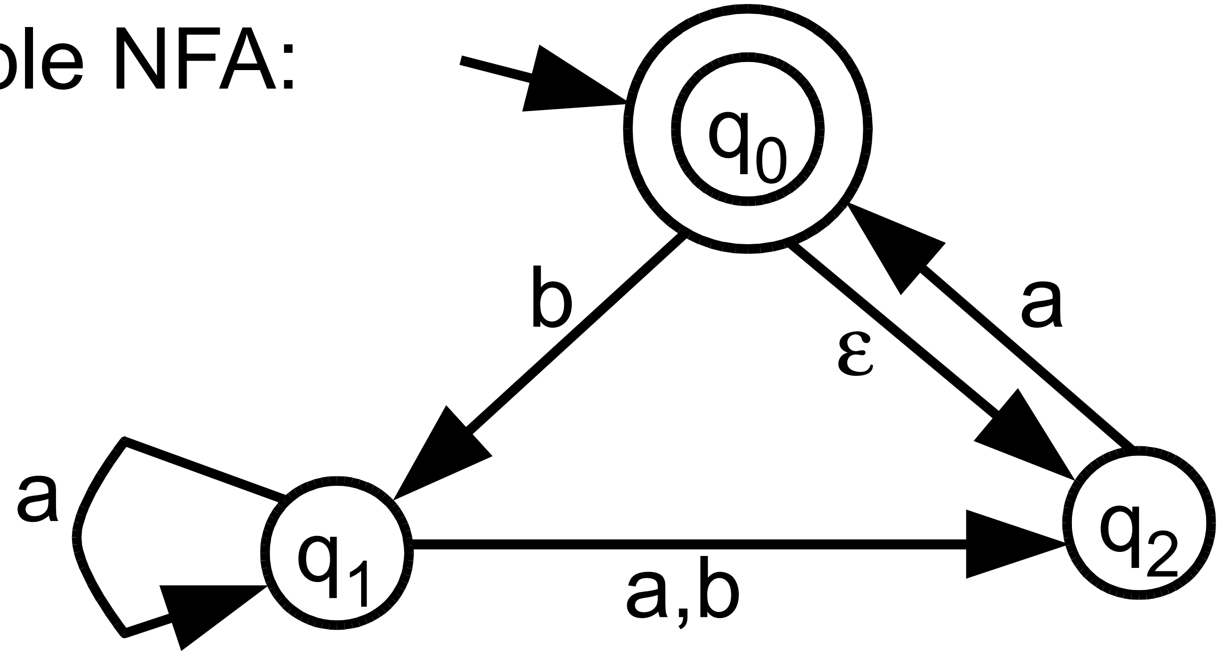

Accepts w = baaa

$w_1 = b$, $w_2 = a$, $w_3 = a$, $w_4 = \varepsilon$, $w_5 = a$

Accepting sequence of 5+1 = 6 states:

$r_0 = q_0$, $r_1 = q_1$, $r_2 = q_2$, $r_3 = q_0$, $r_4 = q_2$, $r_5 =$?

Transitions:

$r_1 \in \delta(r_0,b) = \{q_1\}$ $r_2 \in \delta(r_1,a) = \{q_1,q_2\}$

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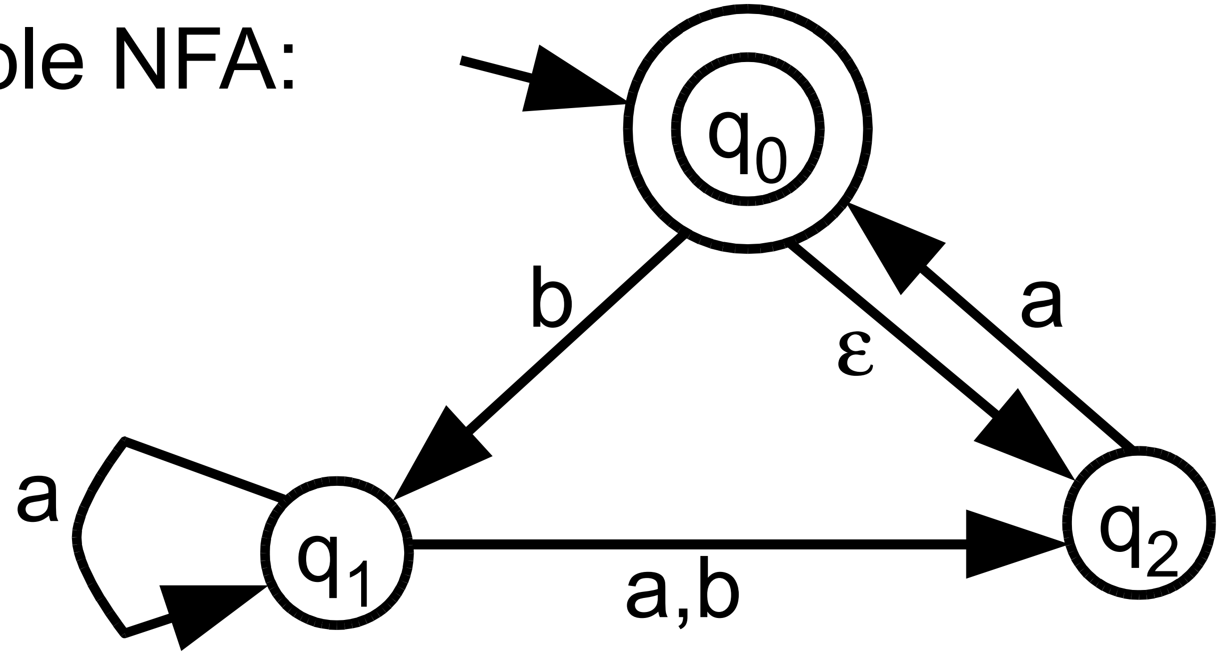

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$r_1 \in \delta(r_0, b) = \{q_1\} \quad r_2 \in \delta(r_1, a) = \{q_1, q_2\}$

$r_3 \in \delta(r_2, a) = \{q_0\} \quad r_4 \in \delta(r_3, \varepsilon) = \{q_2\} \quad r_5 \in \delta(r_4, a) = \{q_0\}$

- NFA are at least as powerful as DFA, because DFA are a special case of NFA

- Are NFA more powerful than DFA?

- Surprisingly, they are not:

- Theorem:

For every NFA N there is DFA M : L(M) = L(N)

- **Theorem**:

For every NFA N there is DFA M : L(M) = L(N)

- **Construction** without ε transitions
- Given NFA N $(Q, \Sigma, \delta, q, F)$
- Construct DFA M $(Q', \Sigma, \delta', q', F')$ where:
- $Q' := \text{Powerset}(Q)$
- $q' = \{q\}$
- $F' = \{ S : S \in Q' \text{ and } S \text{ contains an element of } F\}$
- $\delta'(S, a) := \bigcup_{s \in S} \delta(s,a)$

$$= \{ t : t \in \delta(s,a) \text{ for some } s \in S \}$$

- It remains to deal with ε transitions

- Definition: Let S be a set of states.

E(S) := { q : q can be reached from some state s in S traveling along 0 or more ε transitions }

- We think of following ε transitions at beginning, or right after reading an input symbol in Σ

- Theorem:

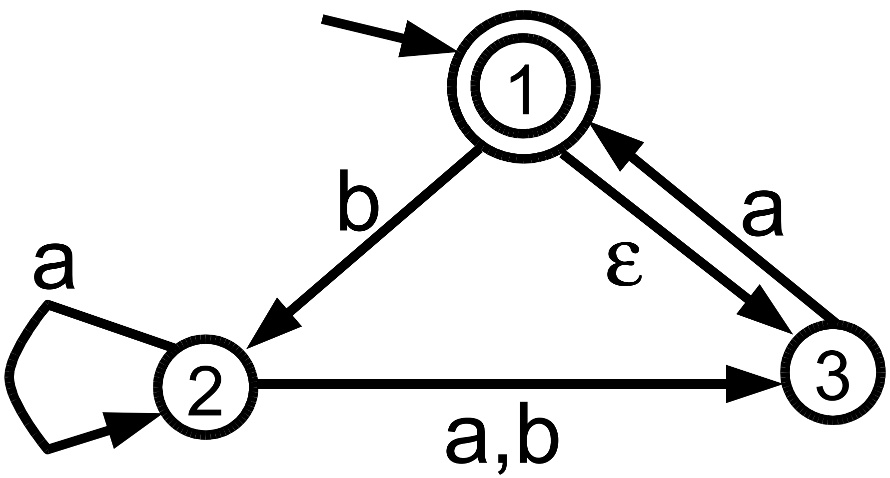
For every NFA N there is DFA M : L(M) = L(N)

- Construction including ε transitions
- Given NFA N $(Q, \Sigma, \delta, q, F)$
- Construct DFA M $(Q', \Sigma, \delta', q', F')$ where:
- $Q' :=$ Powerset(Q)
- $q' = E(\{q\})$
- $F' = \{ S : S \in Q'$ and S contains an element of $F\}$
- $\delta'(S, a) := E(\bigcup_{s \in S} \delta(s,a))$

$$= \{ t : t \in E(\delta (s,a)) \text{ for some } s \in S \}$$

Example: NFA → DFA conversion

NFA

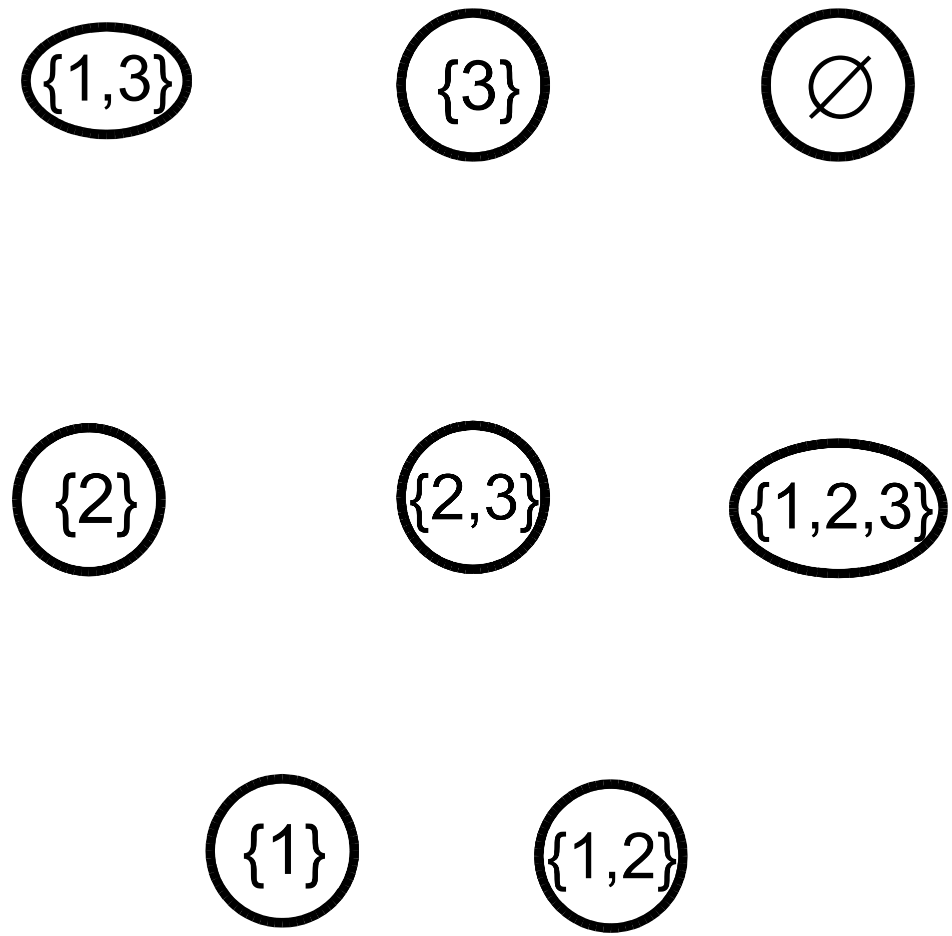


$$Q_{DFA} = \text{Powerset}(Q_{NFA})$$

$$= \text{Powerset}(\{1,2,3\})$$

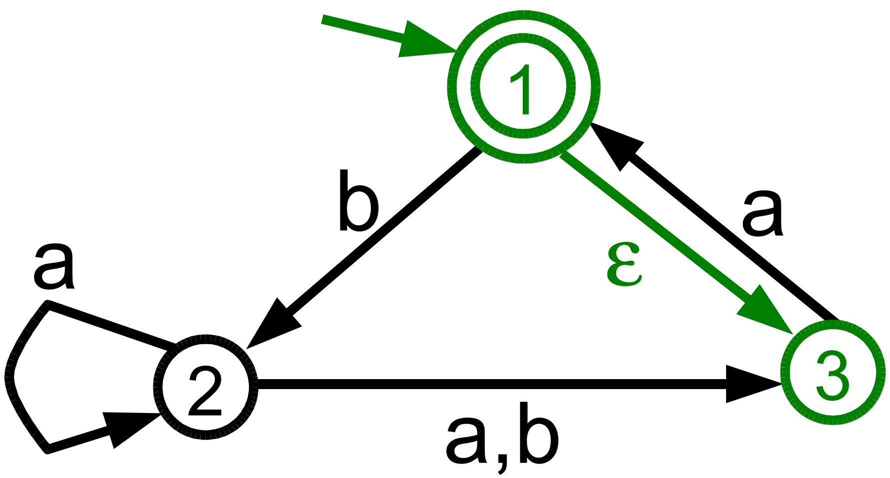
$$= \{\varnothing,\{1\},\{2\},\{3\},\{1,2\}\ldots\}$$

DFA



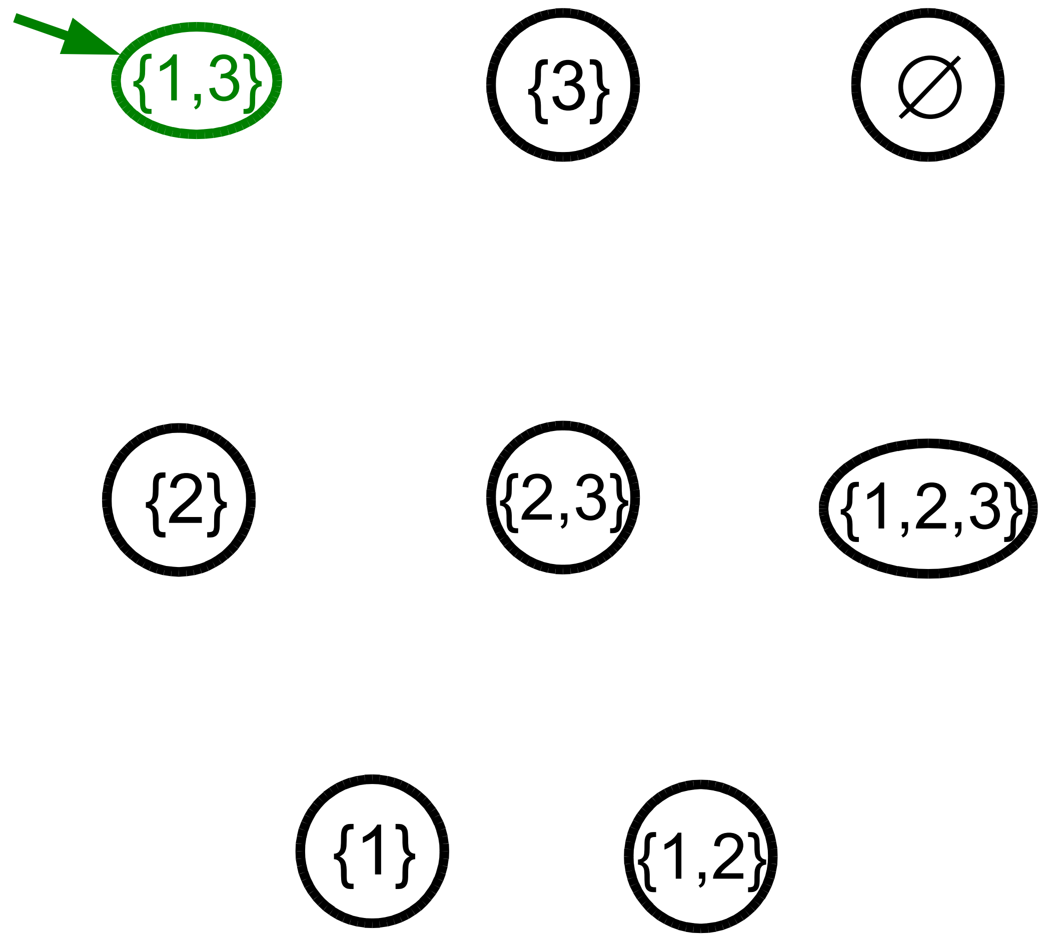
Example: NFA → DFA conversion

NFA



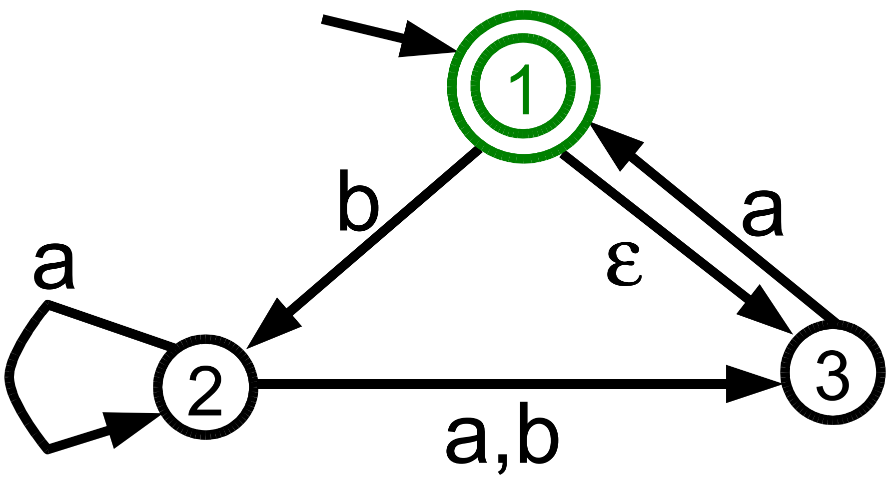
$$q_{DFA} = E(\{q_{NFA}\})$$
$$= E(\{1\})$$
$$= \{1,3\}$$

DFA



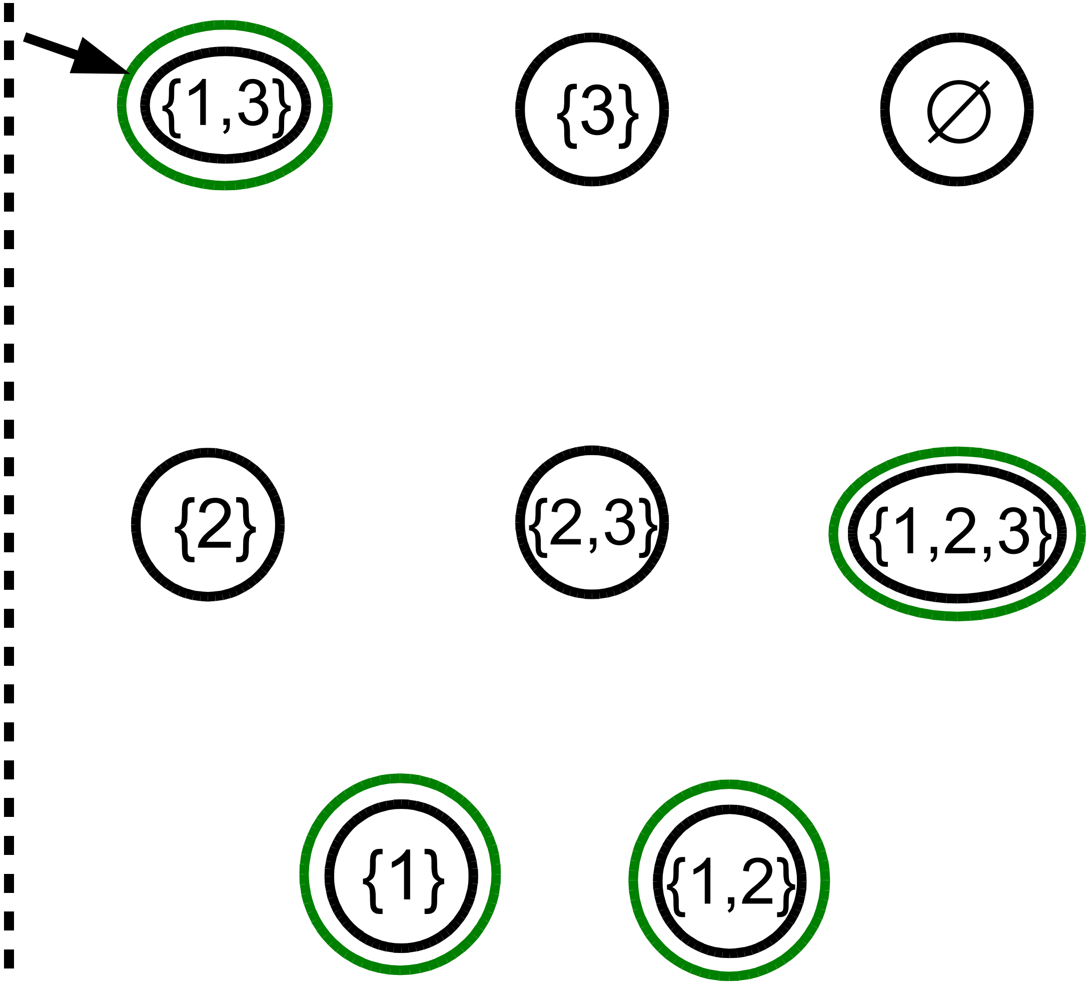
Example: NFA → DFA conversion

NFA



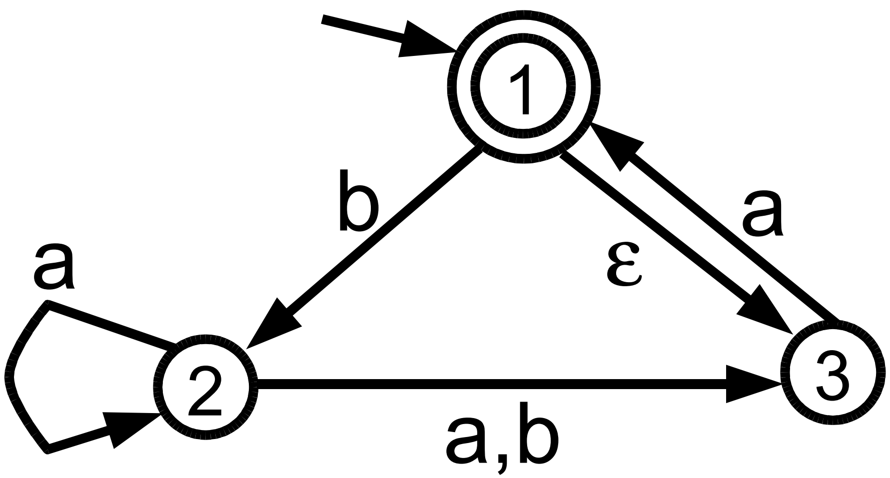
F_{DFA} = {S : S contains an element of F_{NFA}}

DFA



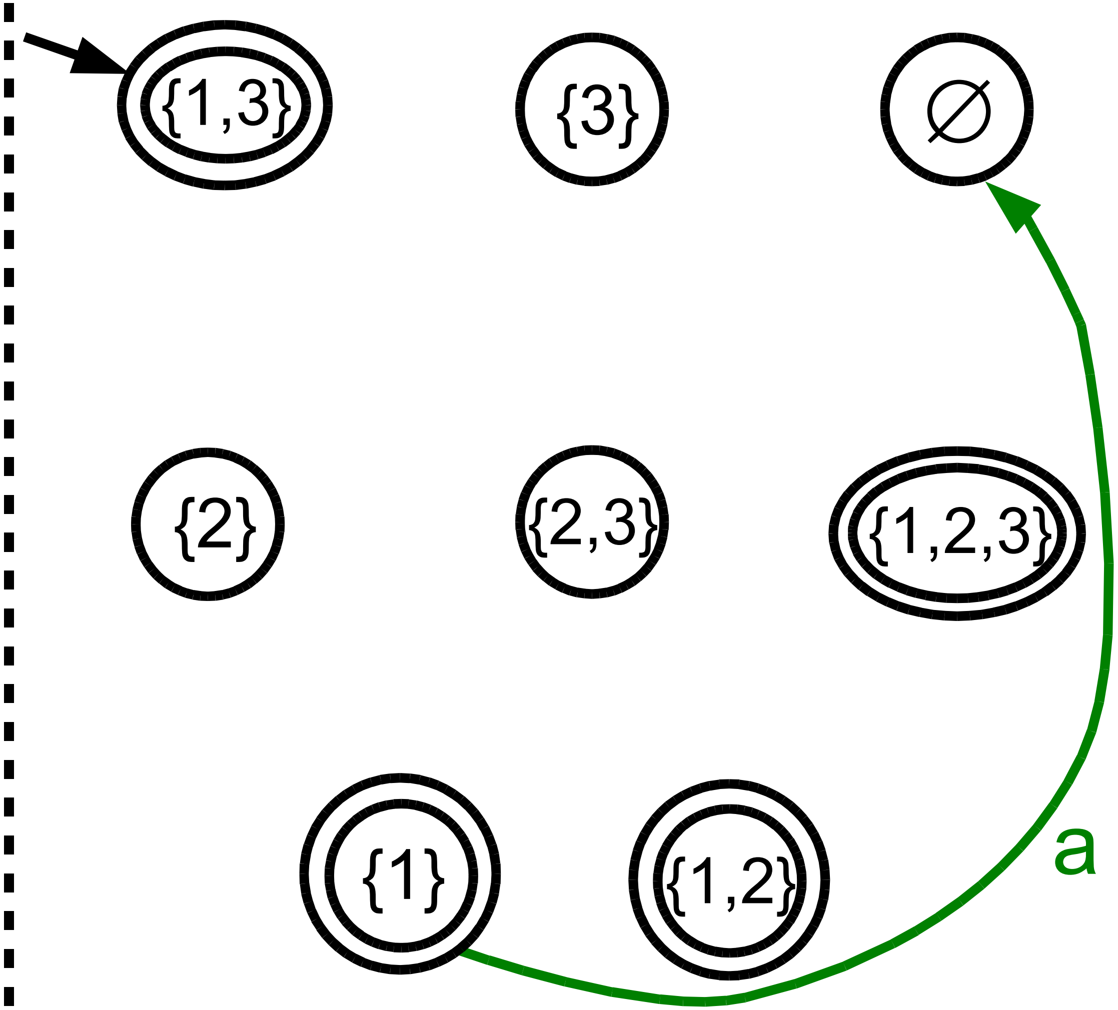
Example: NFA → DFA conversion

NFA

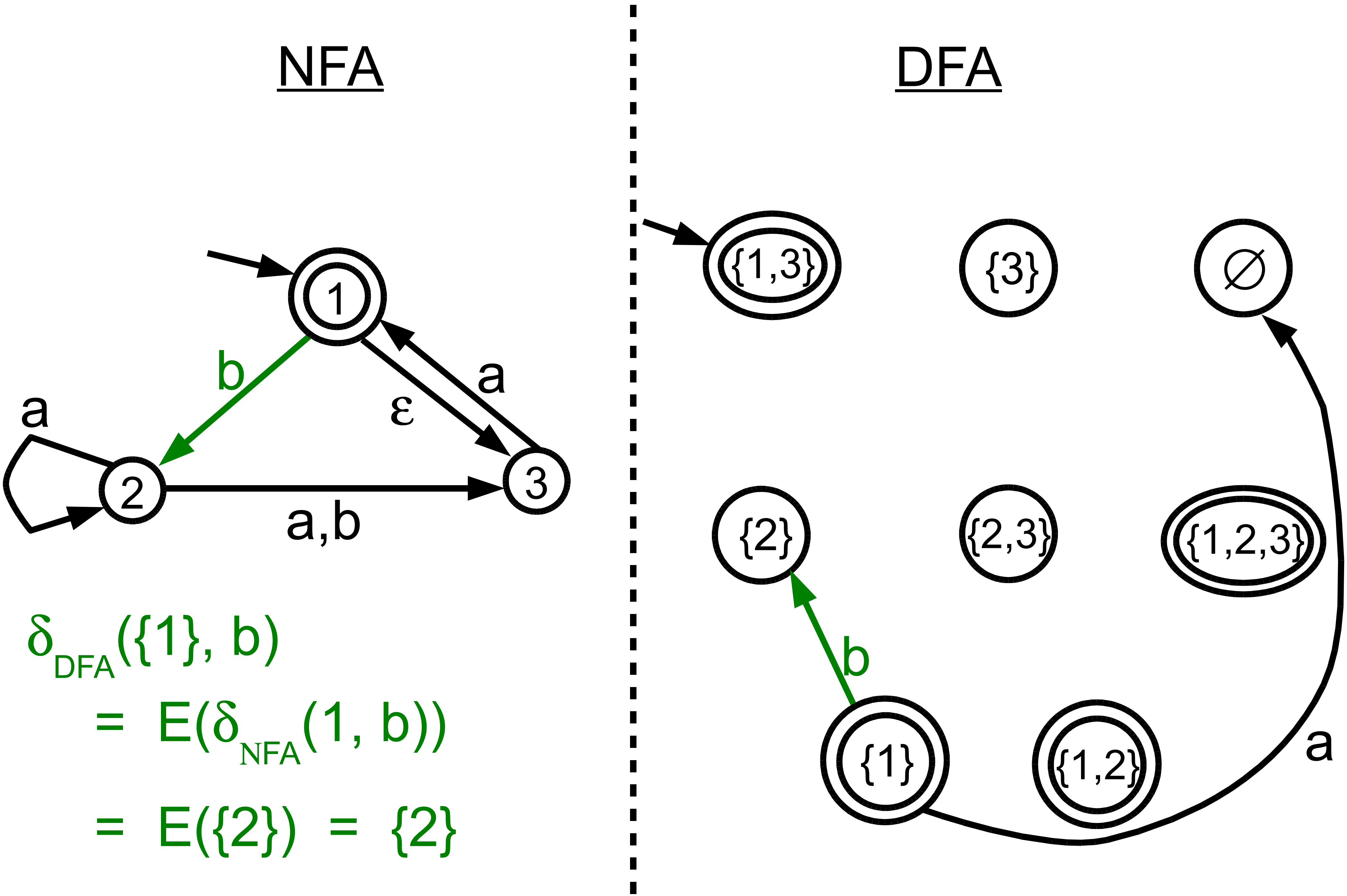


$$\delta_{DFA}(\{1\}, a)$$
$$= E(\delta_{NFA}(1, a))$$
$$= E(\varnothing) = \varnothing$$

DFA



Example: NFA → DFA conversion



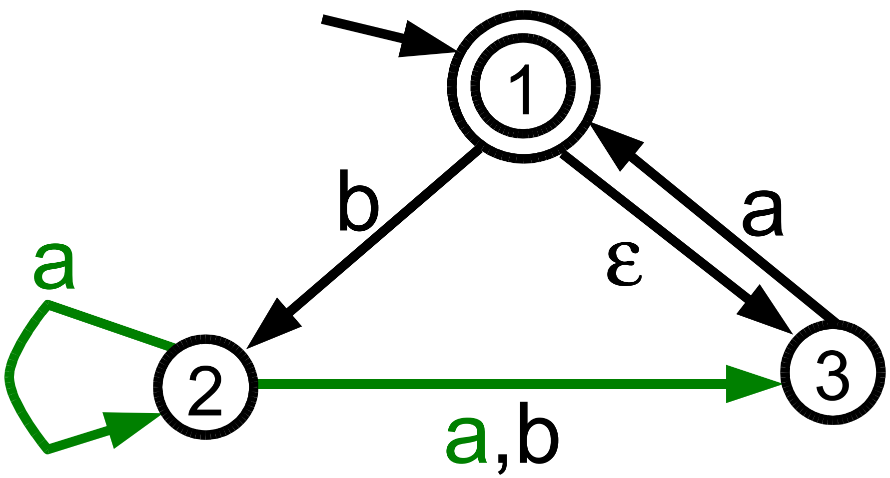
$\delta_{DFA}(\{1\}, b)$

$= E(\delta_{NFA}(1, b))$

$= E(\{2\}) = \{2\}$

Example: NFA → DFA conversion

NFA

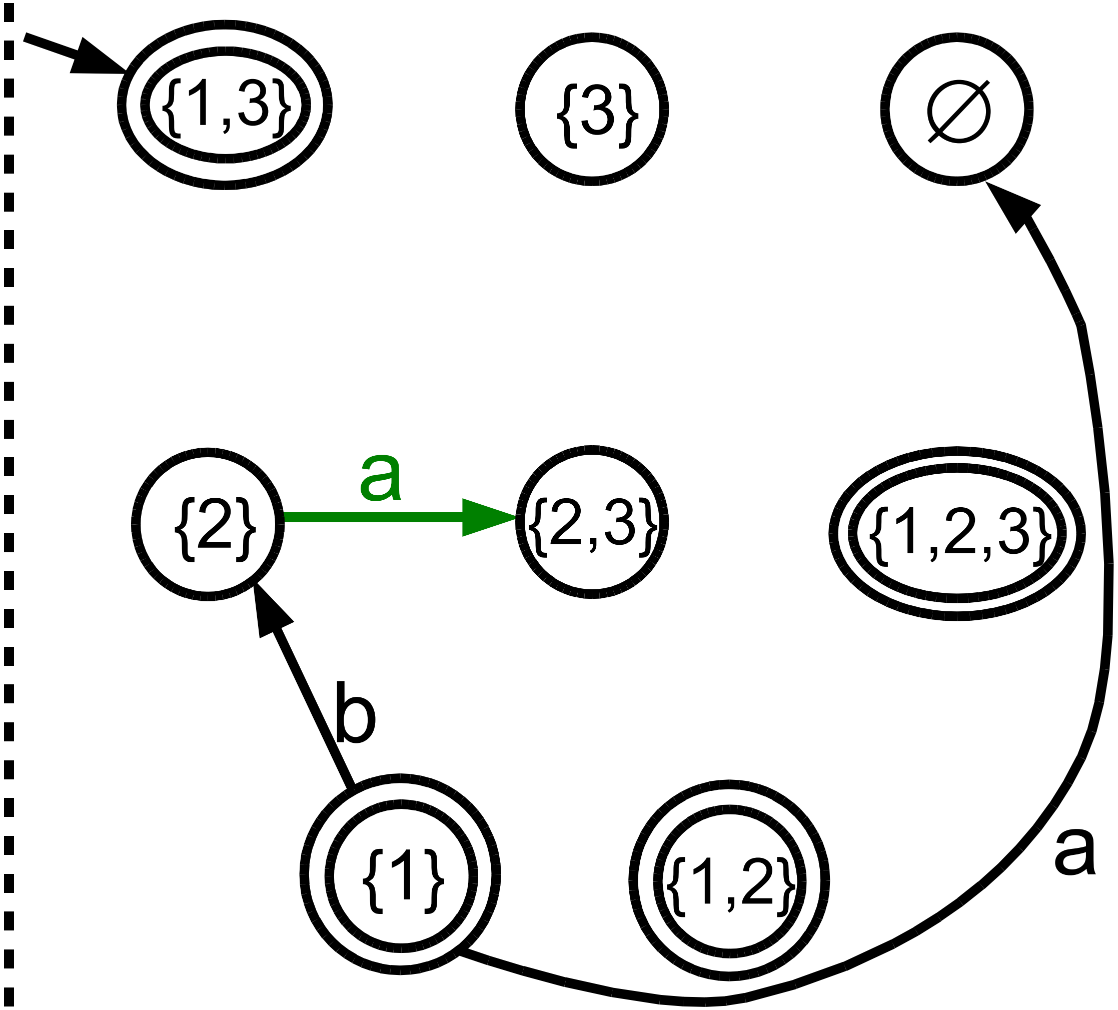


$\delta_{DFA}(\{2\}, a)$

$= E(\delta_{NFA}(2, a))$

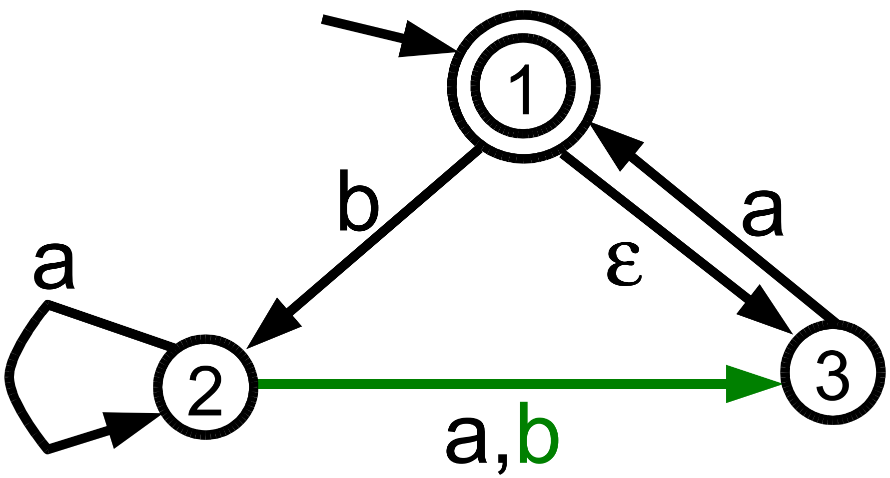
$= E(\{2,3\}) = \{2,3\}$

DFA



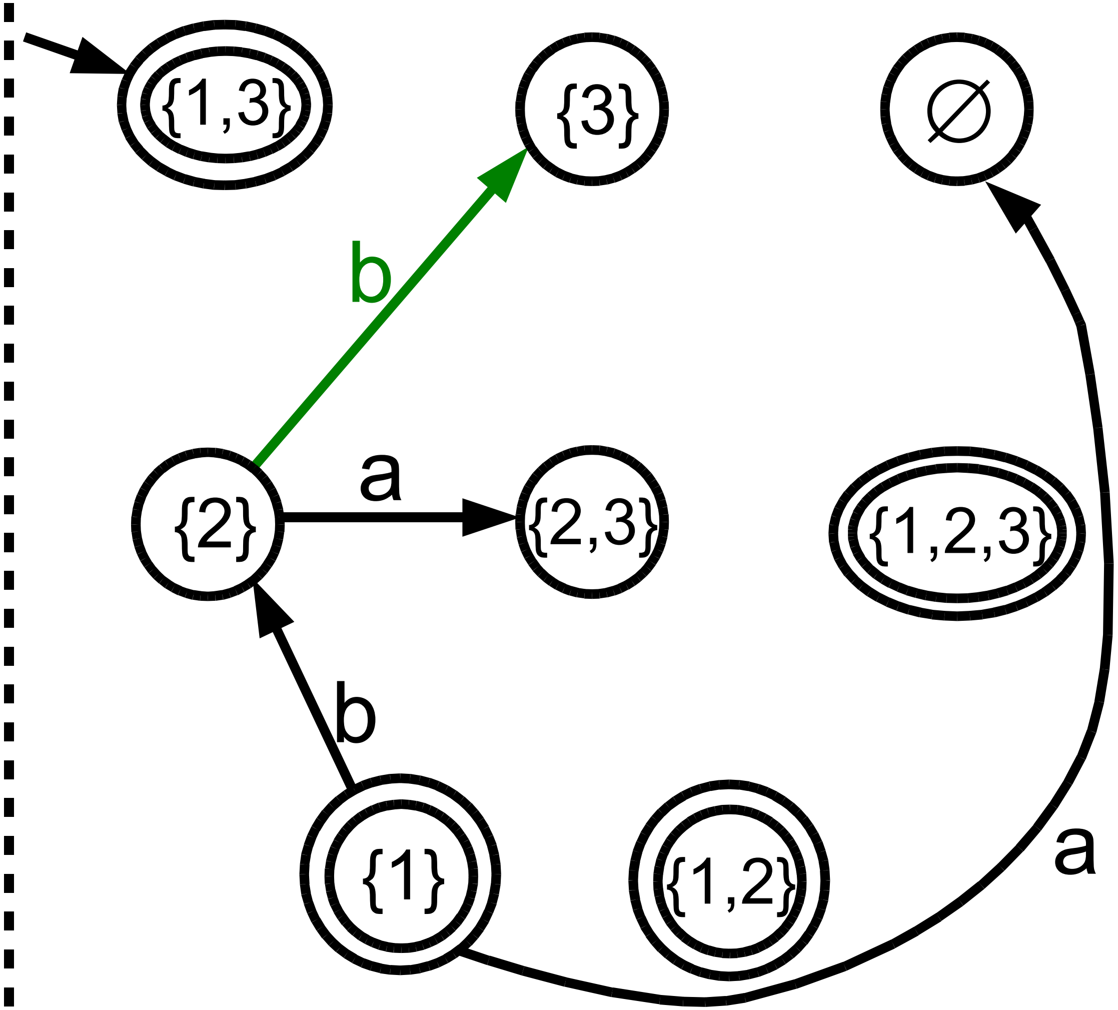
Example: NFA → DFA conversion

NFA

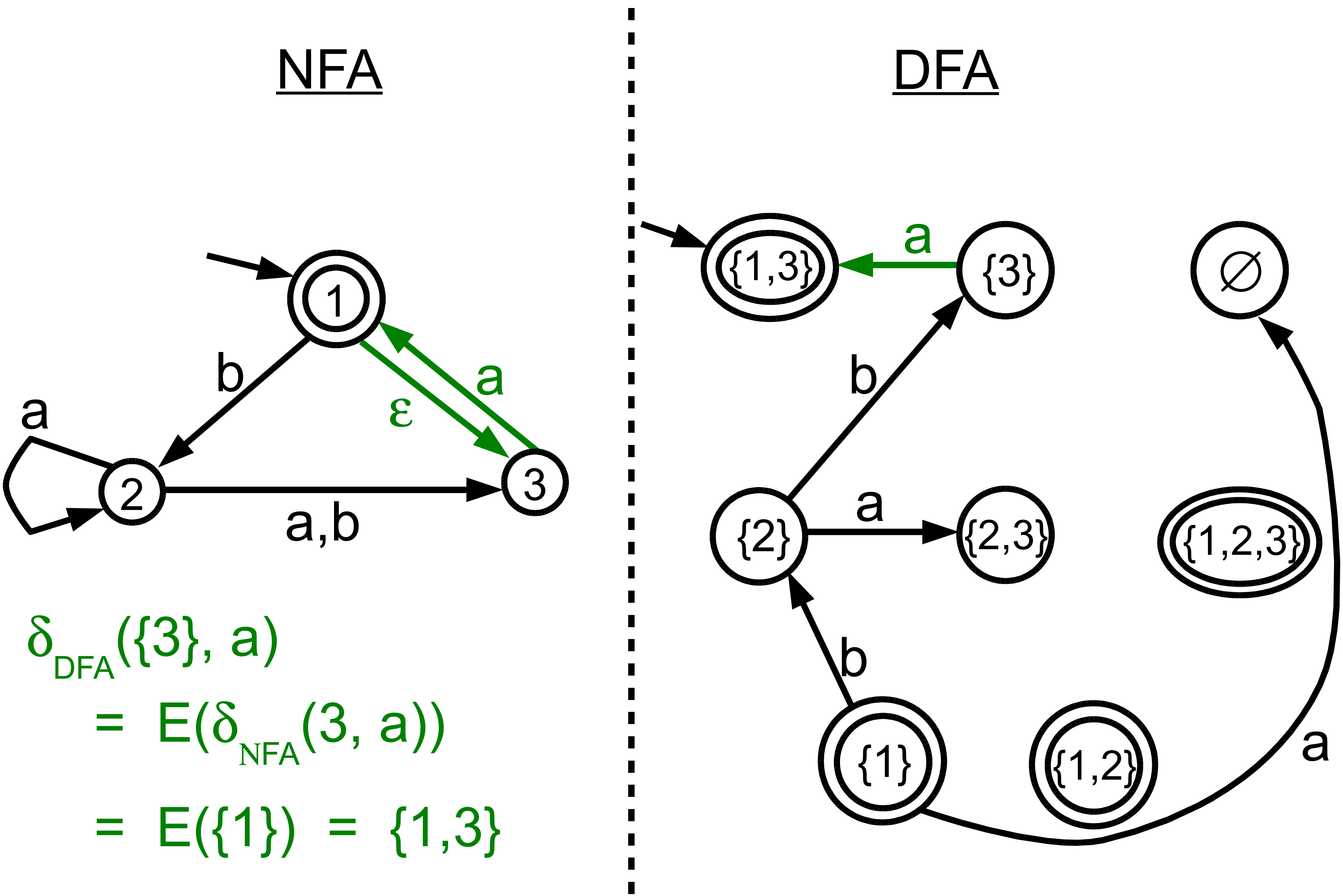


$$\delta_{DFA}(\{2\}, b)$$
$$= E(\delta_{NFA}(2, b))$$
$$= E(\{3\}) = \{3\}$$

DFA



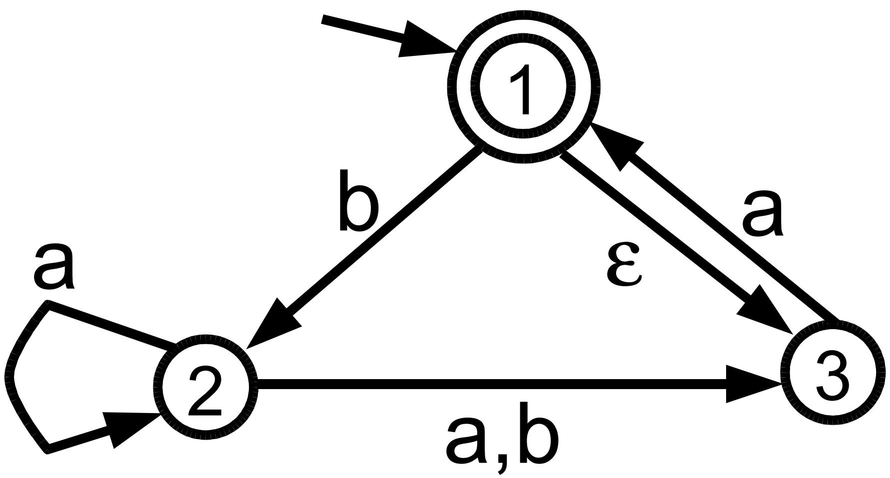
Example: NFA → DFA conversion



$$\delta_{DFA}(\{3\}, a)$$
$$= E(\delta_{NFA}(3, a))$$
$$= E(\{1\}) = \{1,3\}$$

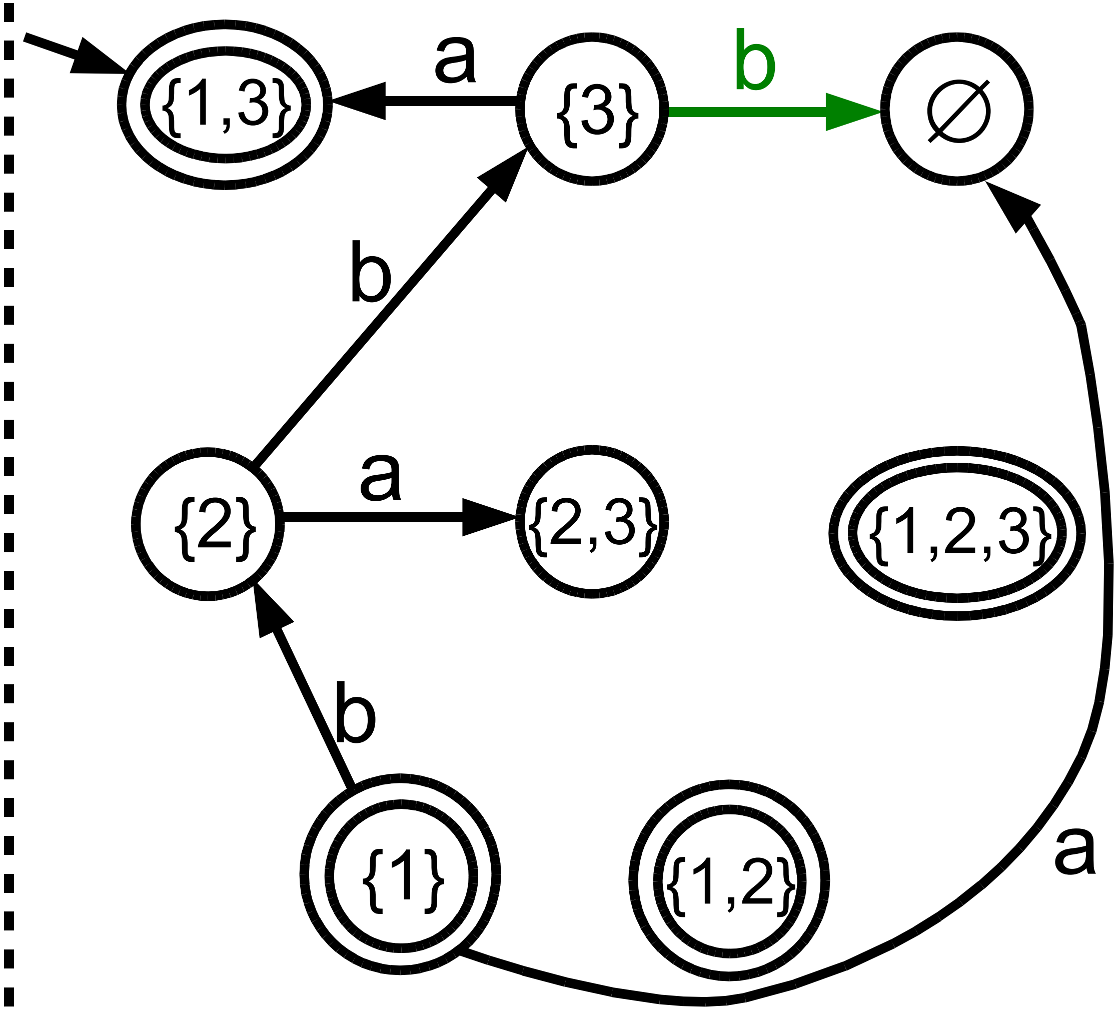
Example: NFA → DFA conversion

NFA

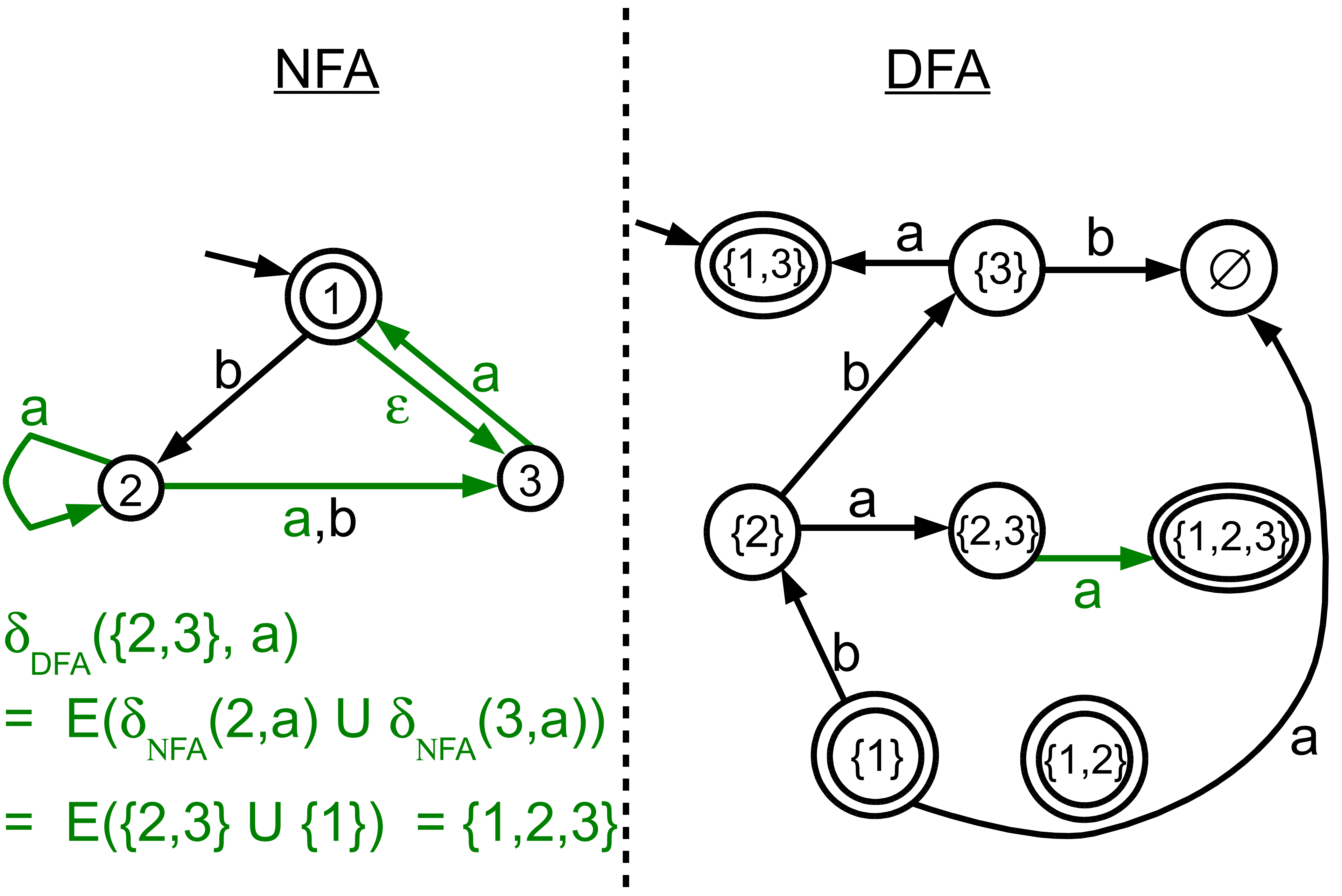


$$\delta_{\text{DFA}}(\{3\}, b)$$
$$= E(\delta_{\text{NFA}}(3, b))$$
$$= E(\varnothing) = \varnothing$$

DFA



Example: NFA → DFA conversion



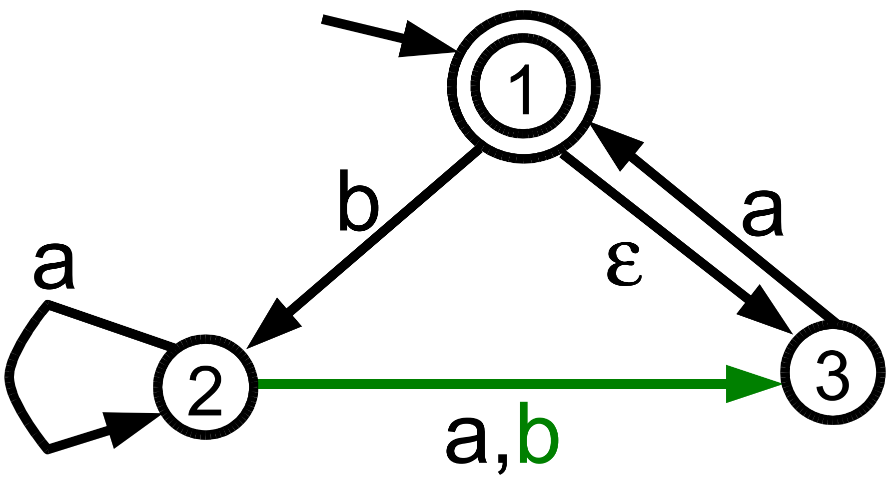
$\delta_{DFA}(\{2,3\}, a)$

$= E(\delta_{NFA}(2,a) \cup \delta_{NFA}(3,a))$

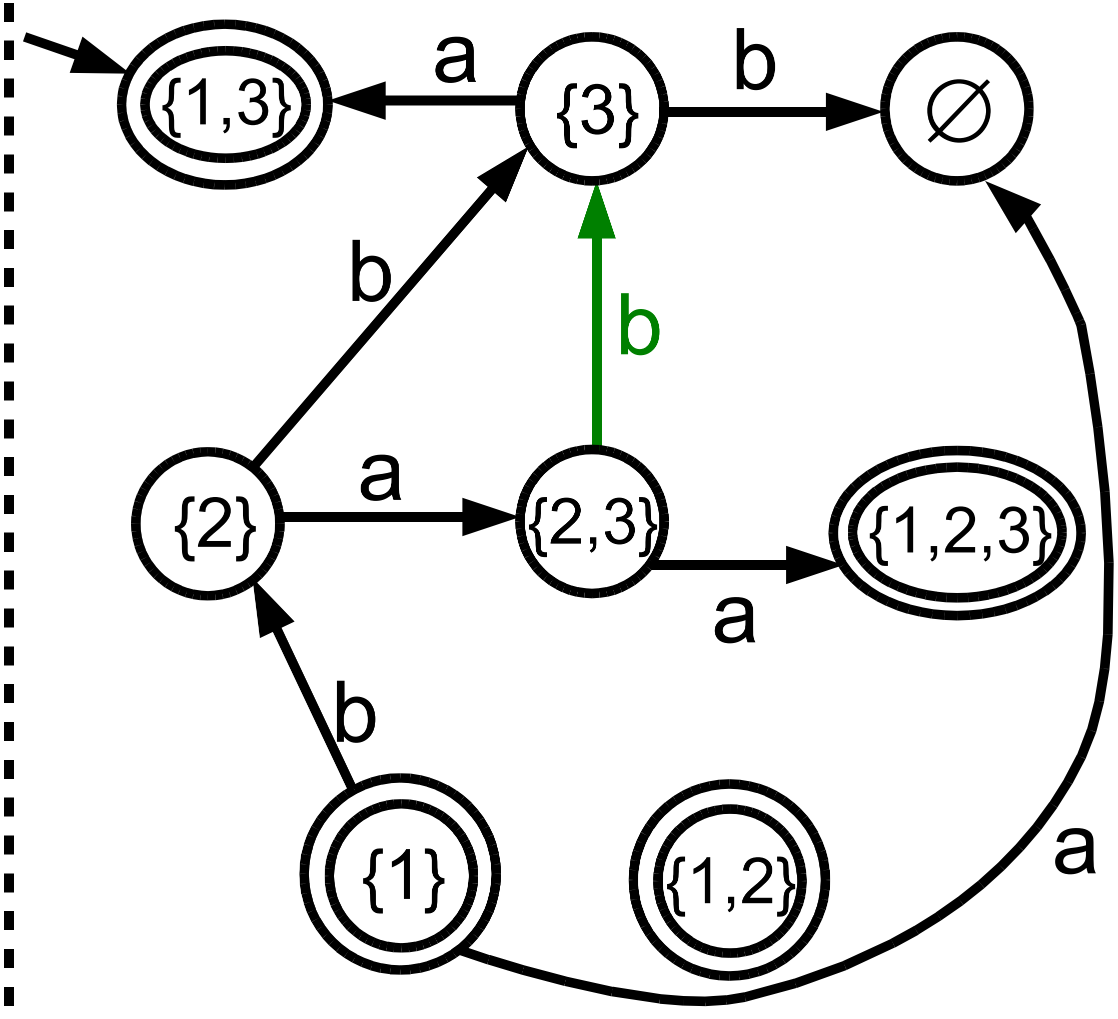
$= E(\{2,3\} \cup \{1\}) = \{1,2,3\}$

Example: NFA → DFA conversion

NFA



DFA

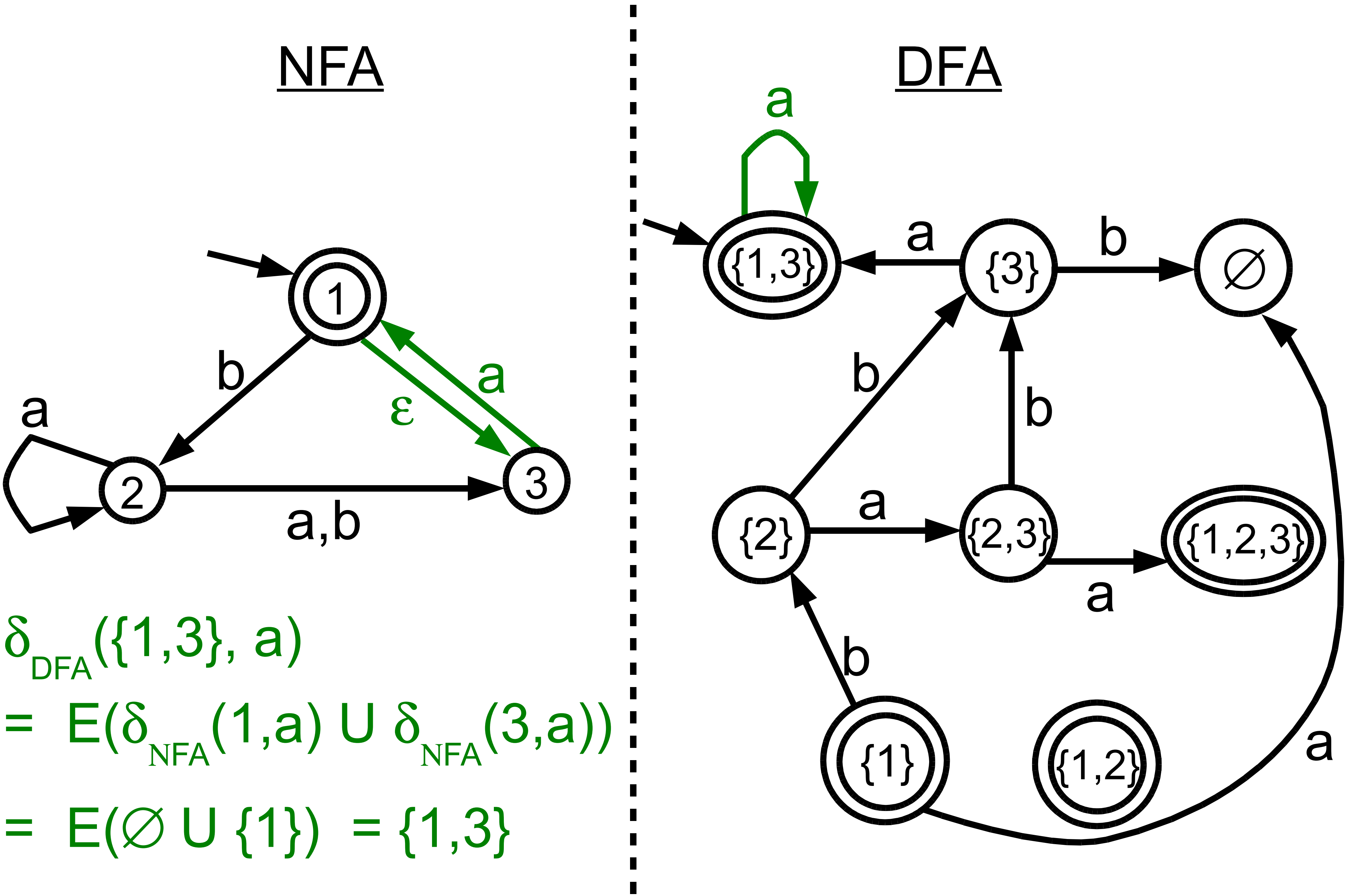


$\delta_{DFA}(\{2,3\}, b)$

$= E(\delta_{NFA}(2,b) \cup \delta_{NFA}(3,b))$

$= E(\{3\} \cup \emptyset) = \{3\}$

Example: NFA → DFA conversion



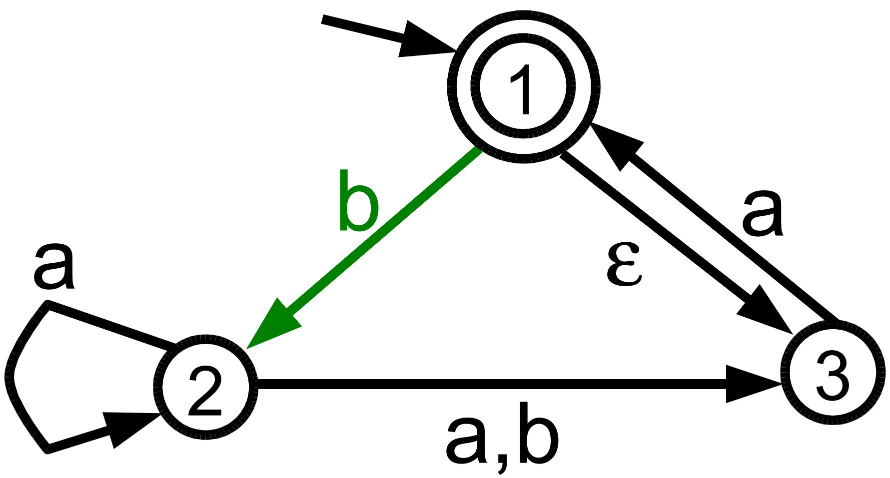
$\delta_{DFA}(\{1,3\}, a)$

$= E(\delta_{NFA}(1,a) \cup \delta_{NFA}(3,a))$

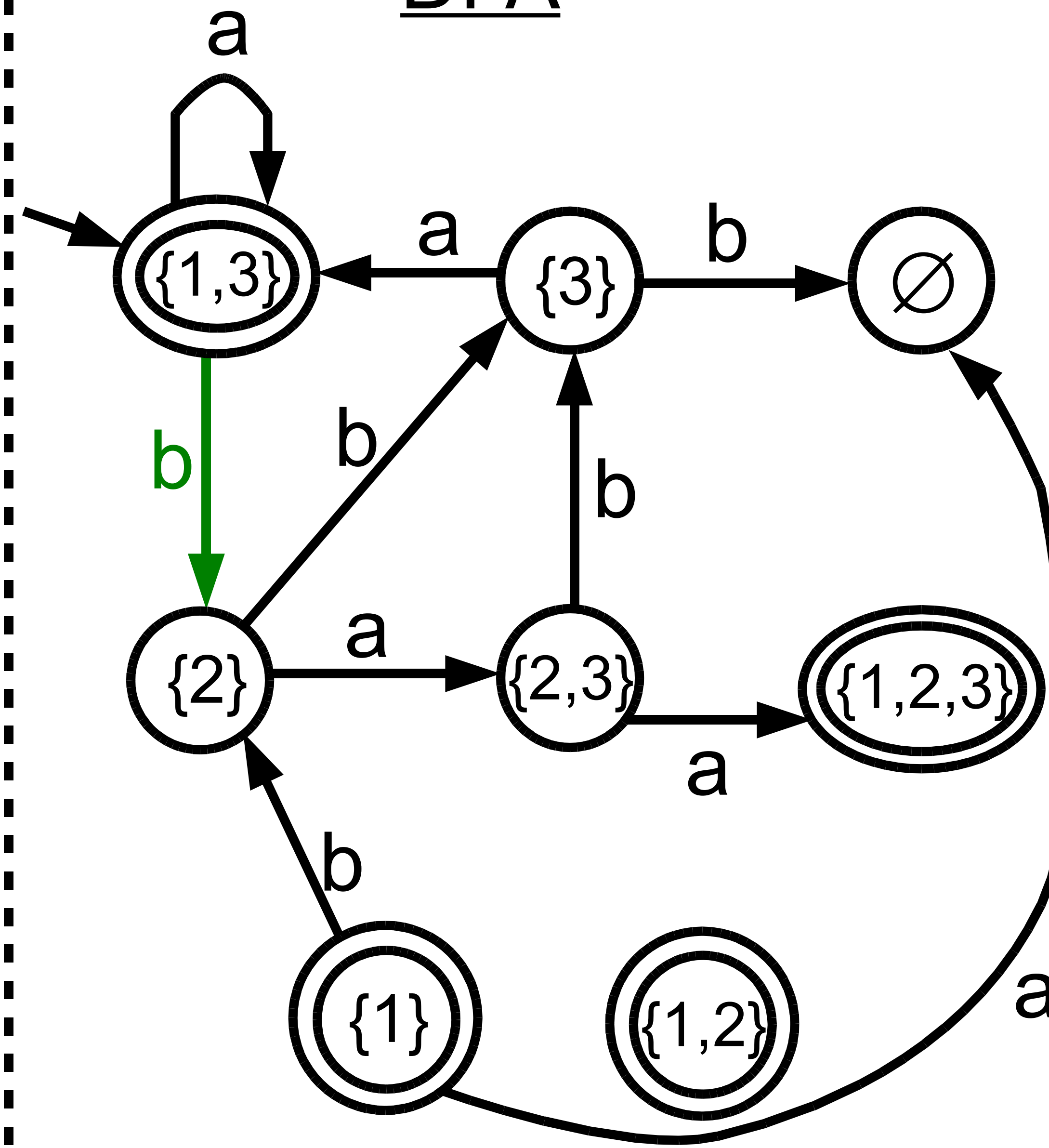
$= E(\varnothing \cup \{1\}) = \{1,3\}$

Example: NFA → DFA conversion

NFA



DFA



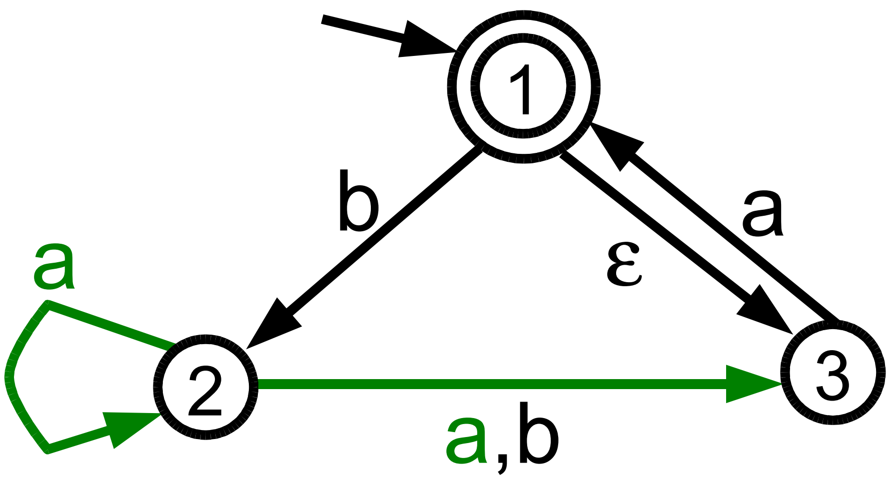
$\delta_{DFA}(\{1,3\}, b)$

$= E(\delta_{NFA}(1,b) \cup \delta_{NFA}(3,b))$

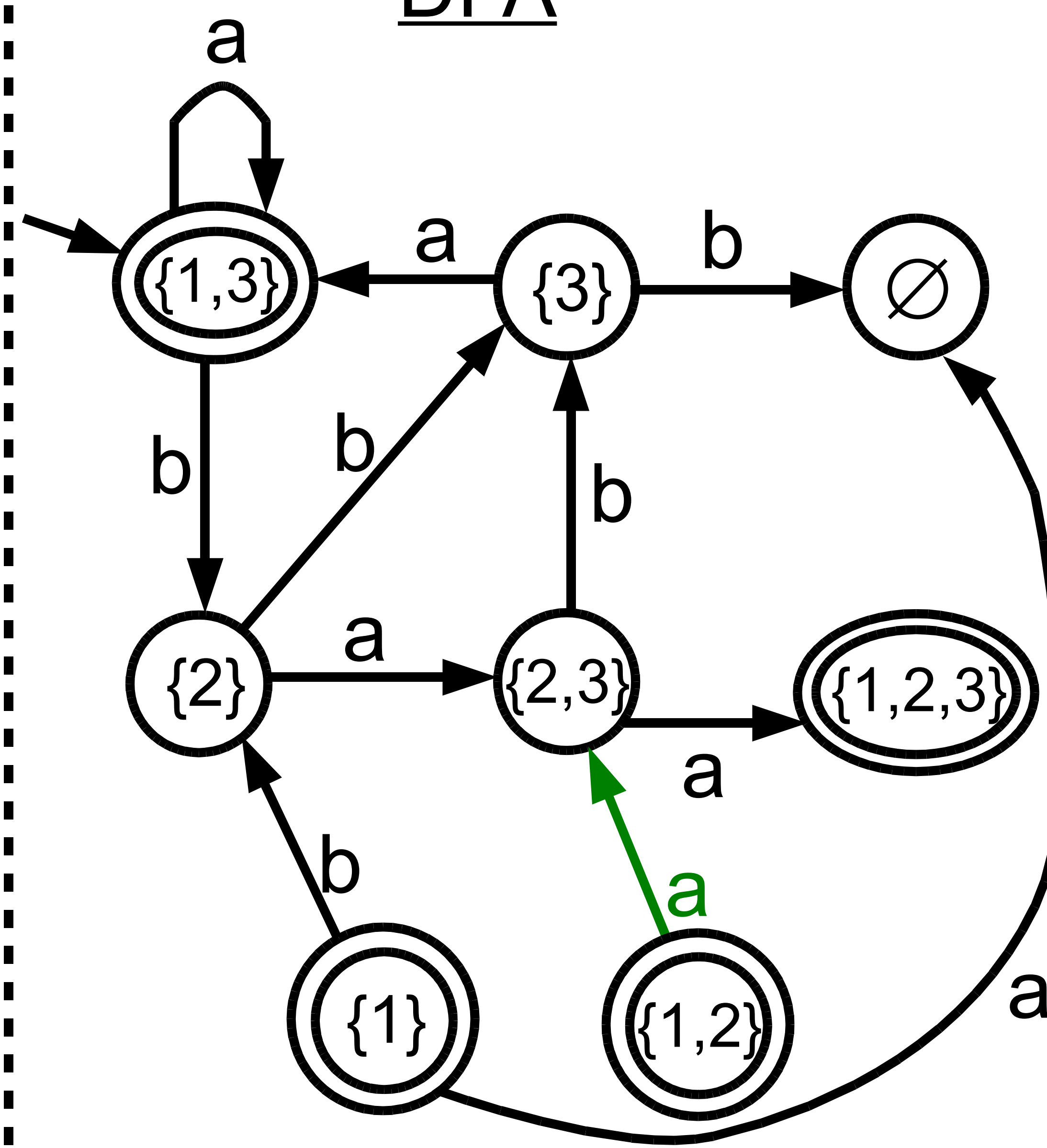
$= E(\{2\} \cup \emptyset) = \{2\}$

Example: NFA → DFA conversion

NFA



DFA



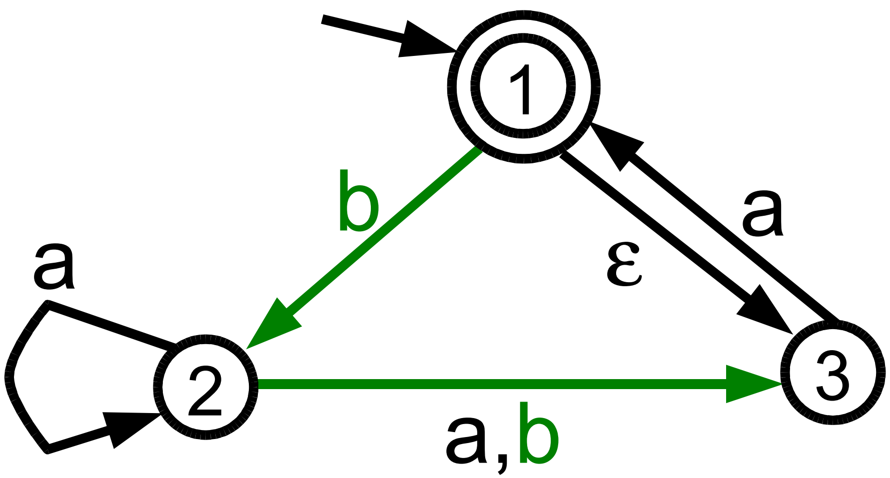
$\delta_{DFA}(\{1,2\}, a)$

$= E(\delta_{NFA}(1,a) \cup \delta_{NFA}(2,a))$

$= E(\emptyset \cup \{2,3\}) = \{2,3\}$

Example: NFA → DFA conversion

NFA

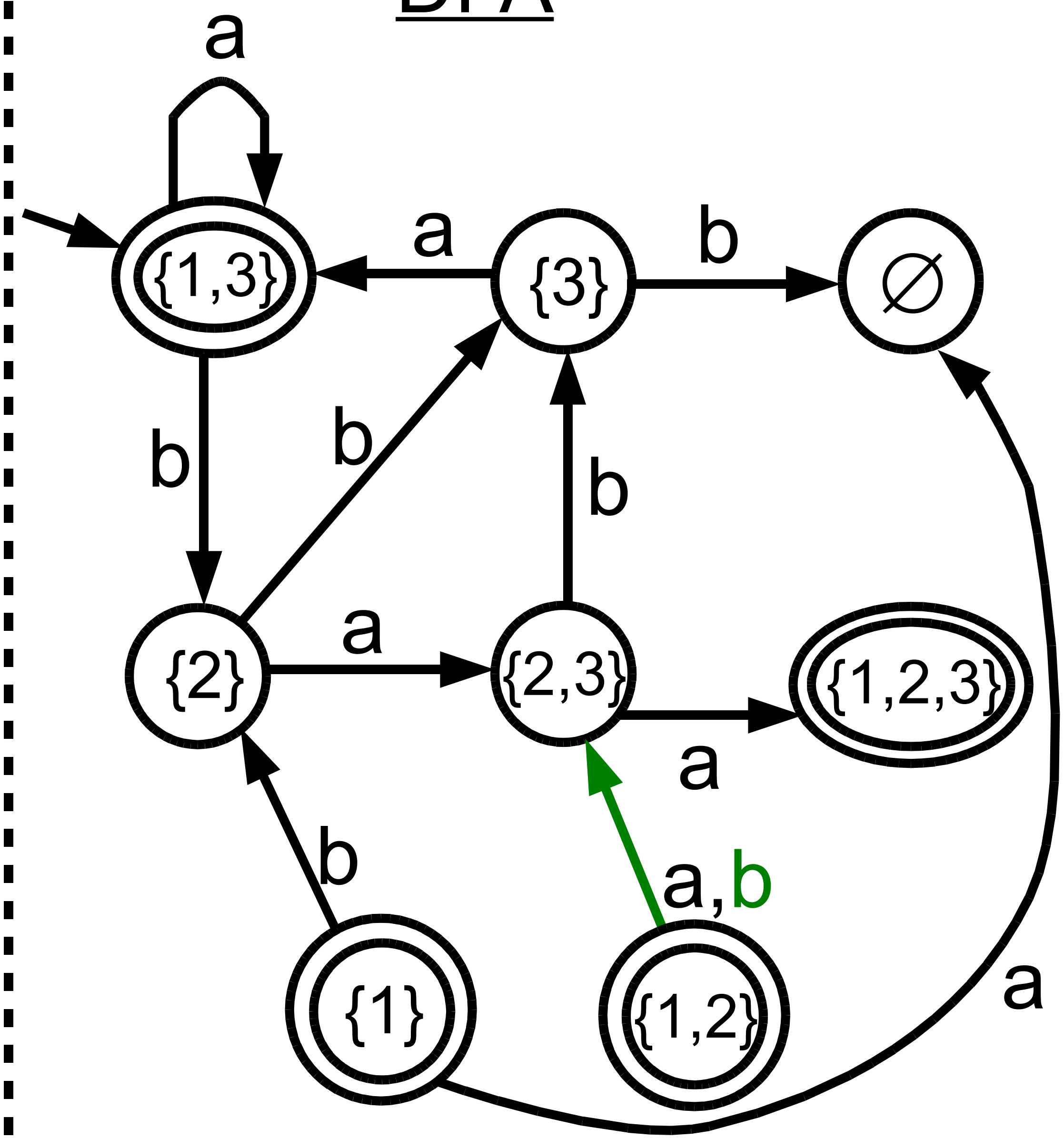


$$\delta_{DFA}(\{1,2\}, b)$$

$$= E(\delta_{NFA}(1,b) \cup \delta_{NFA}(2,b))$$

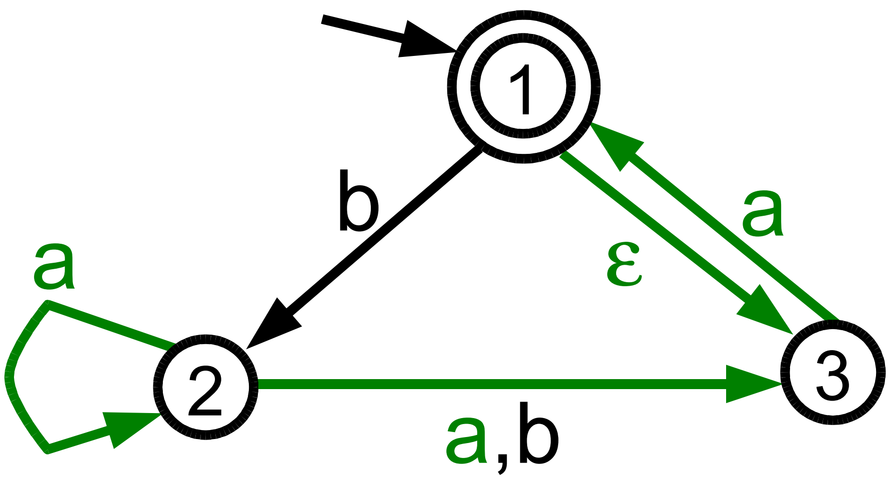
$$= E(\{2\} \cup \{3\}) = \{2,3\}$$

DFA



Example: NFA → DFA conversion

NFA

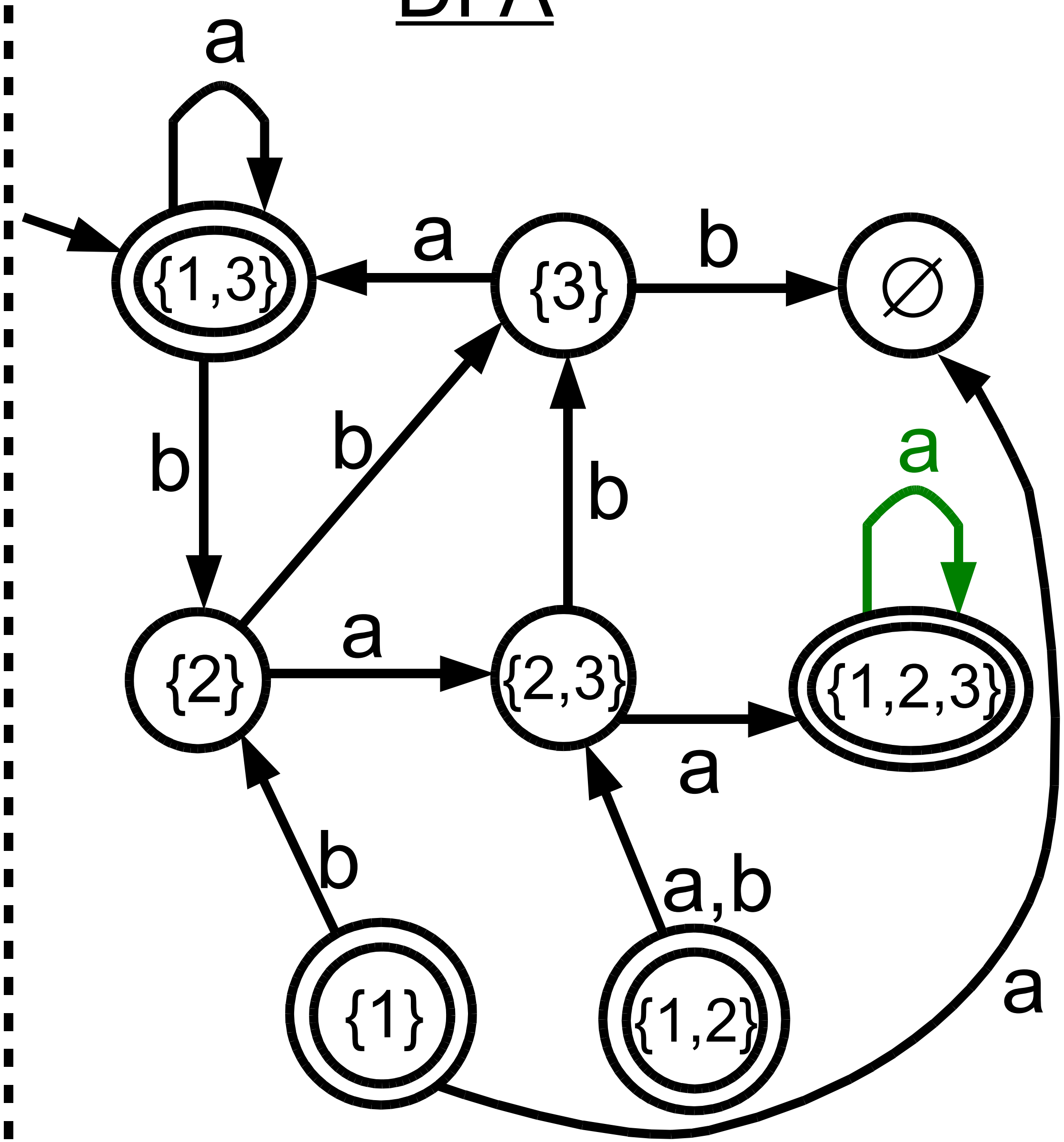


$$\delta_{DFA}(\{1,2,3\}, a)$$

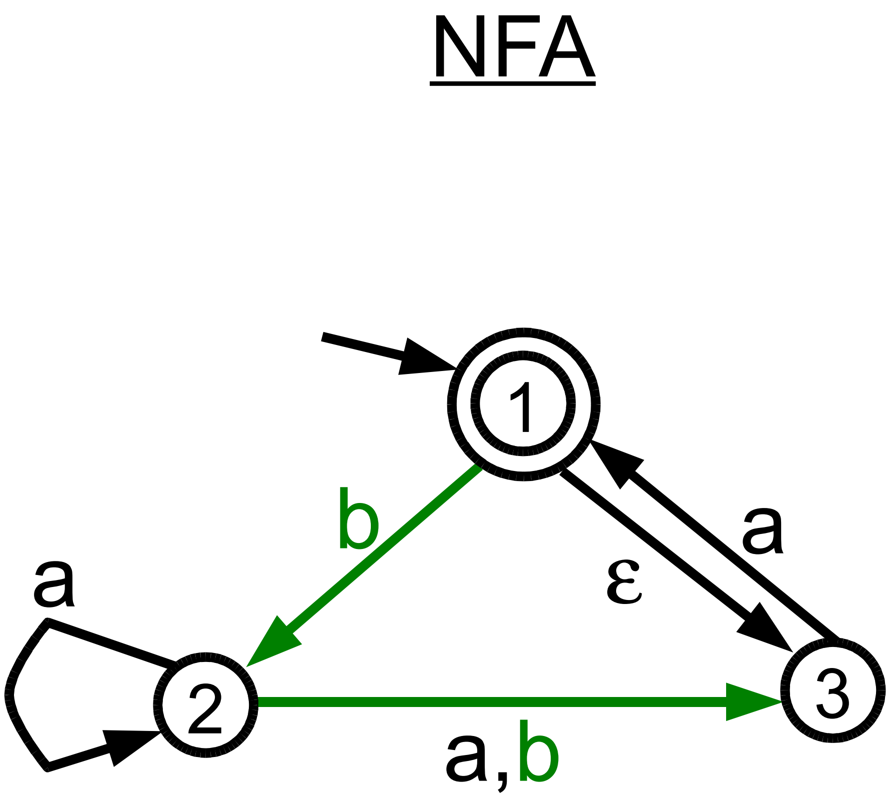
$$= E(\delta_{NFA}(1,a) \cup \delta_{NFA}(2,a) \cup \delta_{NFA}(3,a))$$

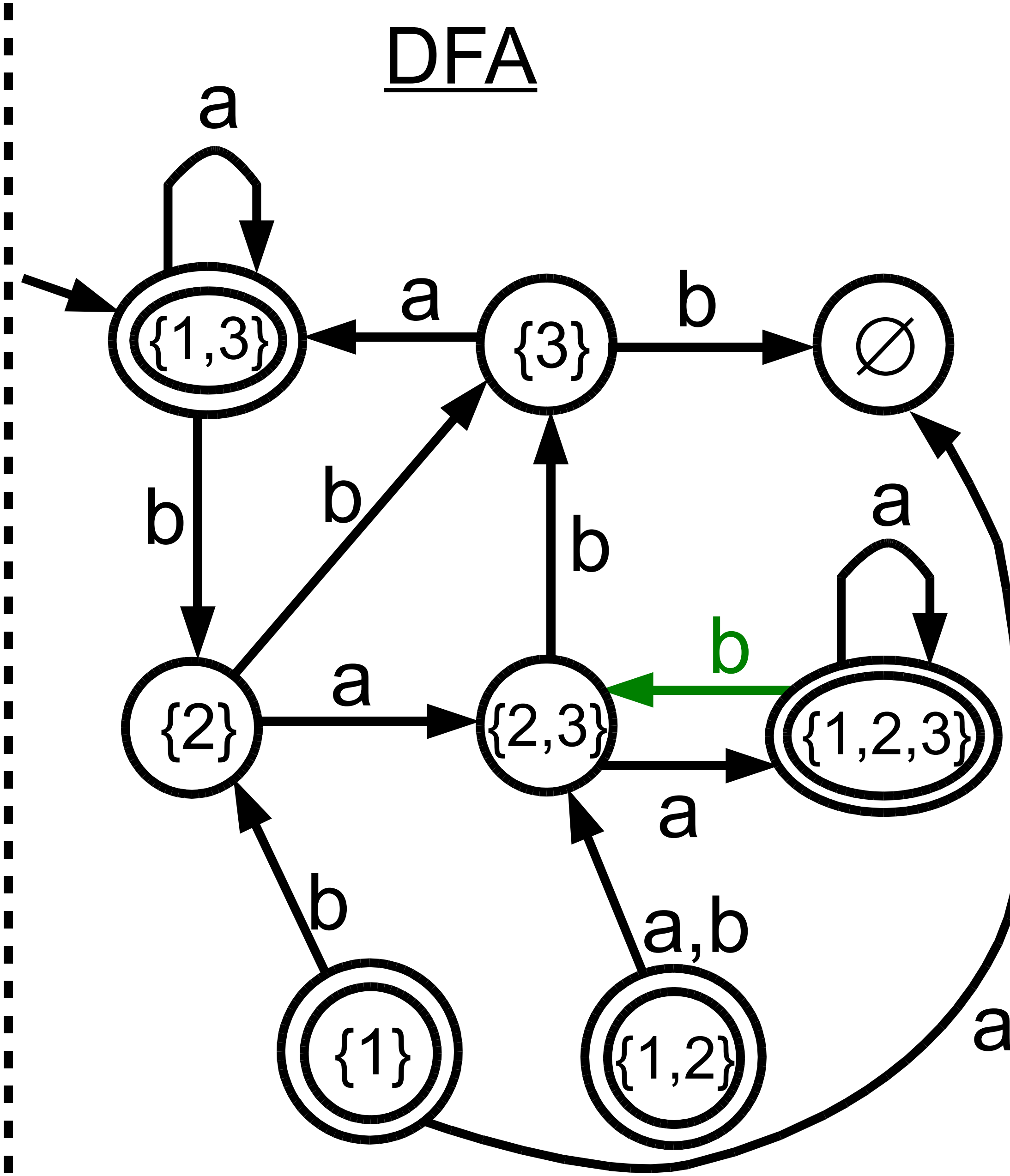
$$= E(\varnothing \cup \{2,3\} \cup \{1\}) = \{1,2,3\}$$

DFA



Example: NFA → DFA conversion



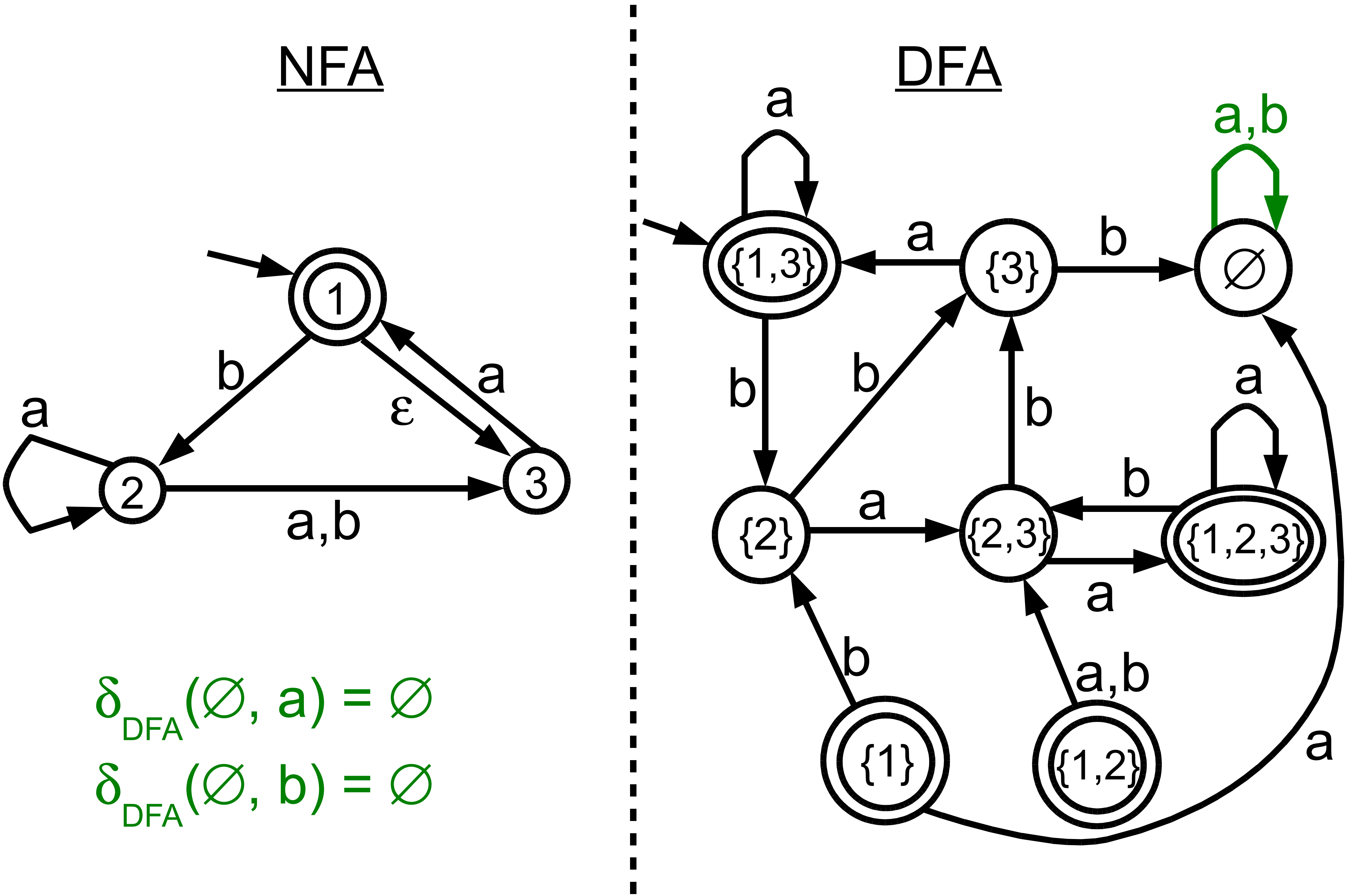


$\delta_{DFA}(\{1,2,3\}, b)$

$= E(\delta_{NFA}(1,b) \cup \delta_{NFA}(2,b) \cup \delta_{NFA}(3,b))$

$= E(\{2\} \cup \{3\} \cup \varnothing) = \{2,3\}$

Example: NFA → DFA conversion

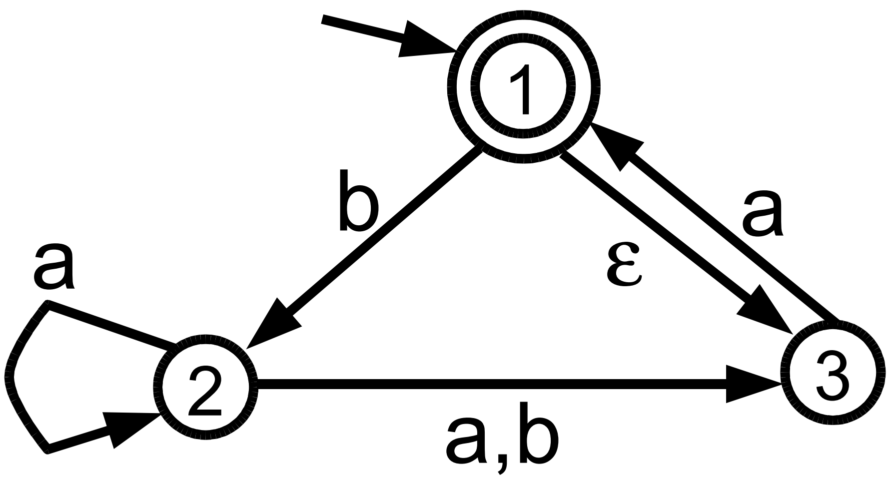


$\delta_{DFA}(\varnothing, a) = \varnothing$

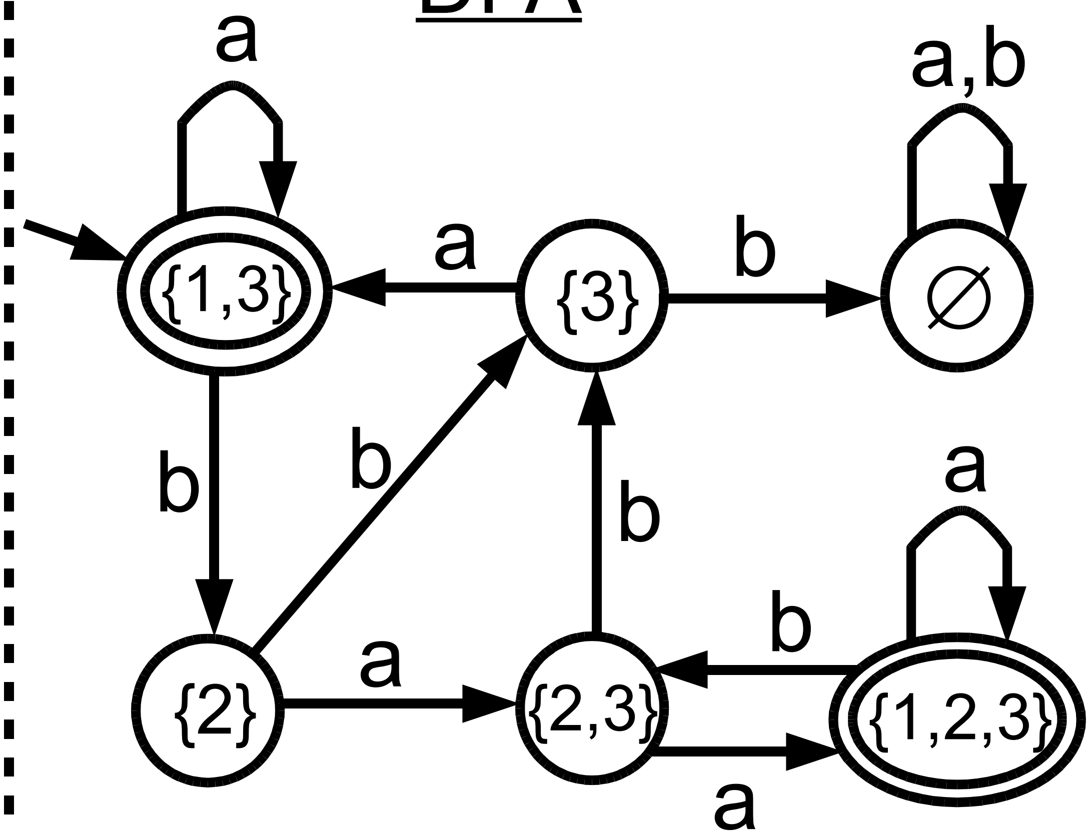
$\delta_{DFA}(\varnothing, b) = \varnothing$

Example: NFA → DFA conversion

NFA



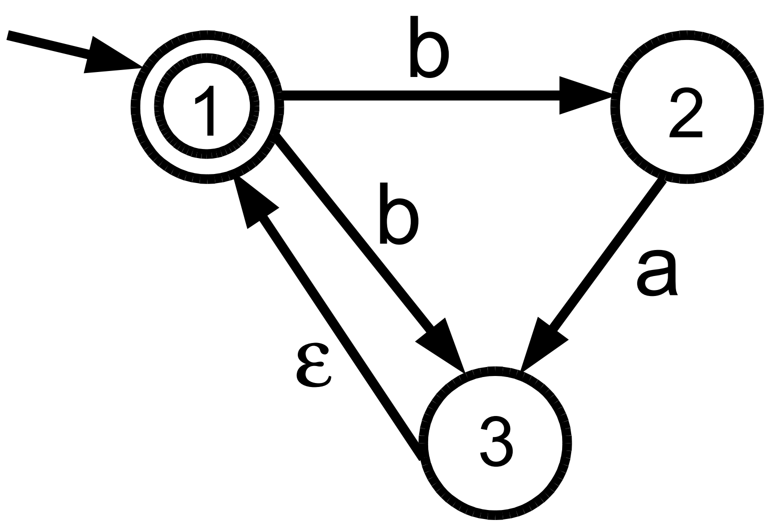
DFA



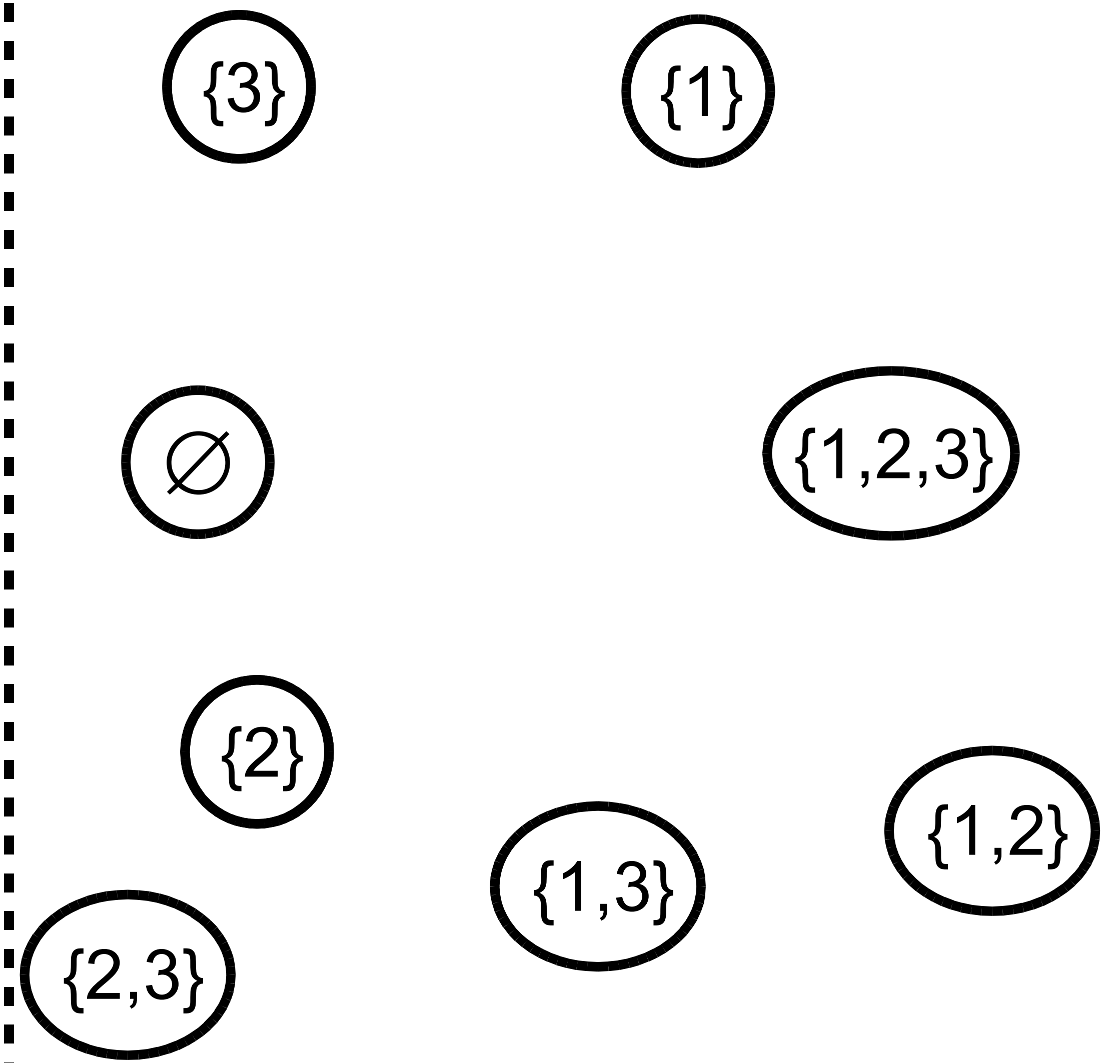
We can delete the unreachable states.

ANOTHER Example: NFA → DFA conversion

NFA



DFA

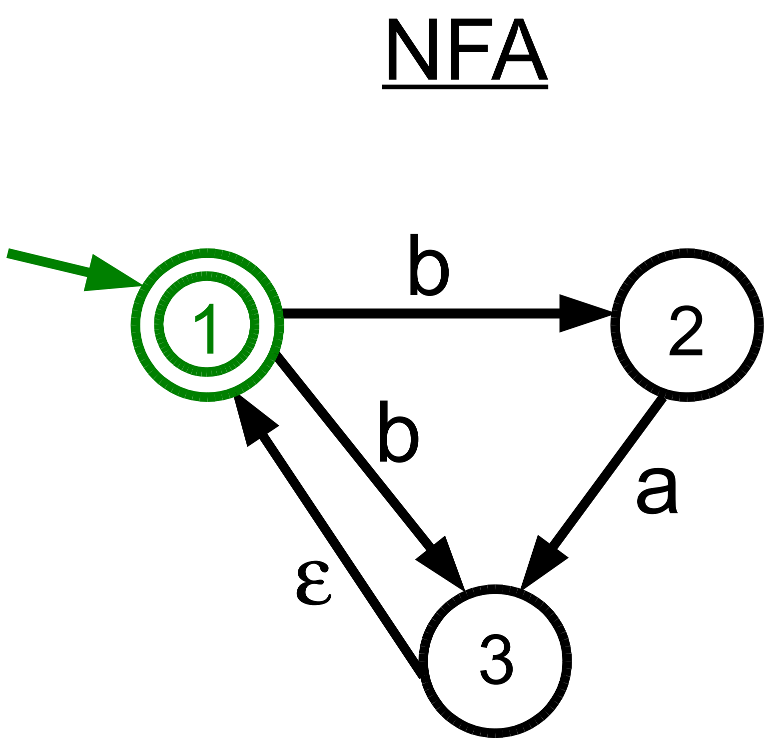


Q_{DFA} = Powerset(Q_{NFA})

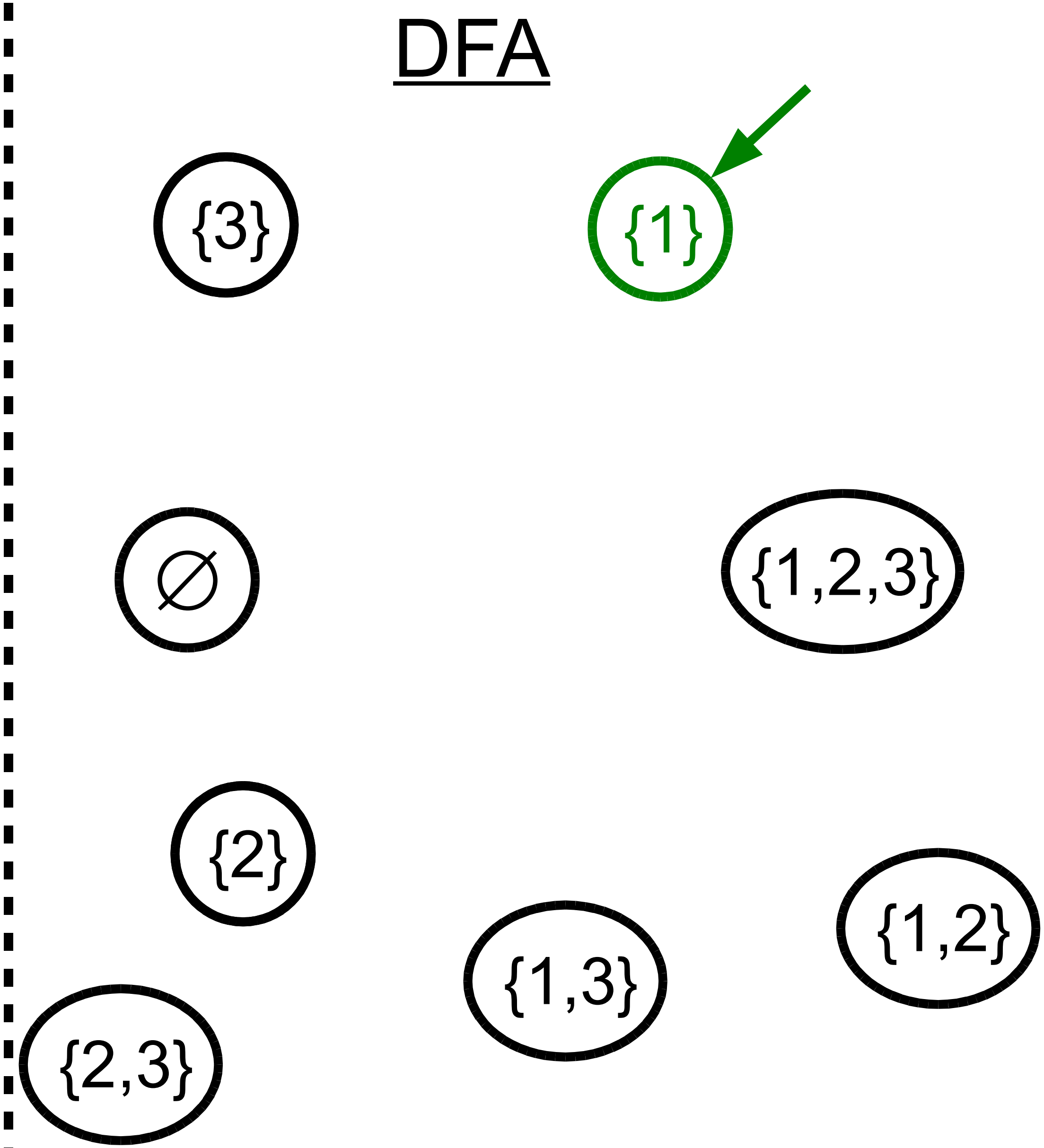
= Powerset({1,2,3})

= {∅,{1},{2},{3},{1,2}...}

ANOTHER Example: NFA → DFA conversion

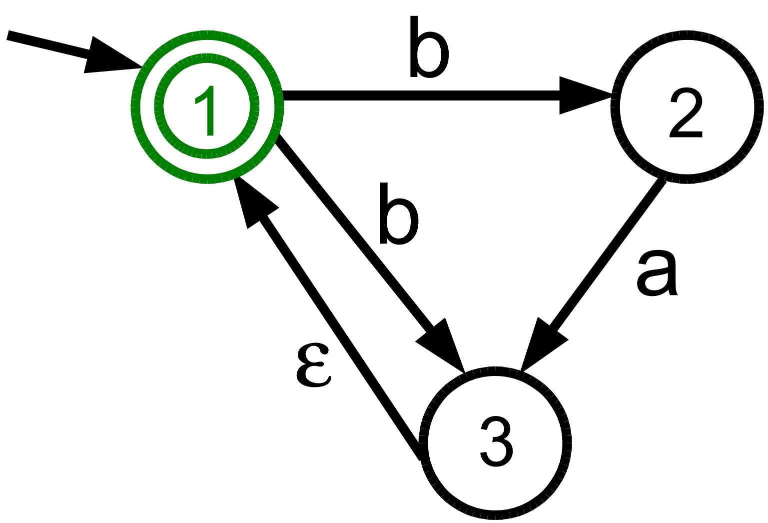


$$q_{DFA} = E(\{q_{NFA}\})$$
$$= E(\{1\})$$
$$= \{1\}$$

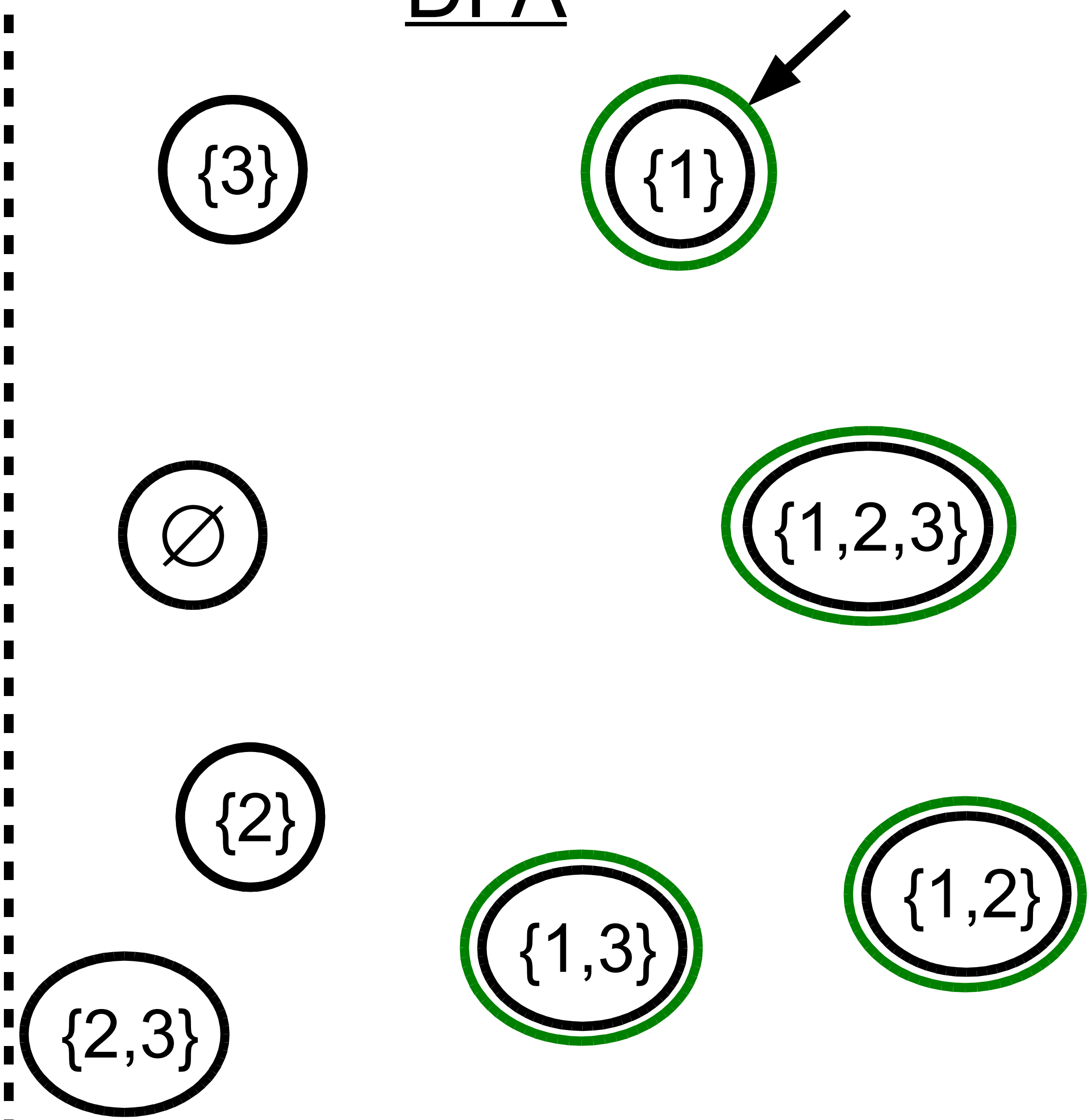


ANOTHER Example: NFA → DFA conversion

NFA

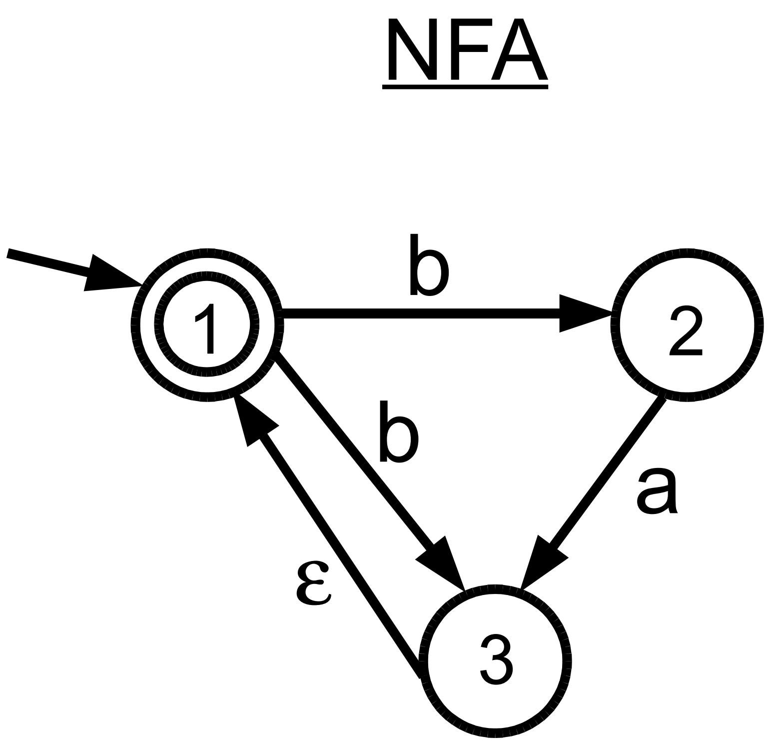


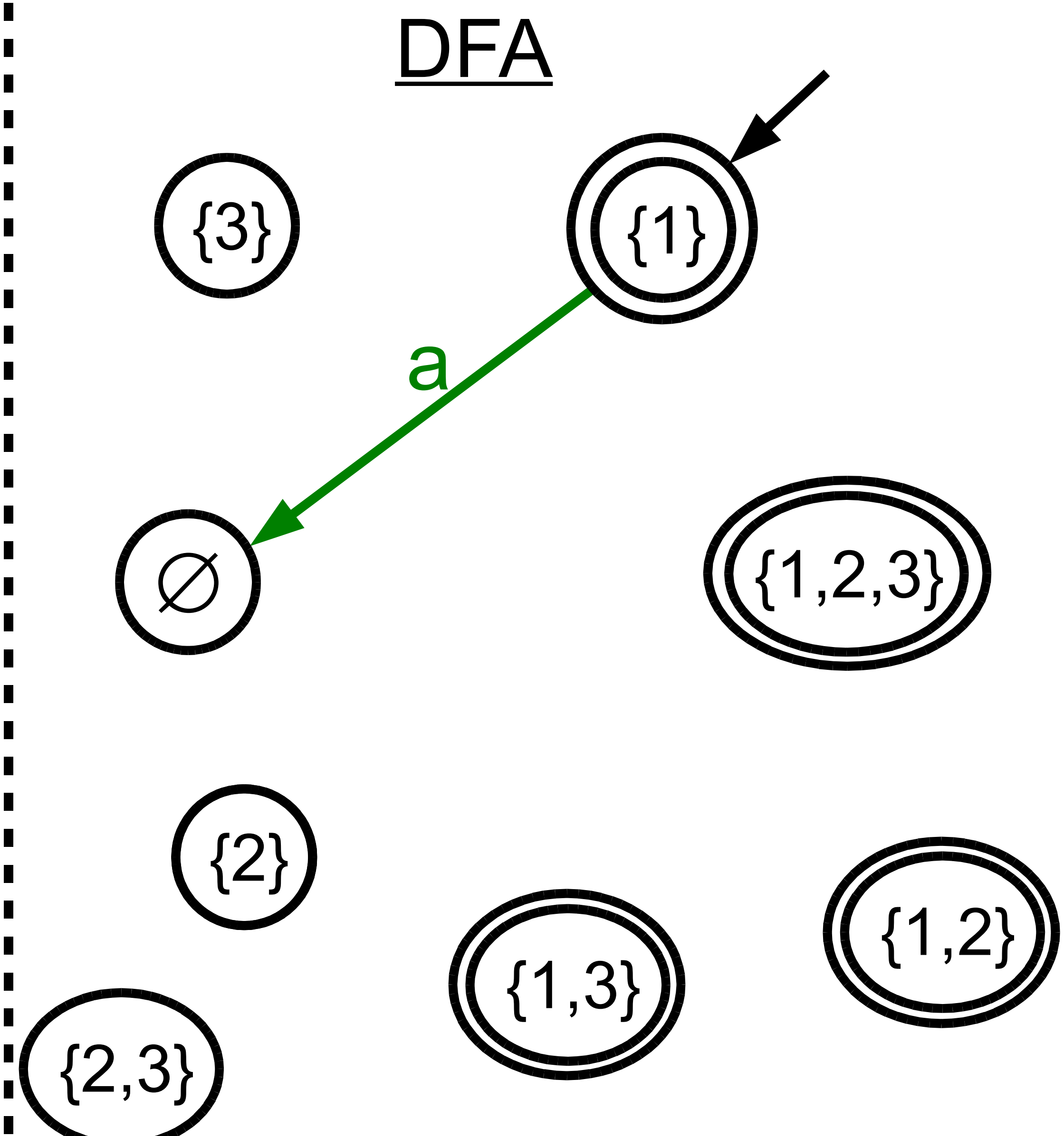
DFA



F_{DFA} = {S : S contains an element of F_{NFA}}

ANOTHER Example: NFA → DFA conversion

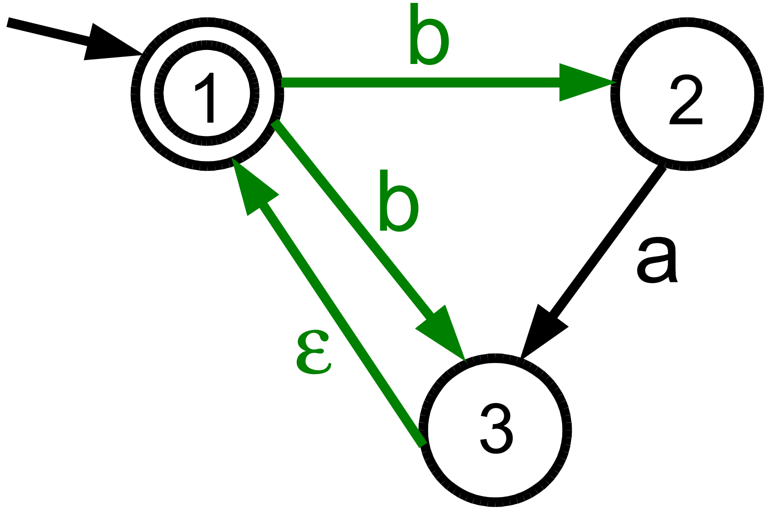




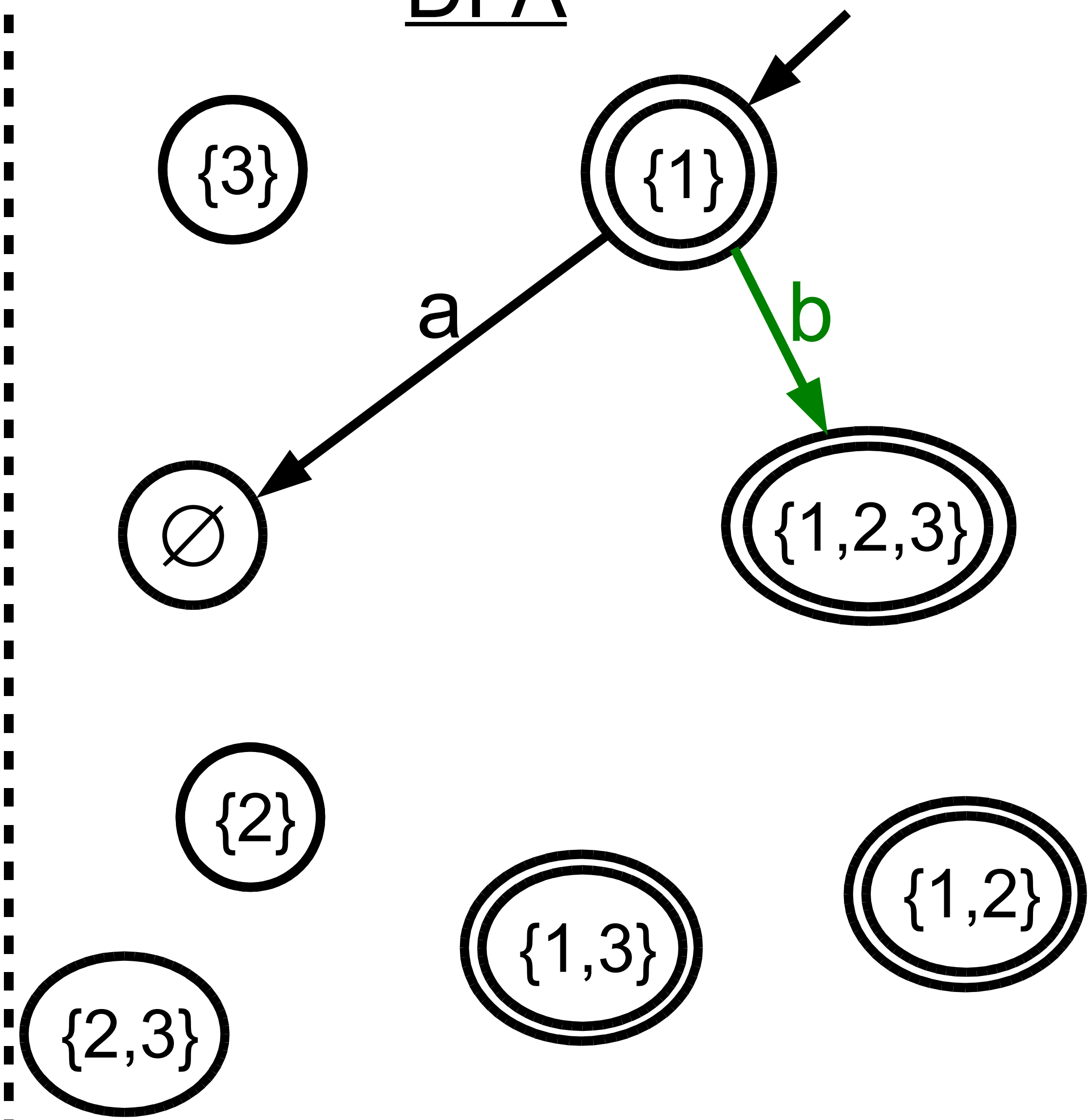
$$\delta_{DFA}(\{1\}, a)$$
$$= E(\delta_{NFA}(1, a))$$
$$= E(\varnothing) = \varnothing$$

ANOTHER Example: NFA → DFA conversion

NFA

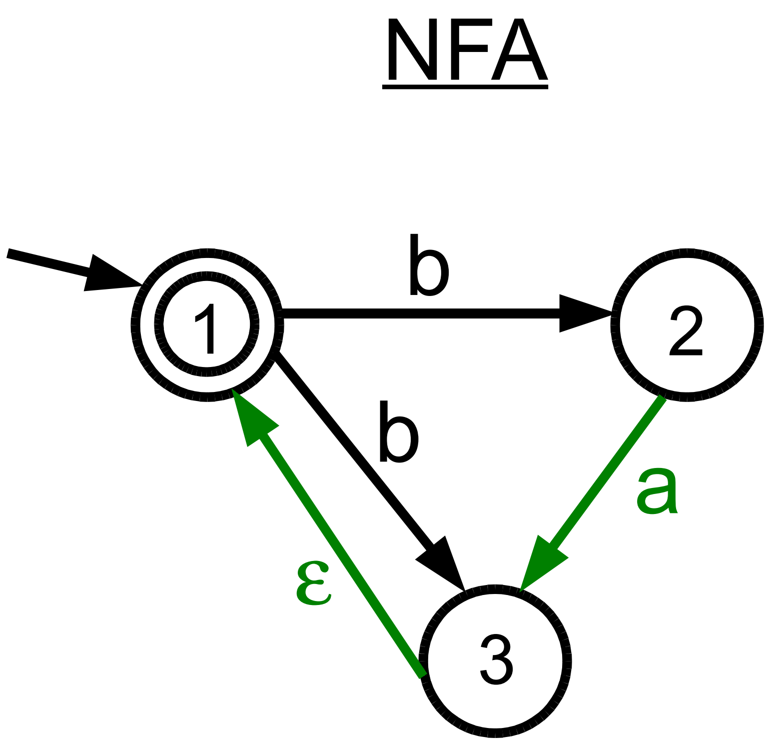


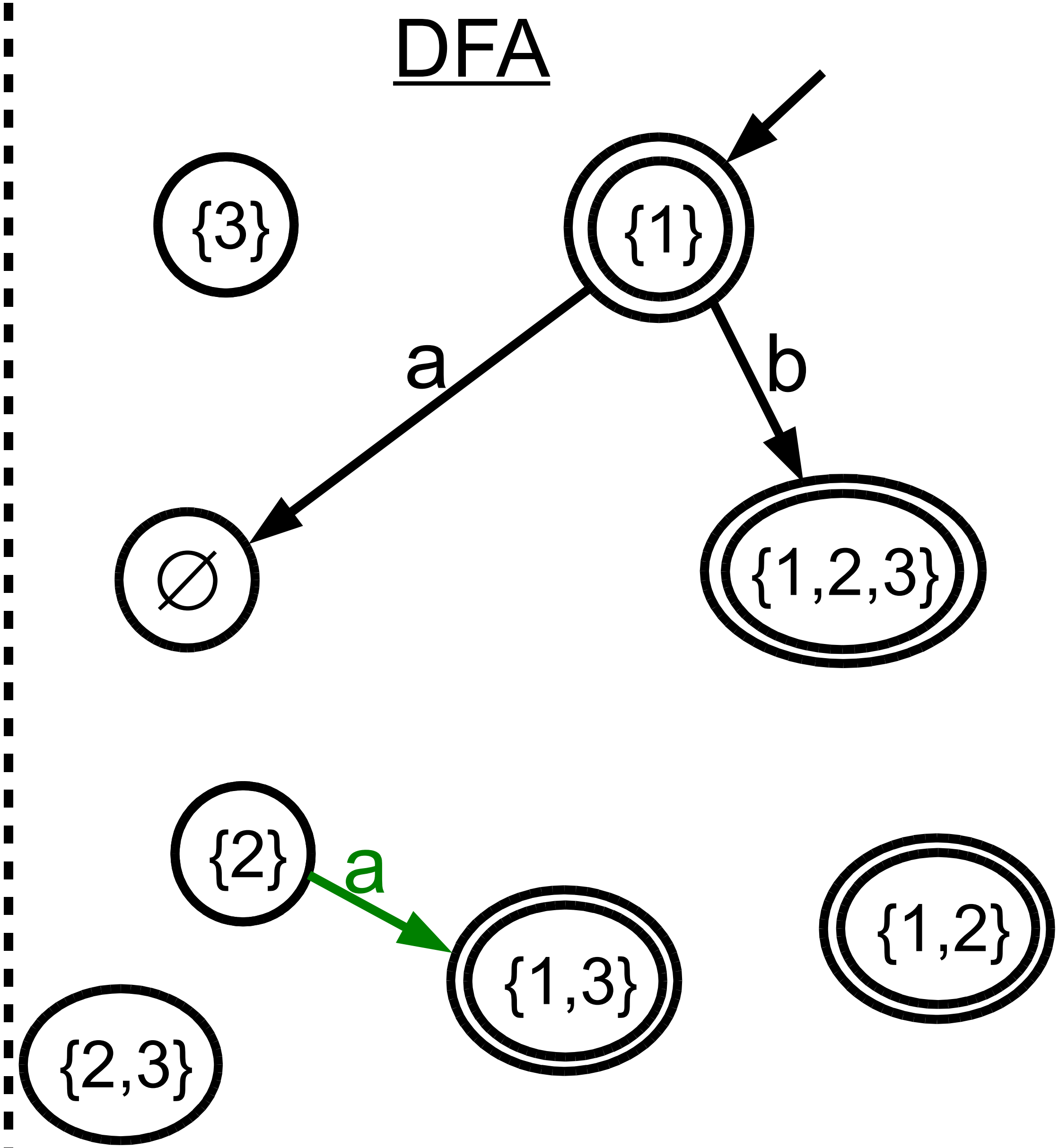
DFA



$$\delta_{DFA}(\{1\}, b)$$
$$= E(\delta_{NFA}(1, b))$$
$$= E(\{2,3\}) = \{1,2,3\}$$

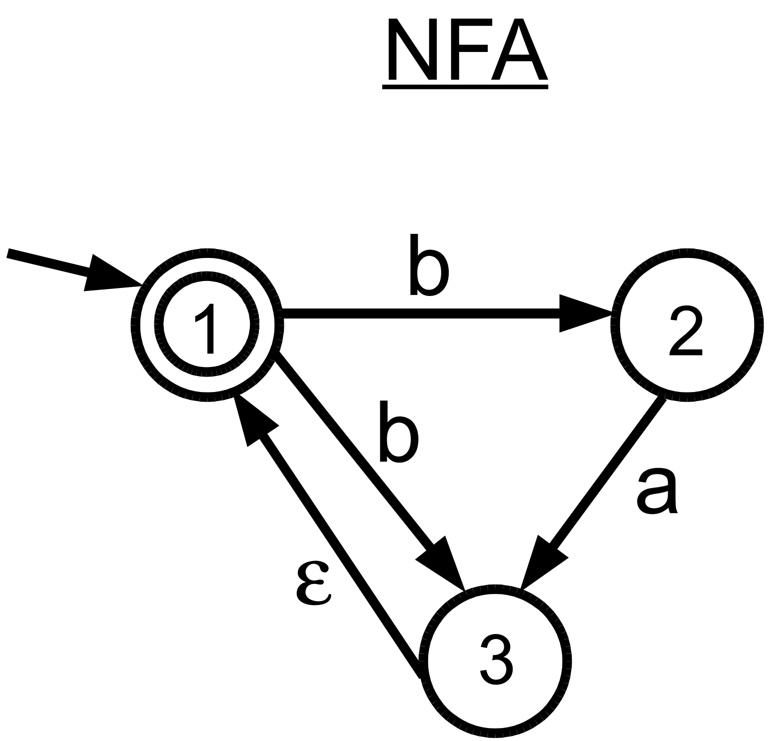
ANOTHER Example: NFA → DFA conversion

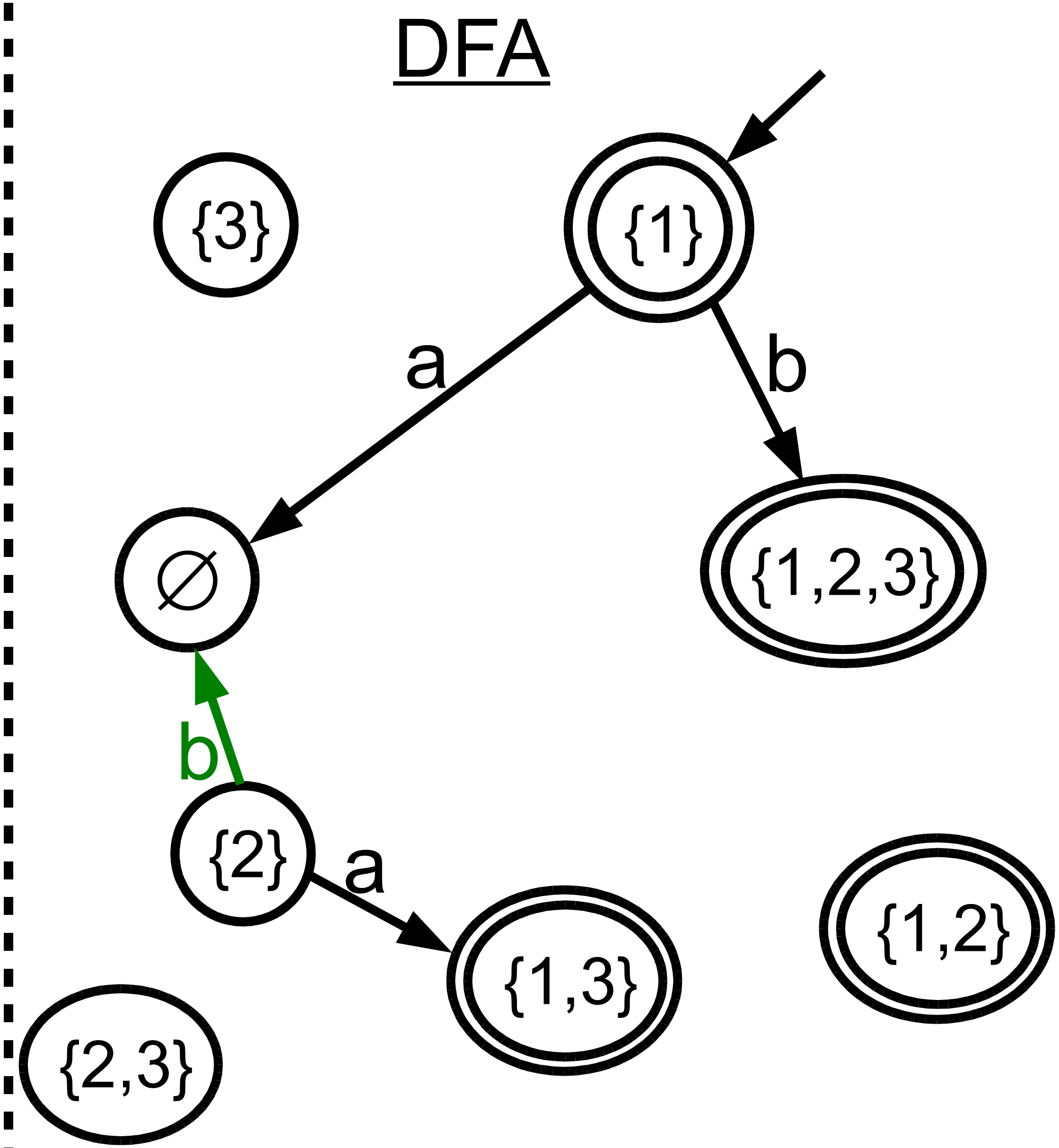




$$\delta_{DFA}(\{2\}, a)$$
$$= E(\delta_{NFA}(2, a))$$
$$= E(\{3\}) = \{1,3\}$$

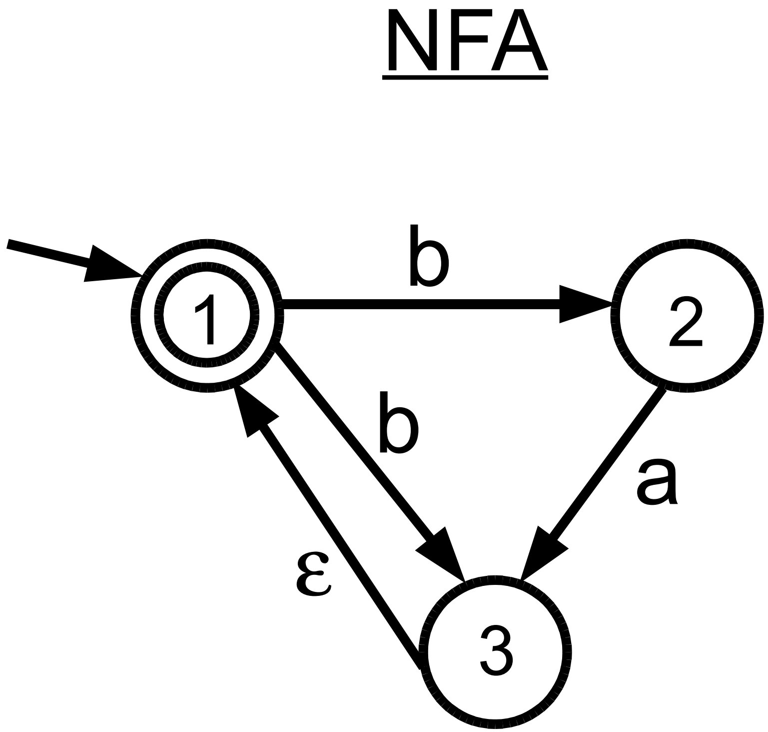
ANOTHER Example: NFA → DFA conversion

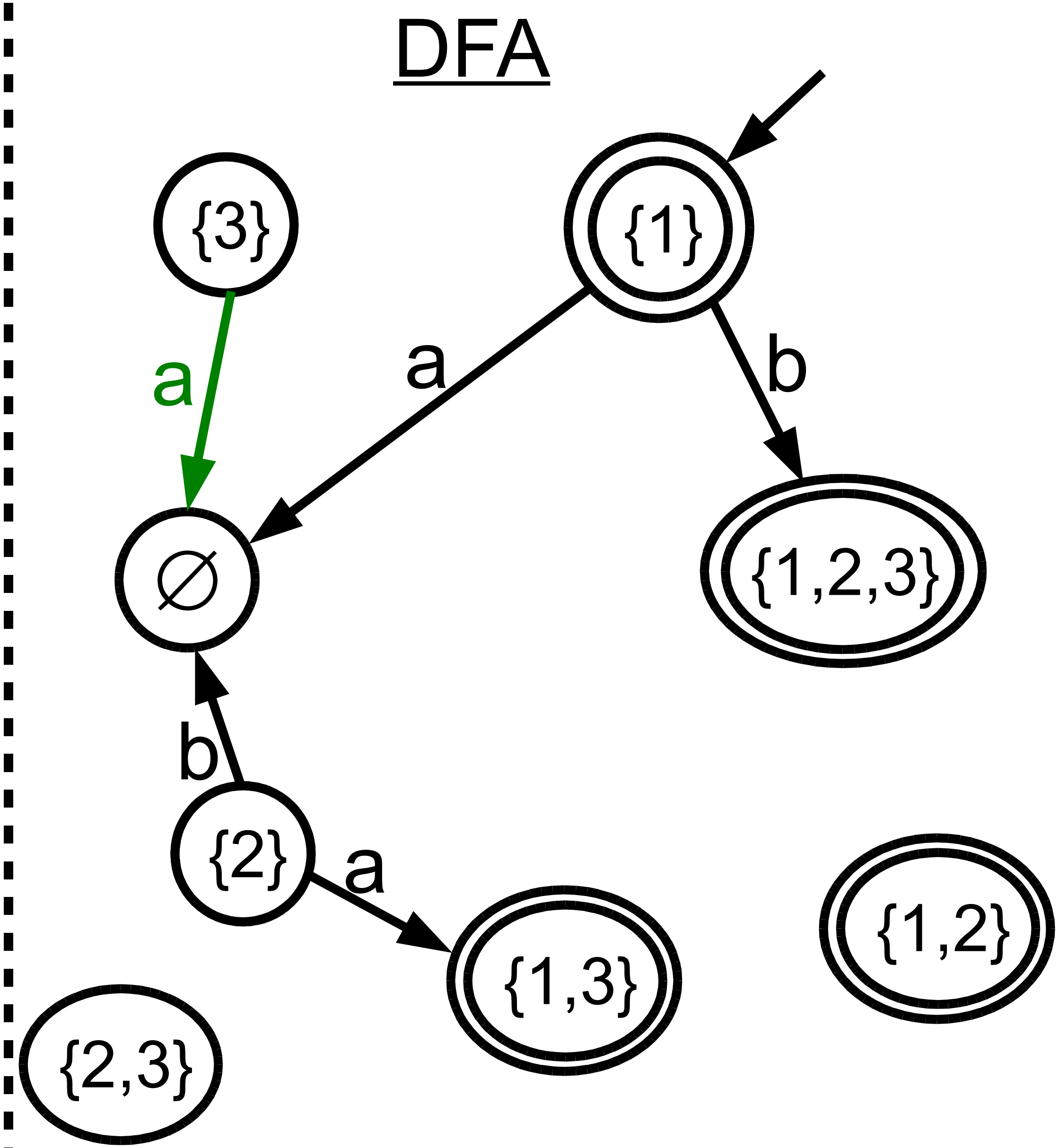




$$\delta_{DFA}(\{2\}, b) = E(\delta_{NFA}(2, b)) = E(\emptyset) = \emptyset$$

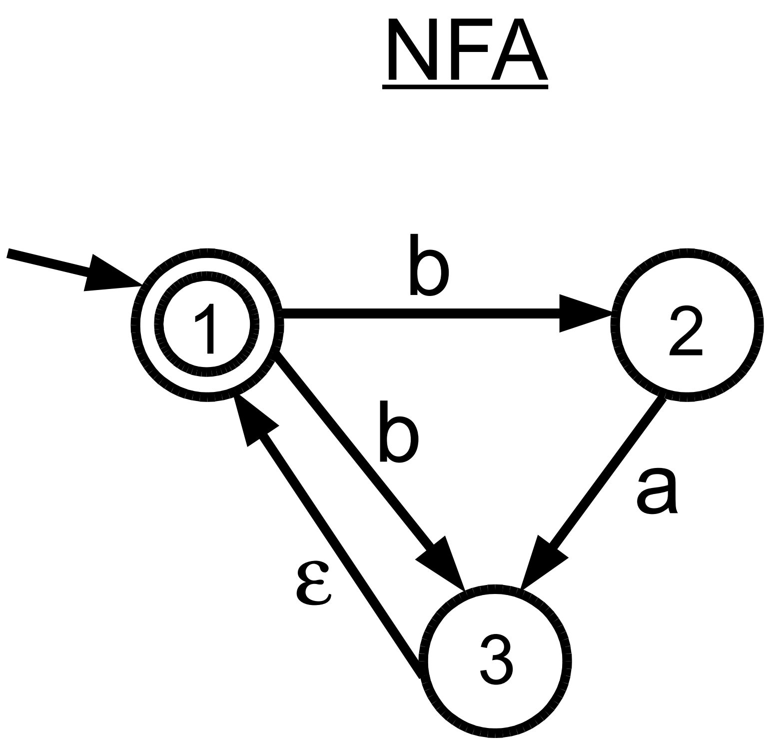
ANOTHER Example: NFA → DFA conversion

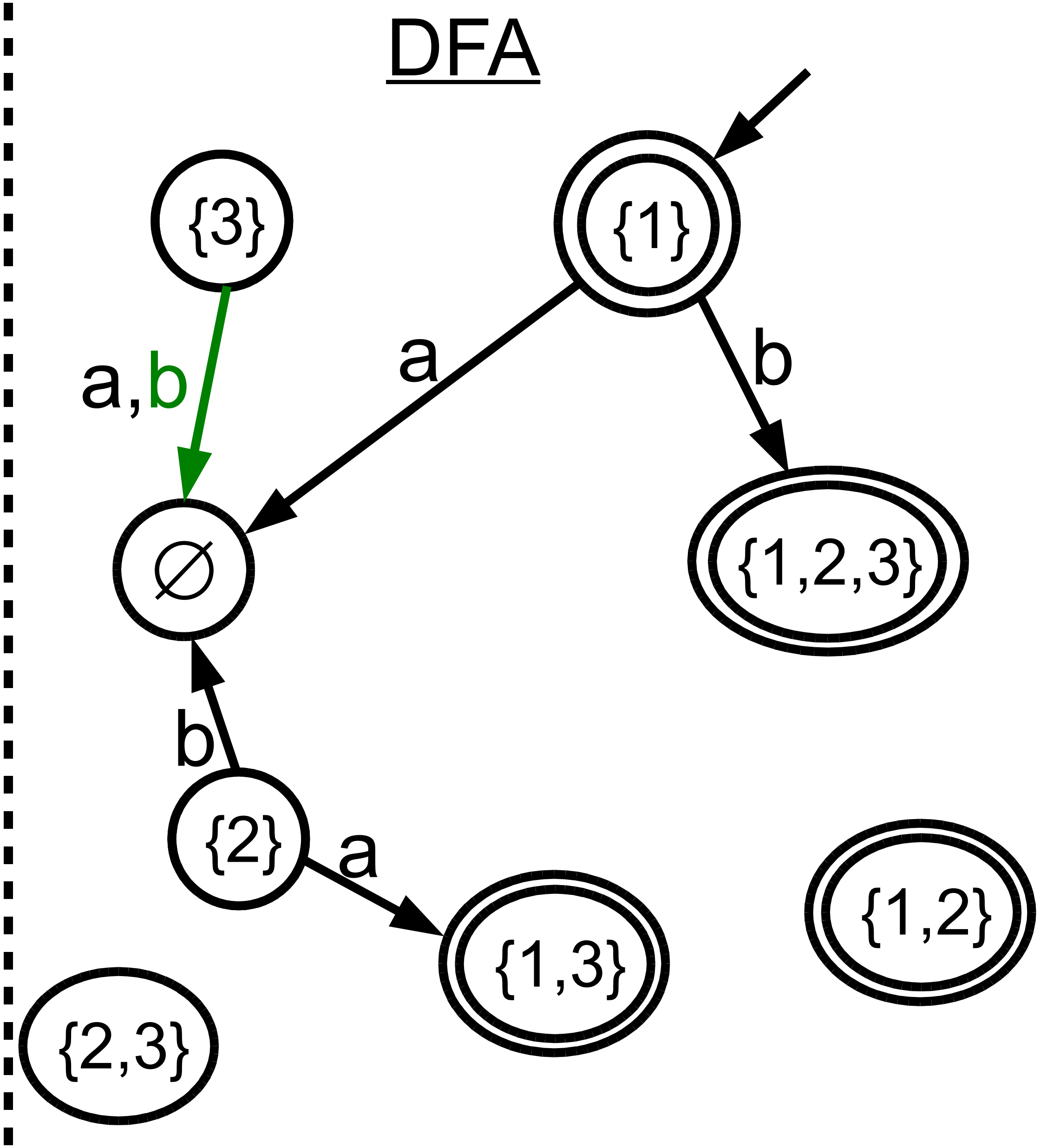




$$\delta_{DFA}(\{3\}, a)$$
$$= E(\delta_{NFA}(3, a))$$
$$= E(\varnothing) = \varnothing$$

ANOTHER Example: NFA → DFA conversion

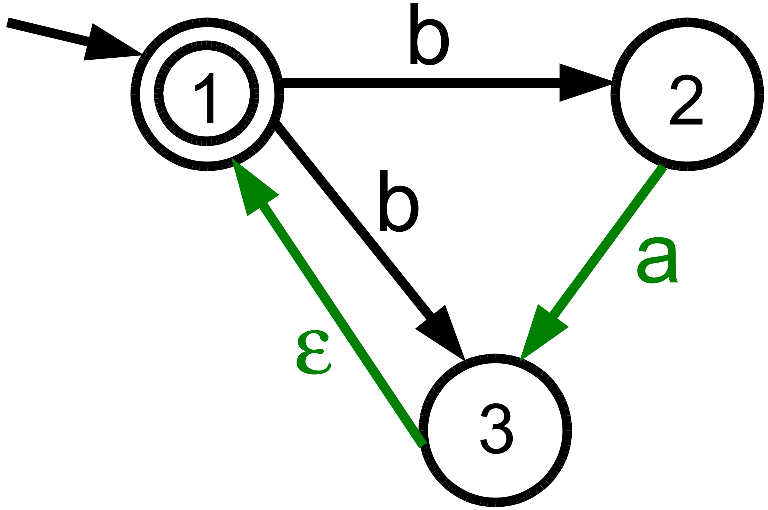




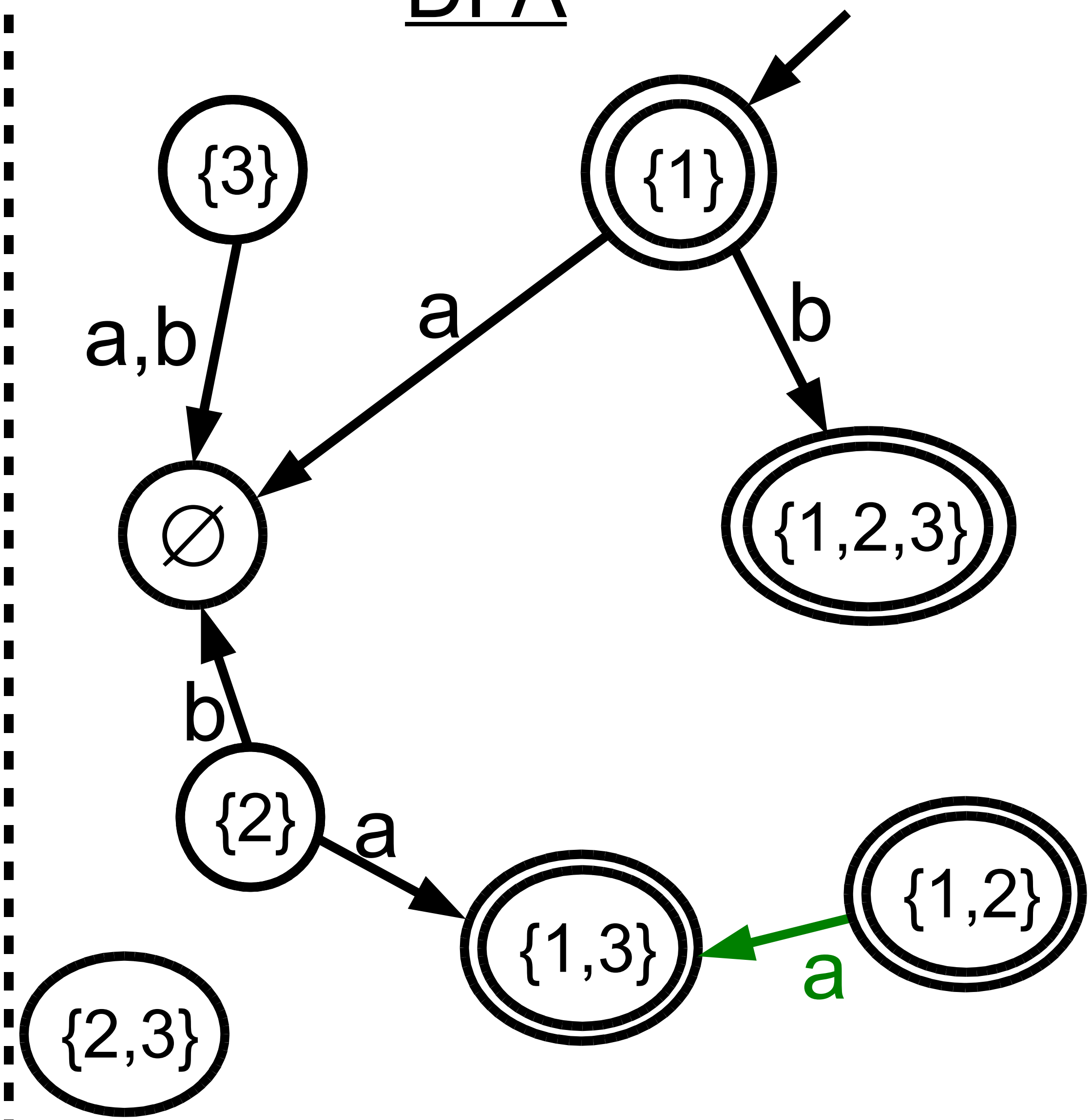
$$\delta_{DFA}(\{3\}, b) = E(\delta_{NFA}(3, b)) = E(\emptyset) = \emptyset$$

ANOTHER Example: NFA → DFA conversion

NFA



DFA



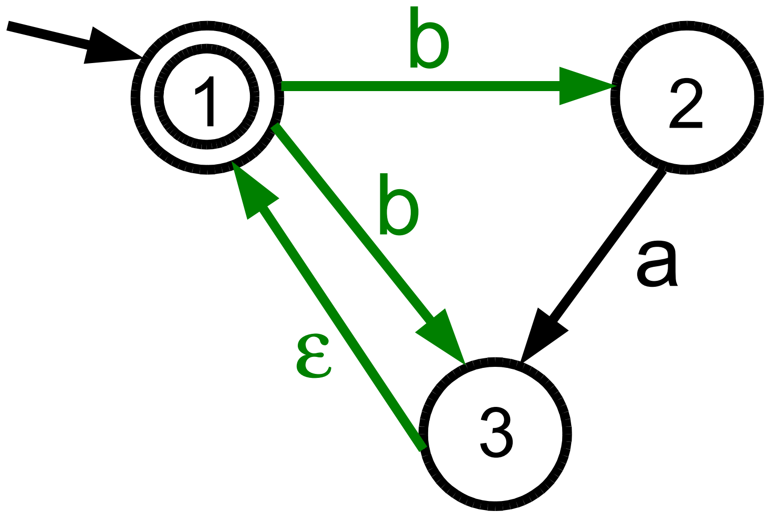
$$\delta_{DFA}(\{1,2\}, a)$$

$$= E(\delta_{NFA}(1,a) \cup \delta_{NFA}(2,a))$$

$$= E(\emptyset \cup \{3\}) = \{1,3\}$$

ANOTHER Example: NFA → DFA conversion

NFA

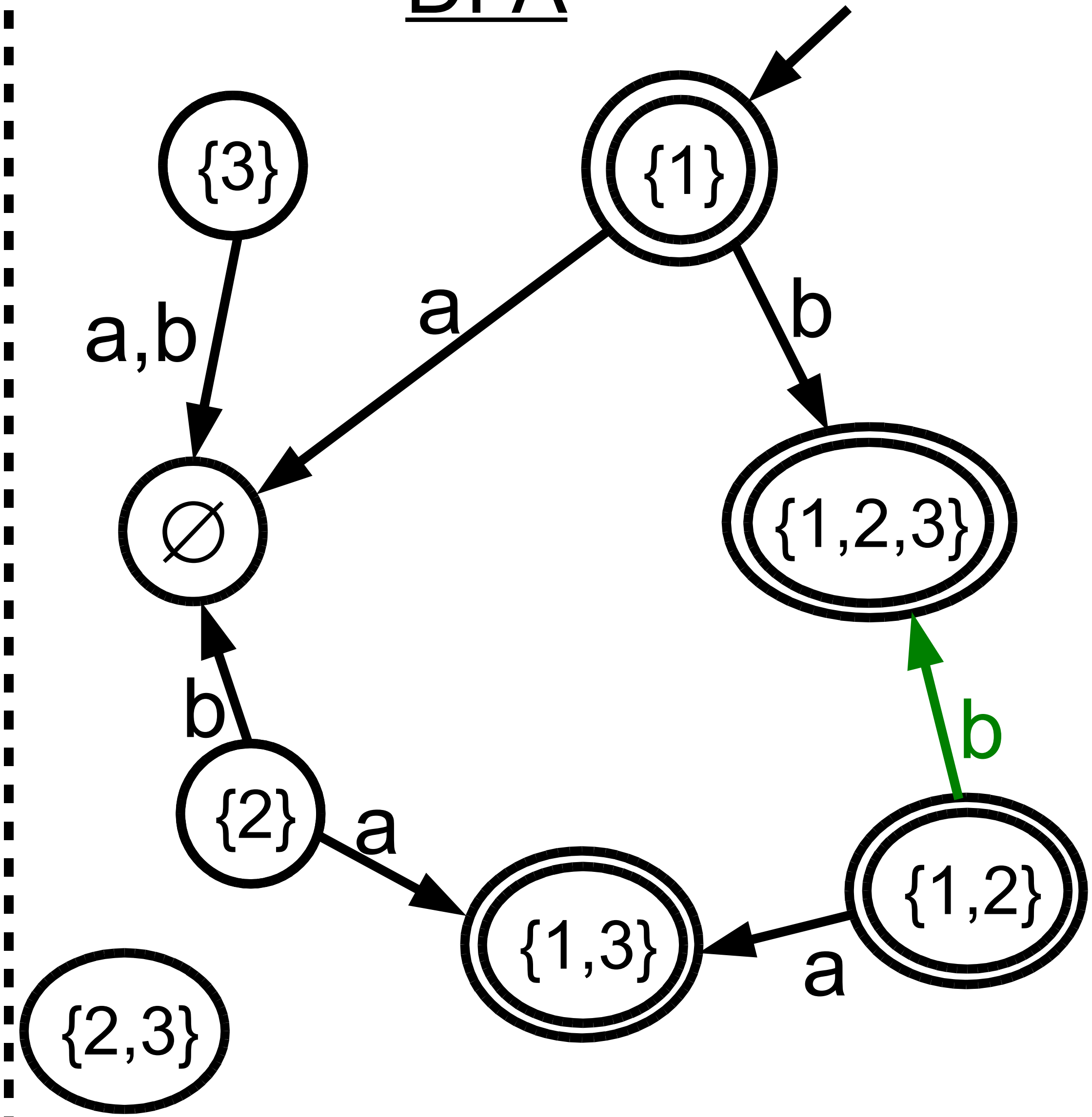


$\delta_{DFA}(\{1,2\}, b)$

$= E(\delta_{NFA}(1,b) \cup \delta_{NFA}(2,b))$

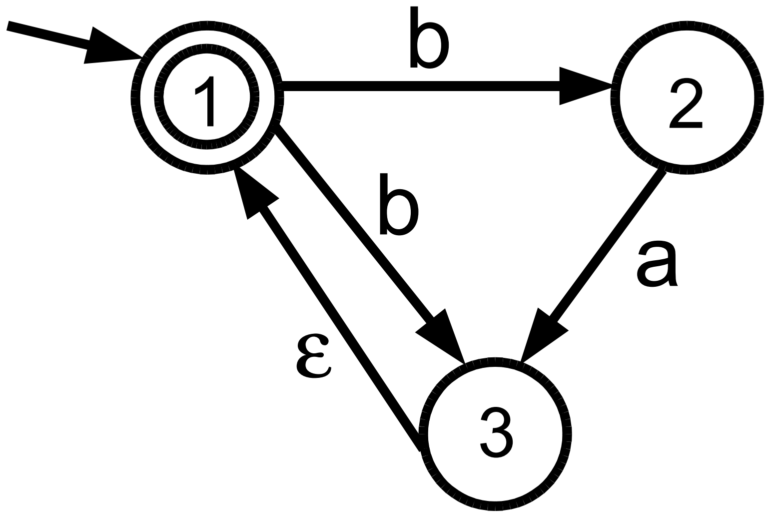
$= E(\{2,3\} \cup \varnothing) = \{1,2,3\}$

DFA

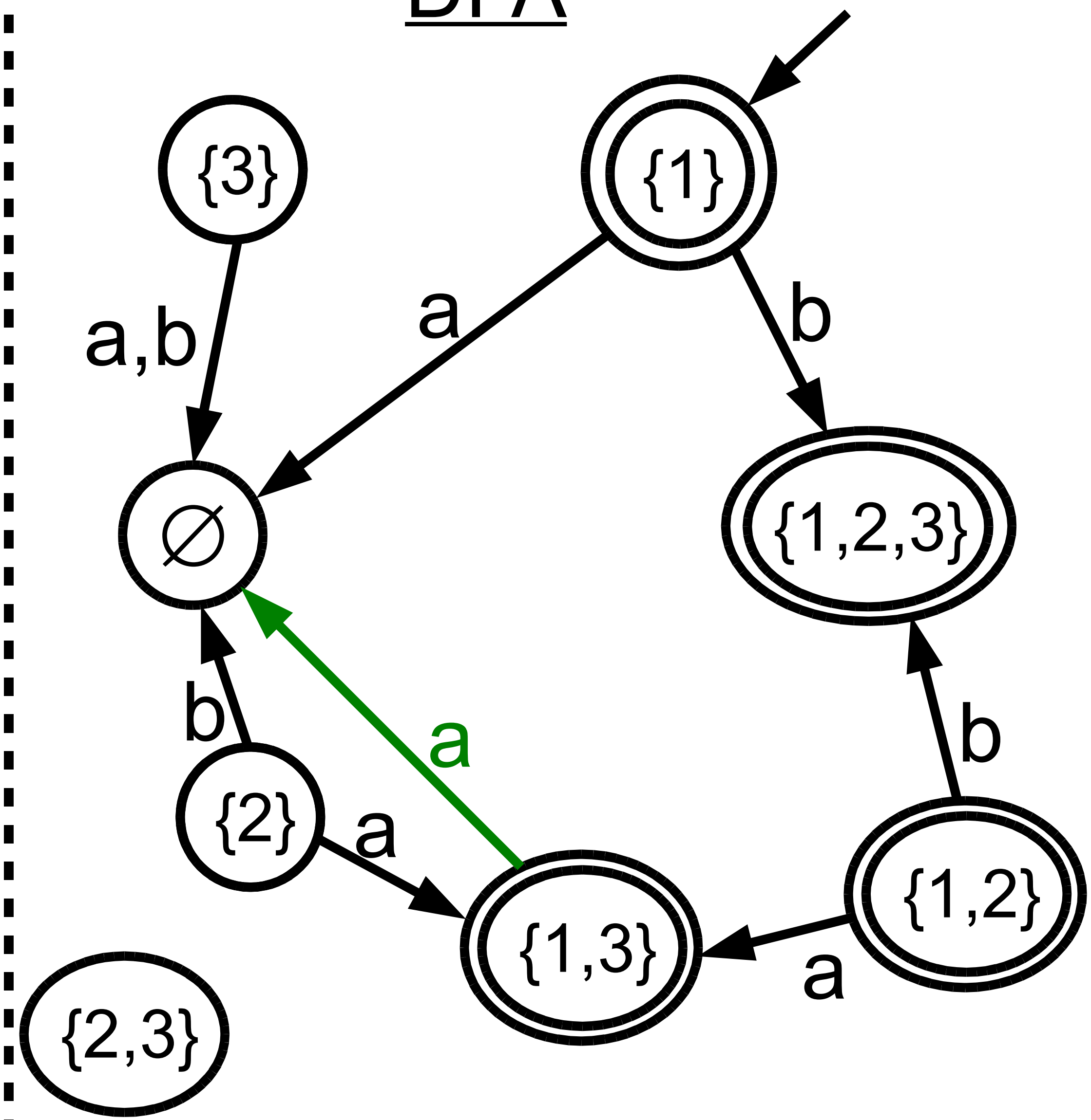


ANOTHER Example: NFA → DFA conversion

NFA



DFA



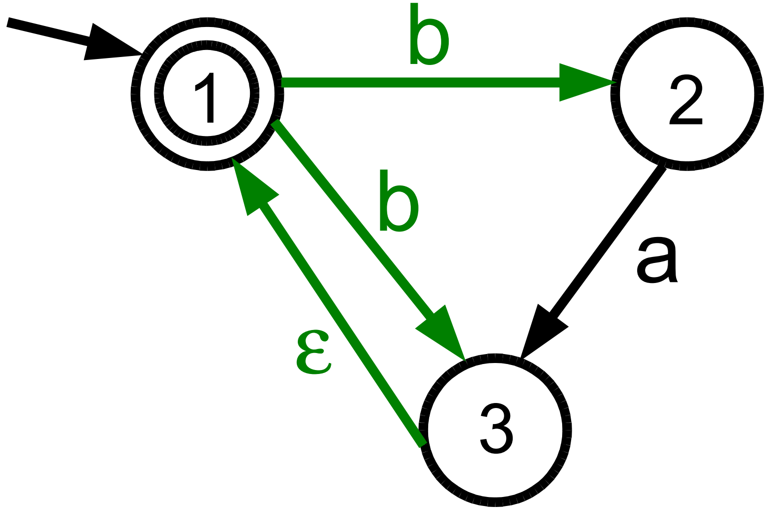
$$\delta_{DFA}(\{1,3\}, a)$$

$$= E(\delta_{NFA}(1,a) \cup \delta_{NFA}(3,a))$$

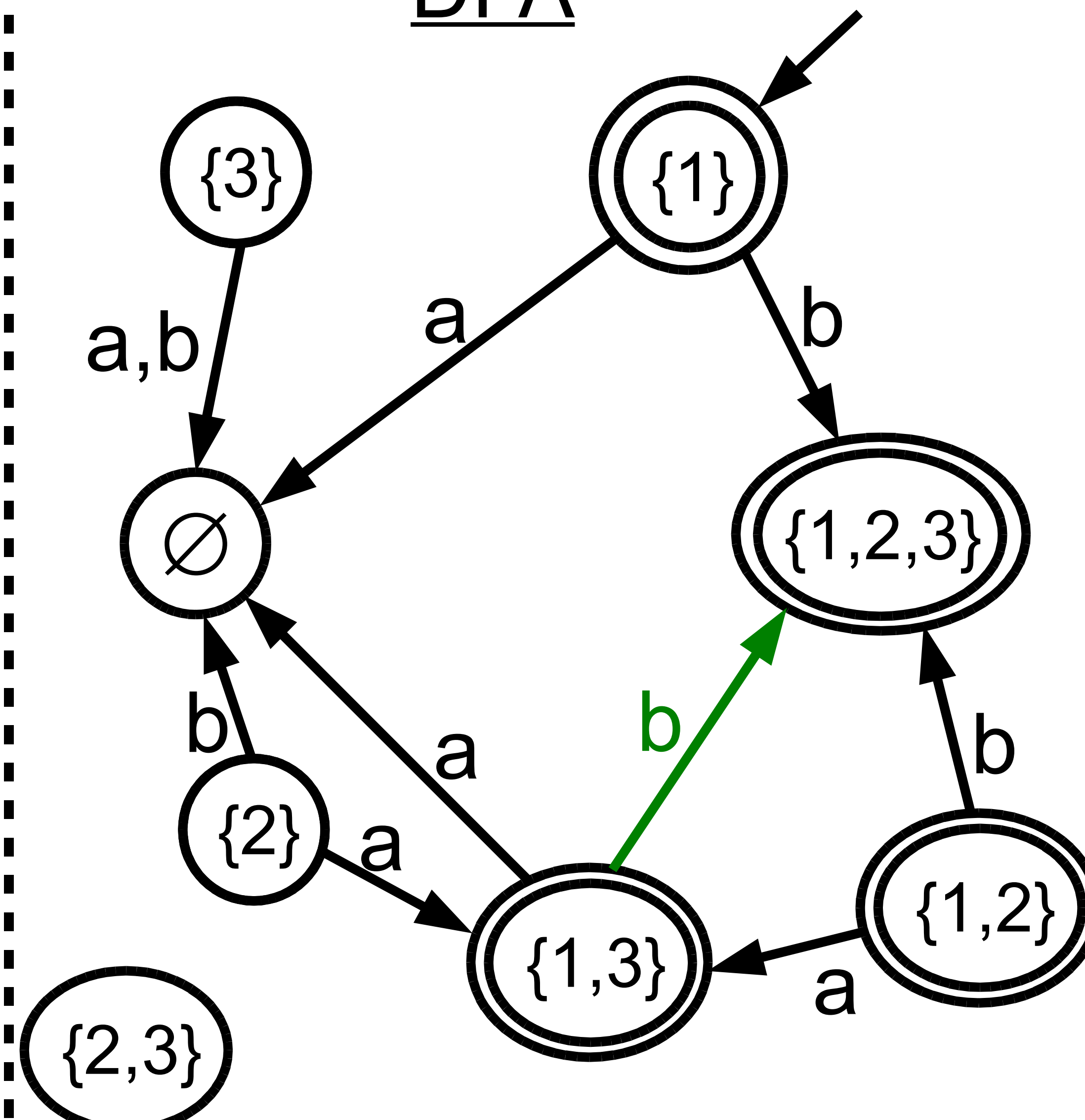
$$= E(\varnothing \cup \varnothing) = \varnothing$$

ANOTHER Example: NFA → DFA conversion

NFA



DFA



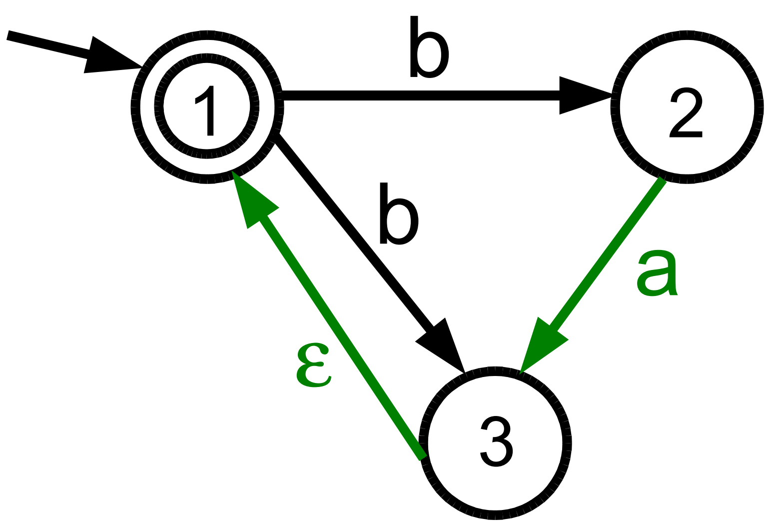
$\delta_{DFA}(\{1,3\}, b)$

$= E(\delta_{NFA}(1,b) \cup \delta_{NFA}(3,b))$

$= E(\{2,3\} \cup \varnothing) = \{1,2,3\}$

ANOTHER Example: NFA → DFA conversion

NFA

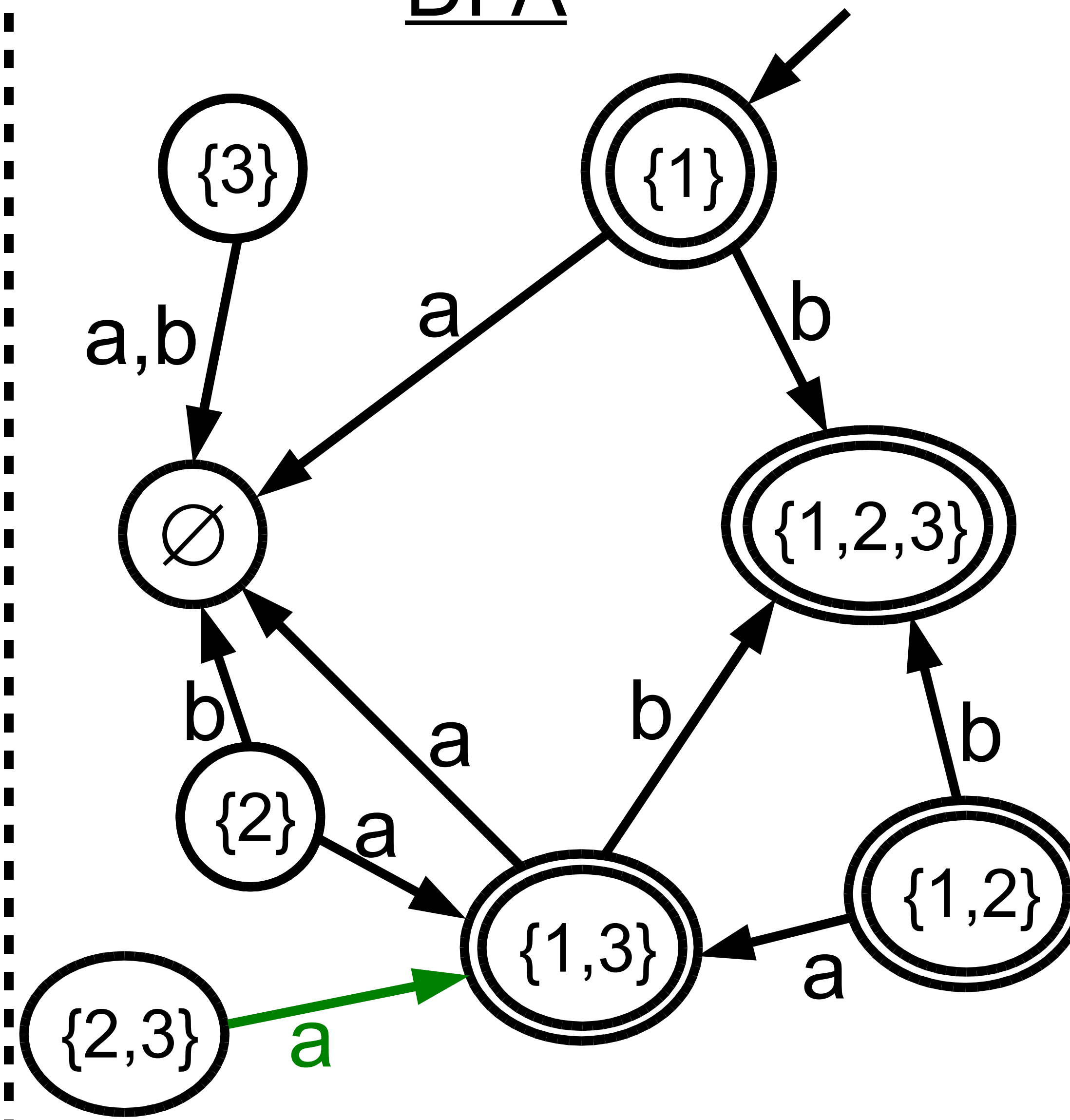


$\delta_{DFA}(\{2,3\}, a)$

$= E(\delta_{NFA}(2,a) \cup \delta_{NFA}(3,a))$

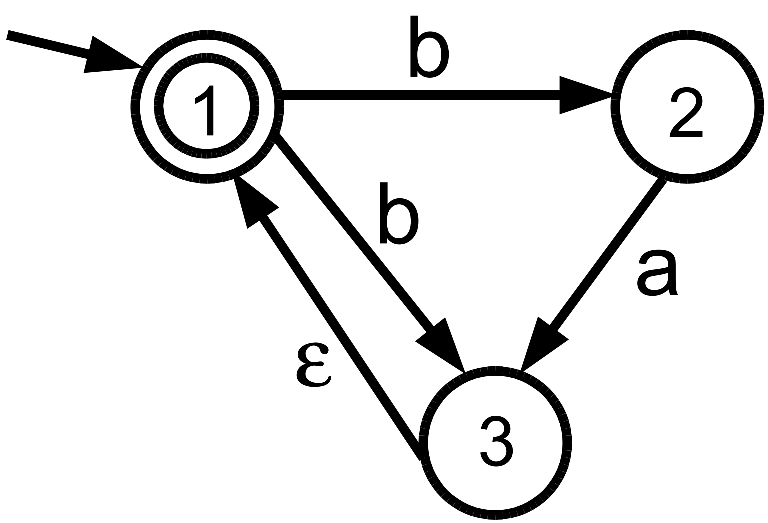
$= E(\{3\} \cup \varnothing) = \{1,3\}$

DFA

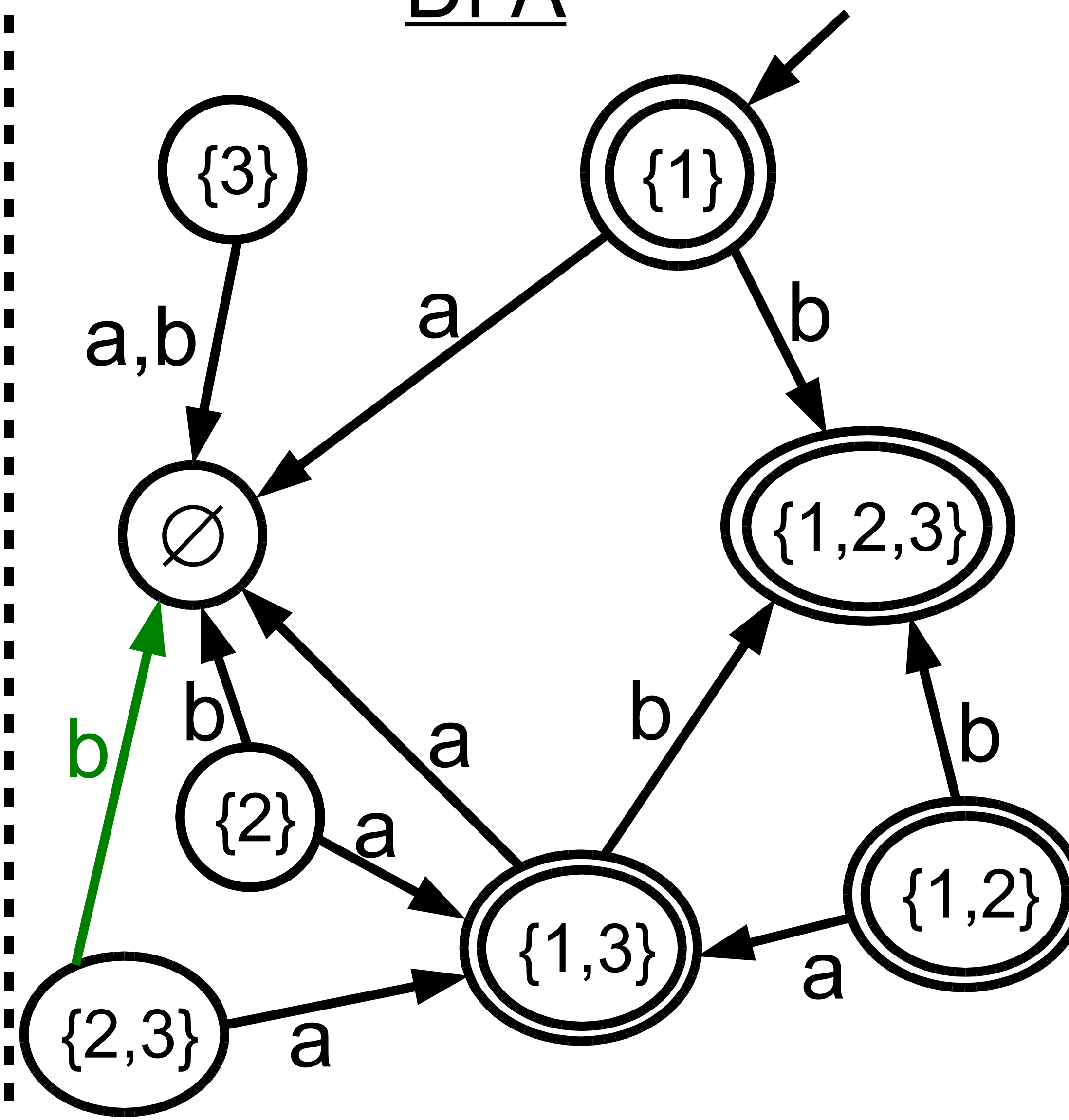


ANOTHER Example: NFA → DFA conversion

NFA



DFA



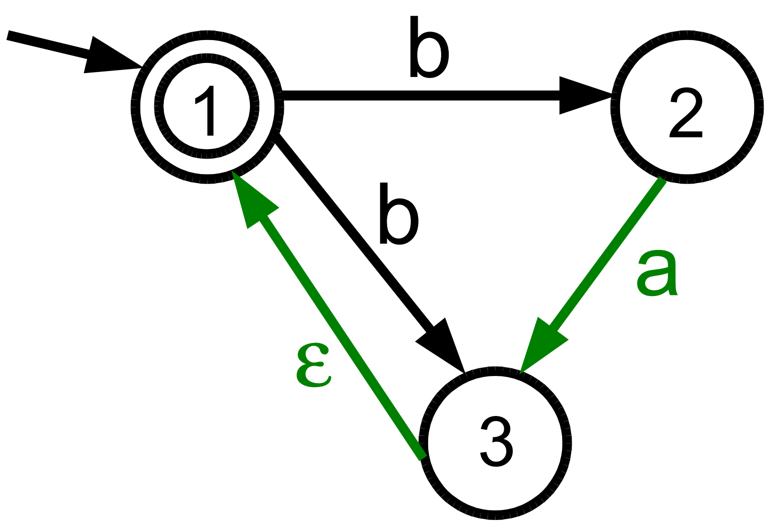
$\delta_{DFA}(\{2,3\}, b)$

$= E(\delta_{NFA}(2,b) \cup \delta_{NFA}(3,b))$

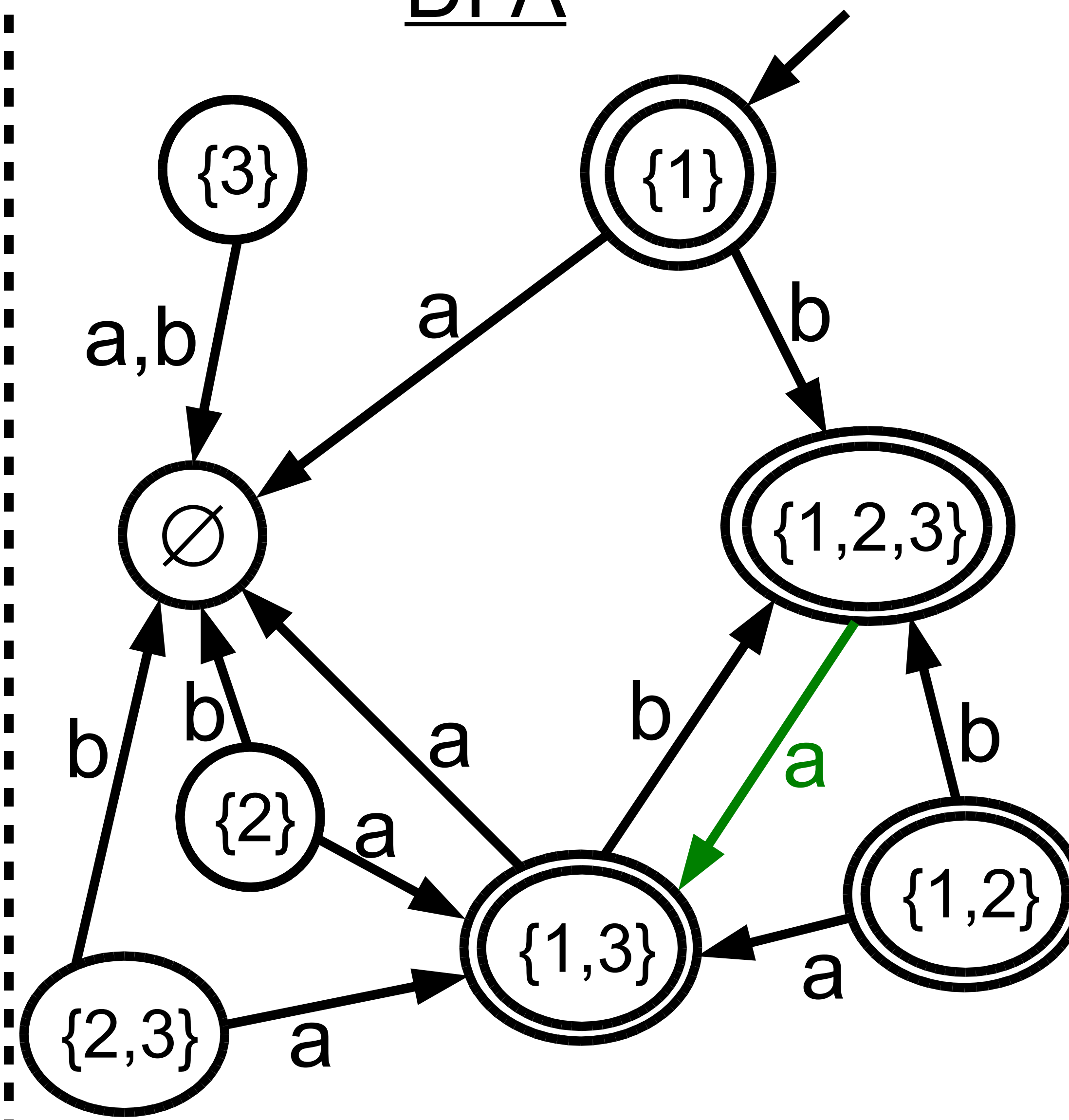
$= E(\varnothing \cup \varnothing) = \varnothing$

ANOTHER Example: NFA → DFA conversion

<u>NFA</u>



<u>DFA</u>

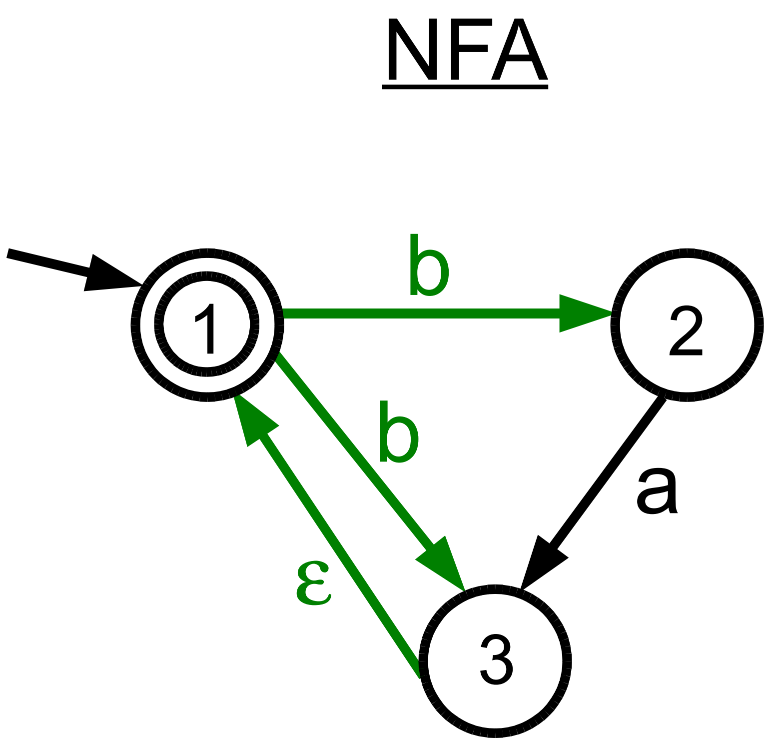


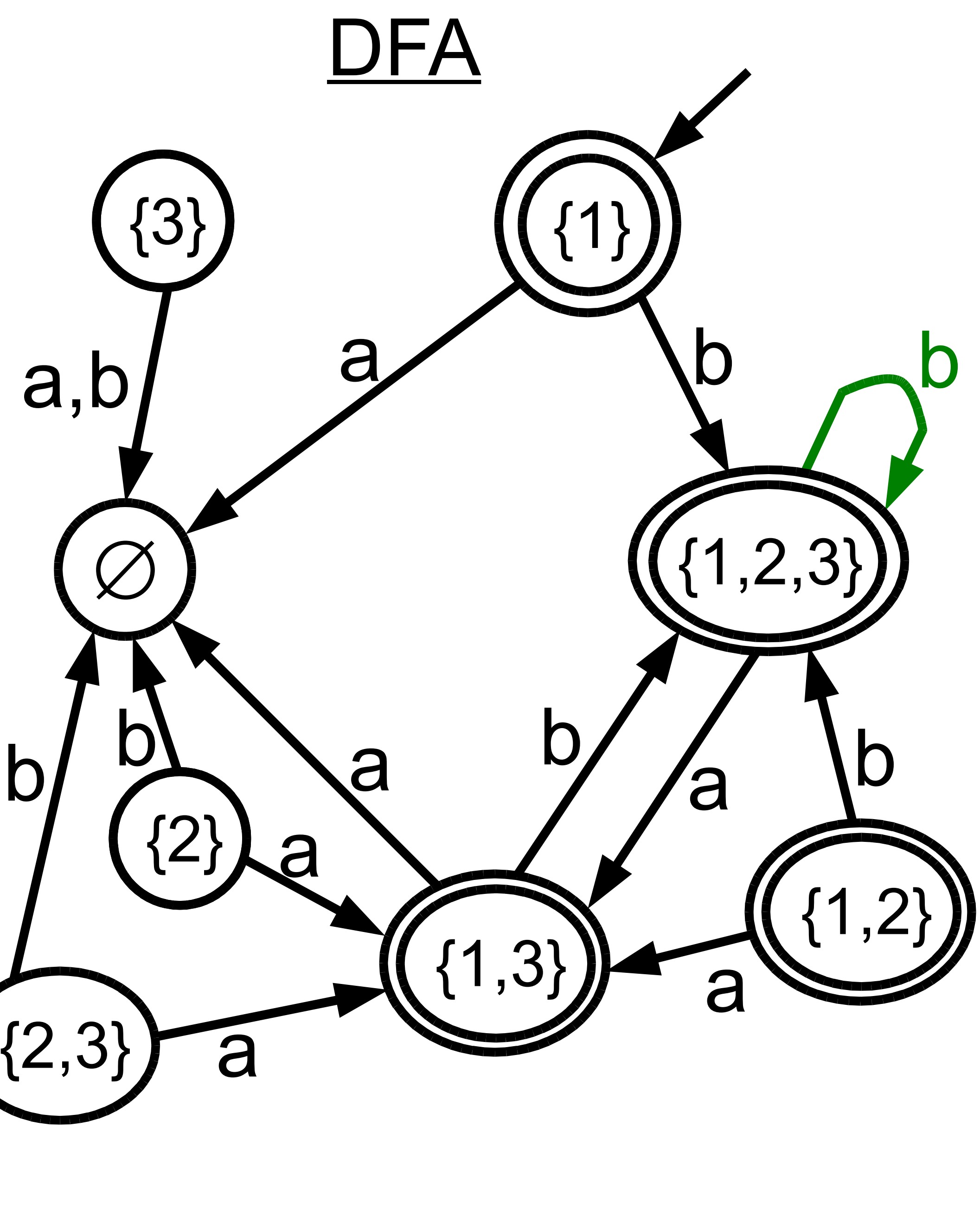
$\delta_{DFA}(\{1,2,3\}, a)$

$= E\big(\delta_{NFA}(1,a) \cup \delta_{NFA}(2,a) \cup \delta_{NFA}(3,a)\big)$

$= E(\varnothing \cup \{3\} \cup \varnothing) = \{1,3\}$

ANOTHER Example: NFA → DFA conversion



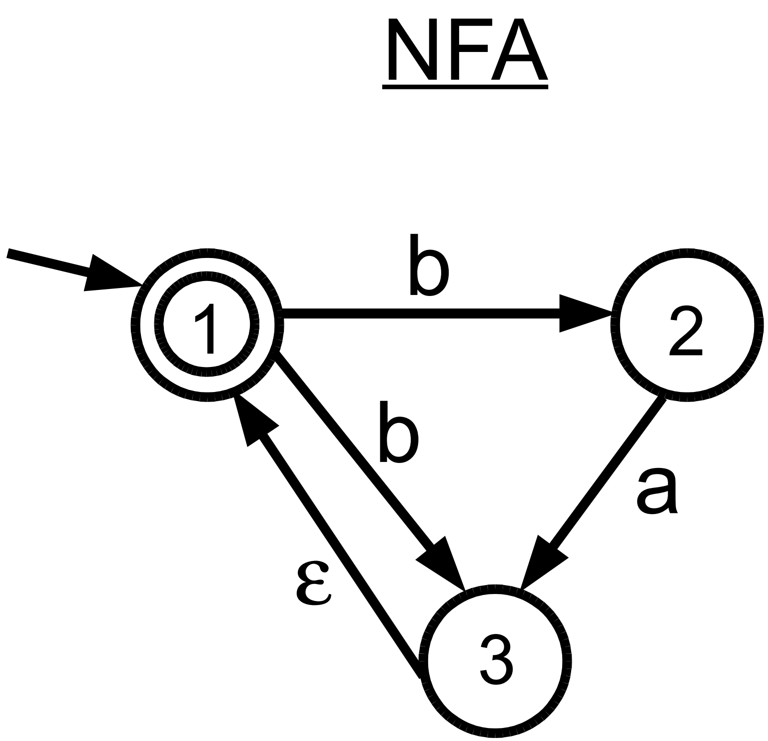


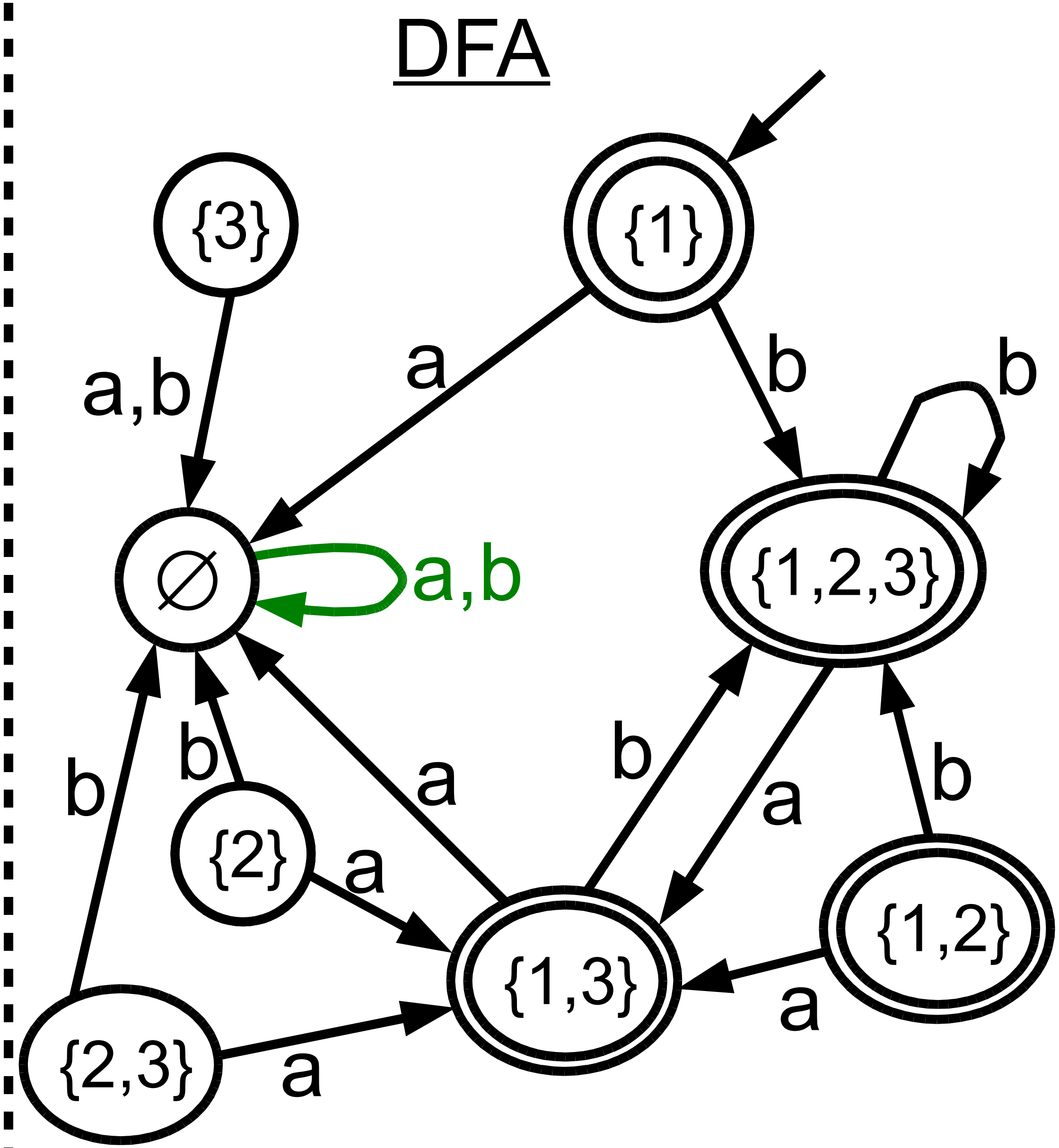
$\delta_{DFA}(\{1,2,3\}, b)$

$=E(\delta_{NFA}(1,b) \cup \delta_{NFA}(2,b) \cup \delta_{NFA}(3,b))$

$=E(\{2,3\} \cup \varnothing \cup \varnothing) = \{1,2,3\}$

ANOTHER Example: NFA → DFA conversion



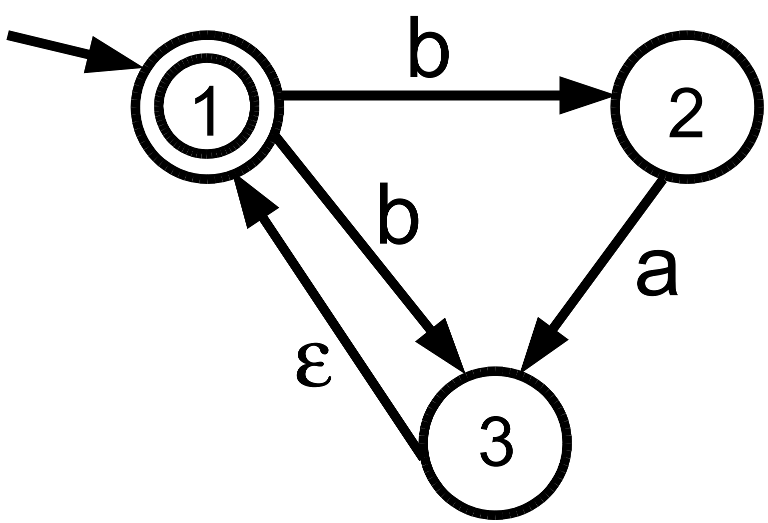


$\delta_{DFA}(\varnothing, a) = \varnothing$

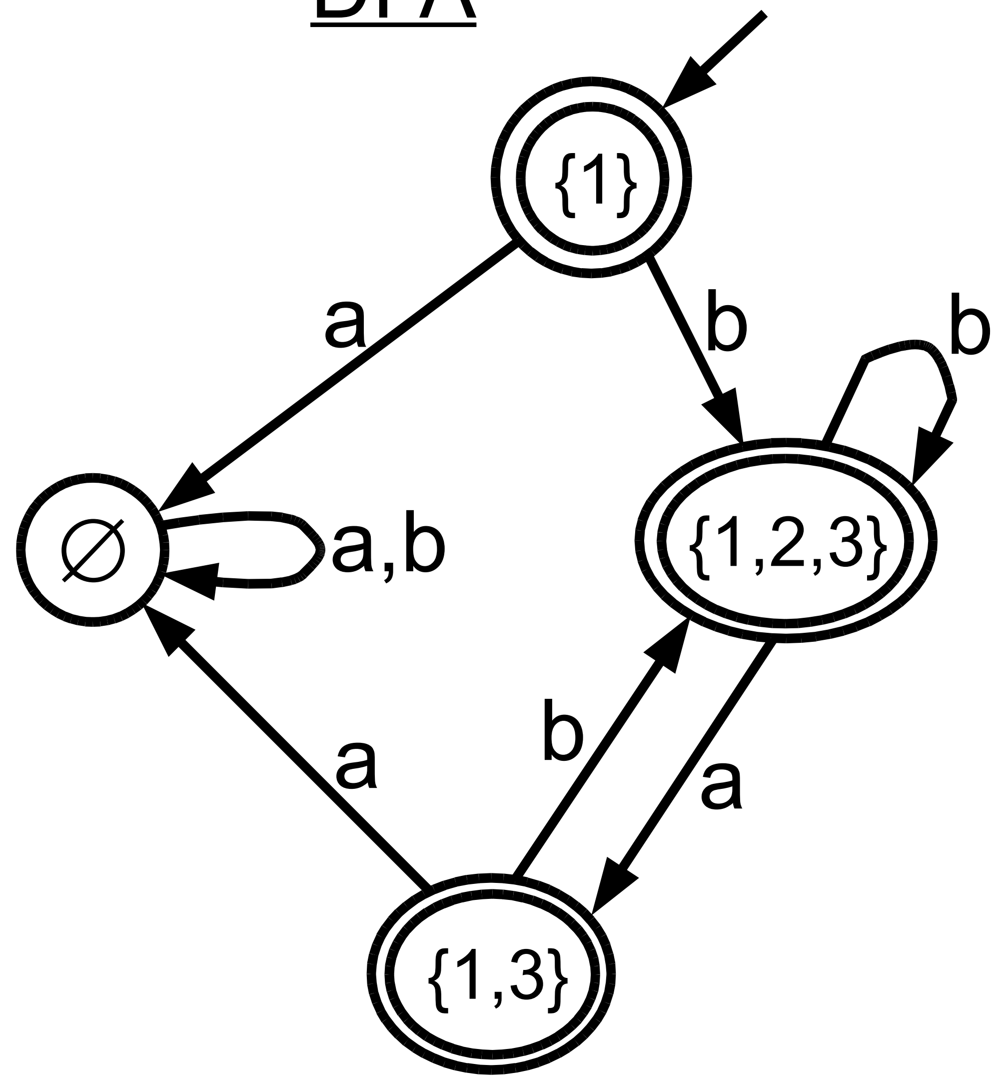
$\delta_{DFA}(\varnothing, b) = \varnothing$

ANOTHER Example: NFA → DFA conversion

NFA



DFA



We can delete the unreachable states.

Summary: NFA and DFA recognize the same languages

We now return to the question:

- Suppose A, B are regular languages, what about
- not A := { w : w is not in A } REGULAR
- A U B := { w : w in A or w in B }
- A o B := { w_1 w_2 : w_1 in A and w_2 in B }
- A* := { w_1 w_2 … w_k : $k \geq 0$, w_i in A for every i }

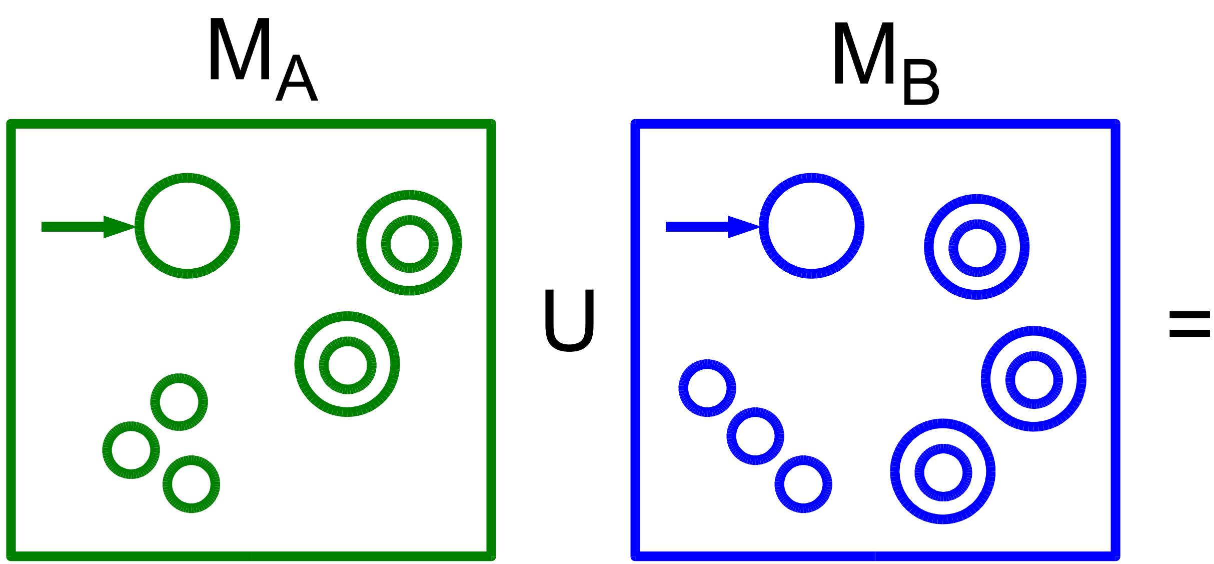
Theorem: If A, B are regular languages, then so is

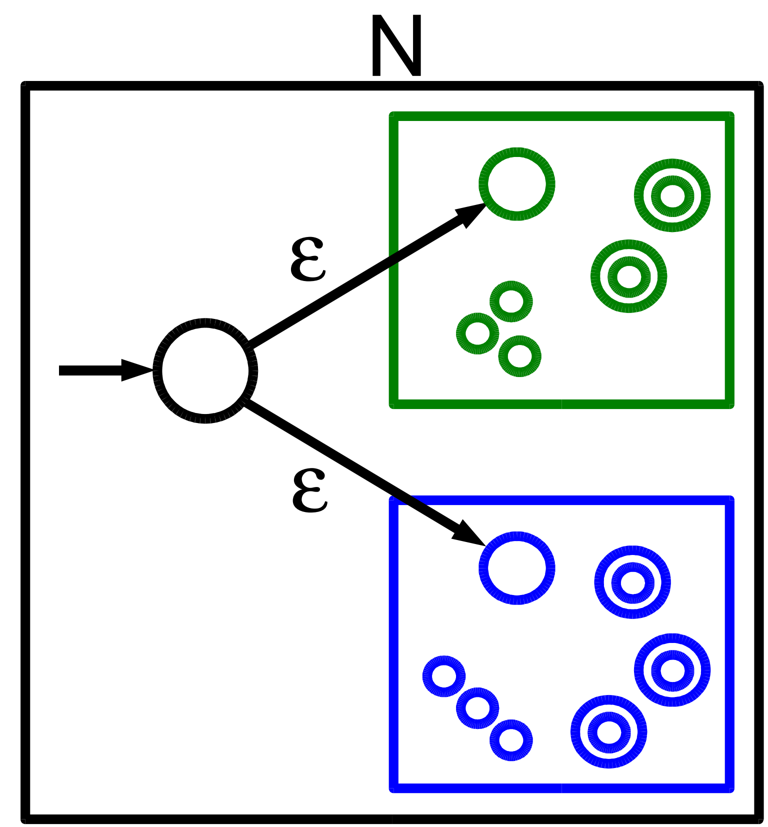
$$A \cup B := \{ w : w \text{ in } A \text{ or } w \text{ in } B \}$$

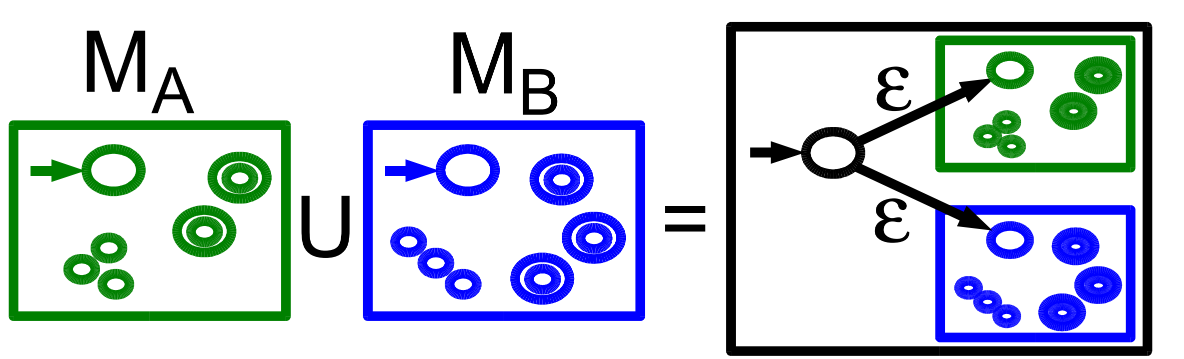
- Proof idea: Given DFA M_A : $L(M_A) = A$,

  DFA M_B : $L(M_B) = B$,

- Construct NFA N : $L(N) = A \cup B$







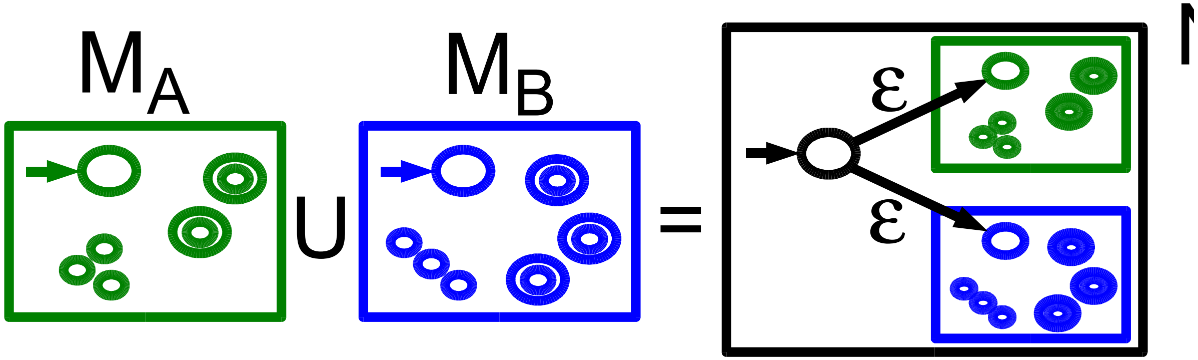
Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$: $L(M_A) = A$,

  DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$: $L(M_B) = B$,

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

- $Q :=$?

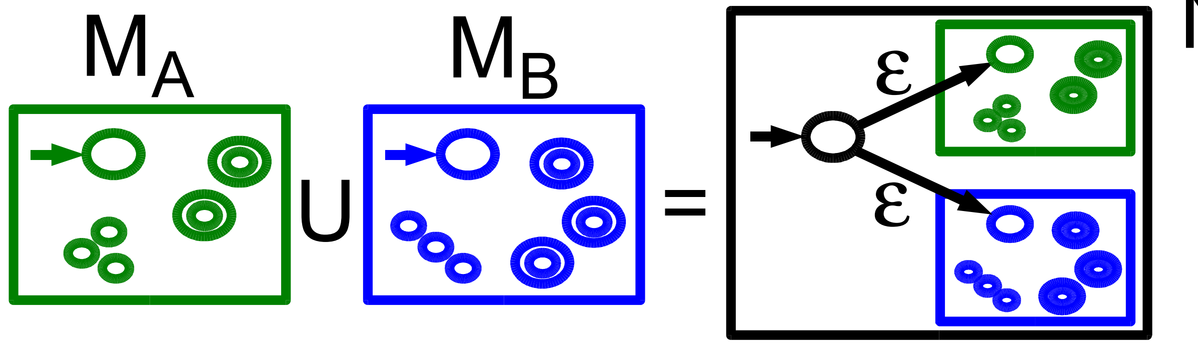


Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

  DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$,

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
- $Q := \{q\} \cup Q_A \cup Q_B$, $F :=$?



Construction:

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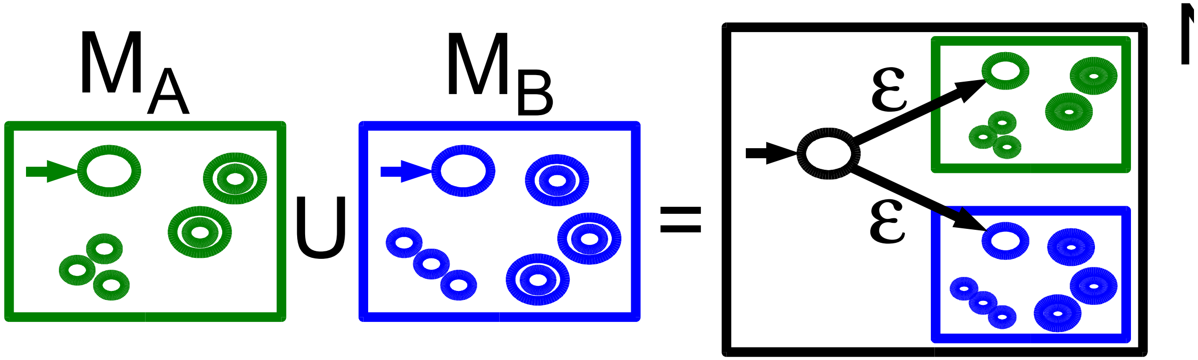
  DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B) : L(M_B) = B$,

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
- $Q := \{q\} \cup Q_A \cup Q_B$, $F := F_A \cup F_B$
- $\delta(r,x) := \{ \delta_A(r,x) \}$ if r in Q_A and $x \neq \varepsilon$
- $\delta(r,x) :=$? if r in Q_B and $x \neq \varepsilon$



Construction:

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- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
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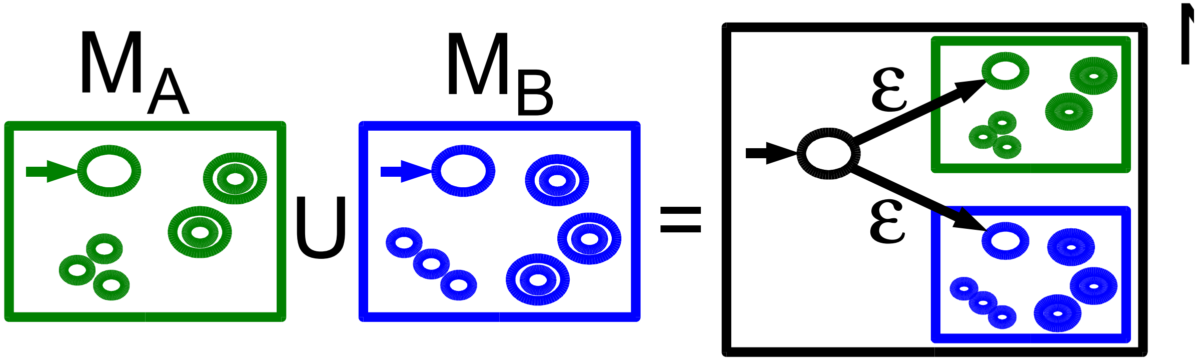
- $\delta(r,x) := \{ \delta_B(r,x) \}$ if r in Q_B and $x \neq \varepsilon$
- $\delta(q,\varepsilon) :=$?



Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$: $L(M_A) = A$,

  DFA $M_B = (Q_B, \Sigma, \delta_B, q_B, F_B)$: $L(M_B) = B$,

- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
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- $\delta(r,x) := \{ \delta_B(r,x) \}$ if r in Q_B and $x \neq \varepsilon$
- $\delta(q,\varepsilon) := \{q_A, q_B\}$
- We have $L(N) = A \cup B$

Example

Is $L = \{w \text{ in } \{0,1\}^* : |w| \text{ is divisible by 3 OR}$

$w \text{ starts with a 1}\}$ regular?

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OR is like U, so try to write $L = L_1 \cup L_2$

where L_1, L_2 are regular

Example

Is $L = \{w \text{ in } \{0,1\}^* : |w| \text{ is divisible by 3 OR}$
w starts with a $1\}$ regular?

OR is like U, so try to write $L = L_1 \cup L_2$
where L_1, L_2 are regular

$L_1 = \{w : |w| \text{ is div. by } 3\}$ $L_2 = \{w : w \text{ starts with a } 1\}$

Example

Is $L = \{w \text{ in } \{0,1\}^* : |w| \text{ is divisible by 3 OR}$
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OR is like U, so try to write $L = L_1 \text{ U } L_2$
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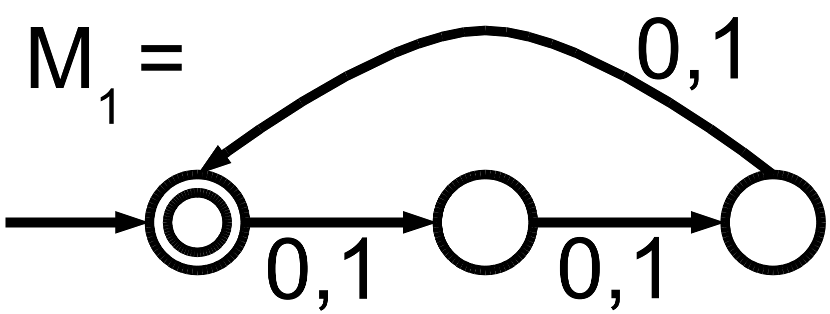
$L_1 = \{w : |w| \text{ is div. by } 3\}$ $L_2 = \{w : w \text{ starts with a } 1\}$

$M_1 =$



$L(M_1) = L_1$

Example

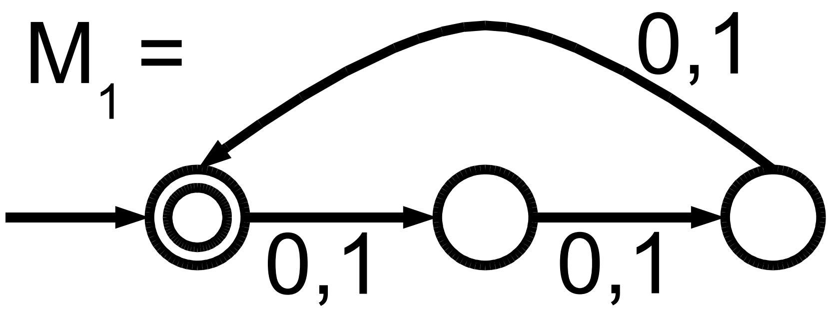
Is $L = \{w \text{ in } \{0,1\}^* : |w| \text{ is divisible by 3 OR}$
$w \text{ starts with a 1}\}$ regular?

OR is like U, so try to write $L = L_1 \cup L_2$

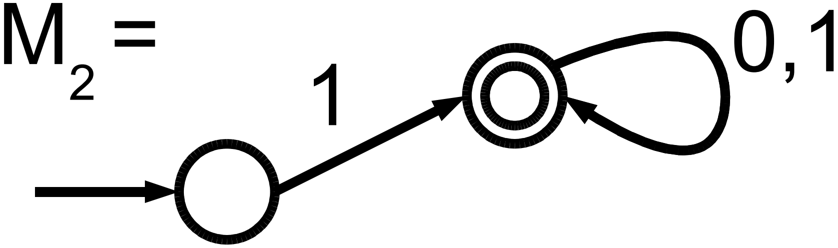
where L_1, L_2 are regular

$L_1 = \{w : |w| \text{ is div. by 3}\}$ $L_2 = \{w : w \text{ starts with a 1}\}$

$M_1 =$



$M_2 =$



$L(M_1) = L_1$ $L(M_2) = L_2$

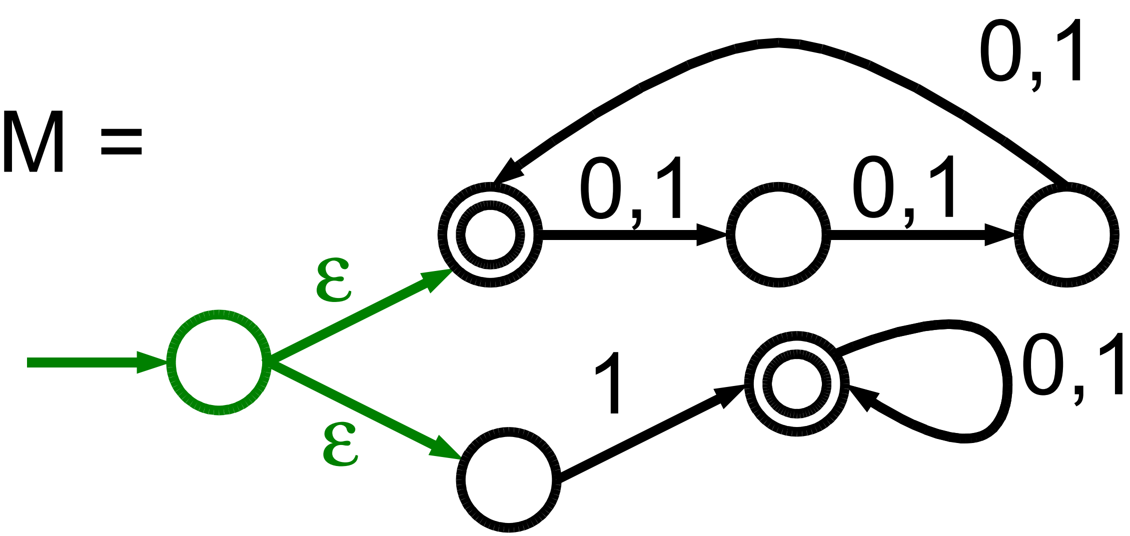
Example

Is $L = \{w \text{ in } \{0,1\}^* : |w| \text{ is divisible by 3 OR}$
$w \text{ starts with a } 1\}$ regular?

OR is like U, so try to write $L = L_1 \cup L_2$
where L_1, L_2 are regular

$L_1 = \{w : |w| \text{ is div. by } 3\}$ $L_2 = \{w : w \text{ starts with a } 1\}$

M =



$L(M) = L(M_1) \cup L(M_2)$

$= L_1 \cup L_2$

$= L$

$\Rightarrow$ L is regular.

We now return to the question:

- Suppose A, B are regular languages, then
- not A := { w : w is not in A } REGULAR
- A U B := { w : w in A or w in B } REGULAR
- A o B := { w_1 w_2 : w_1 in A and w_2 in B }
- A* := { w_1 w_2 … w_k : $k \geq 0$, w_i in A for every i }

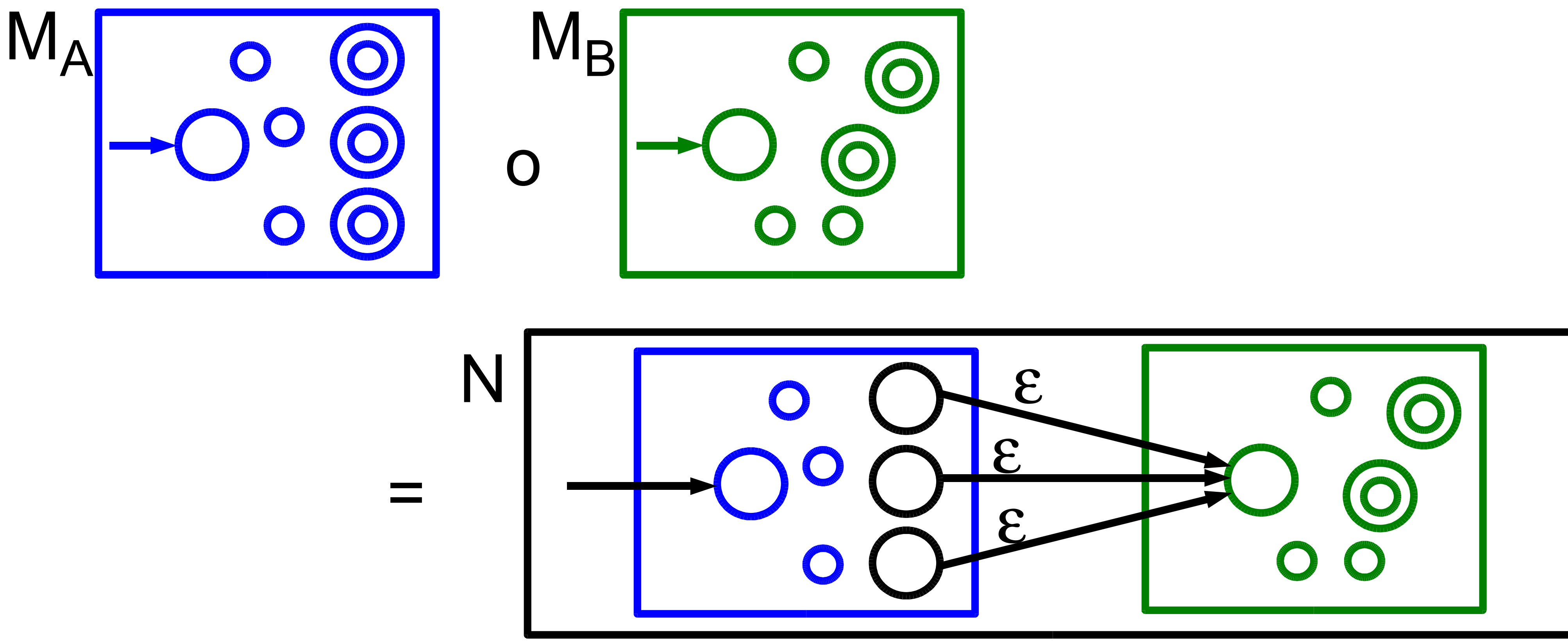
Theorem: If A, B are regular languages, then so is

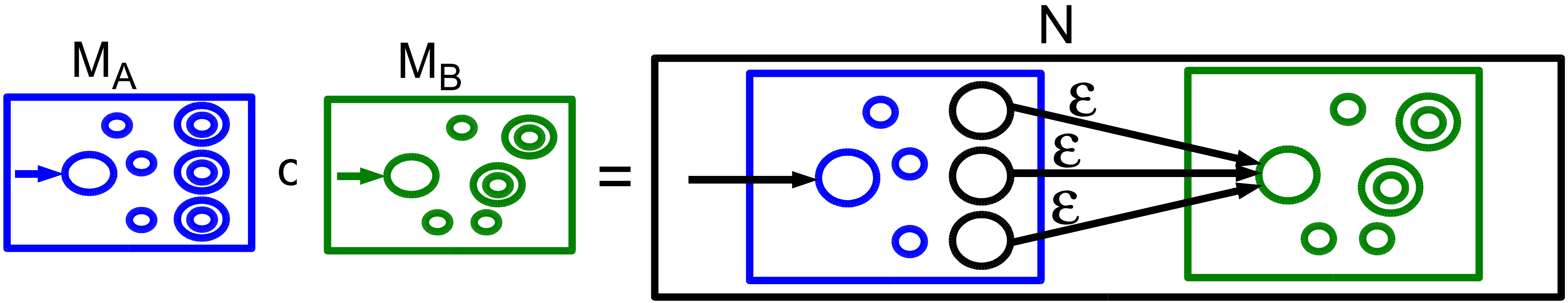
A o B := { w : w = xy for some

x in A and y in B }.

- Proof idea: Given DFAs M_A, M_B for A, B

construct NFA N : L(N) = A o B.





Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

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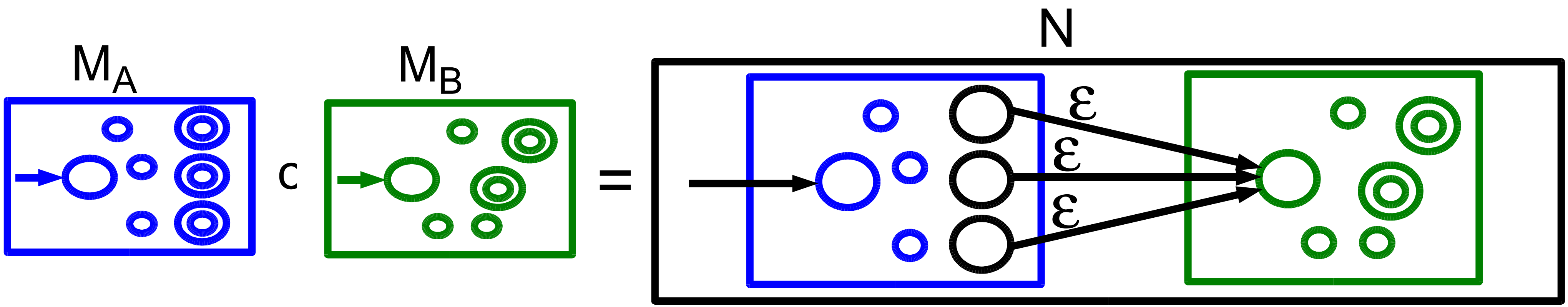

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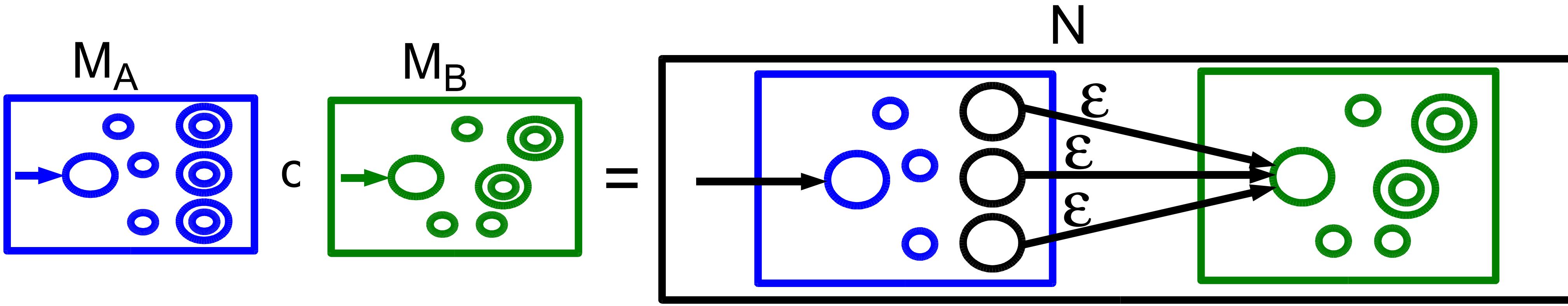

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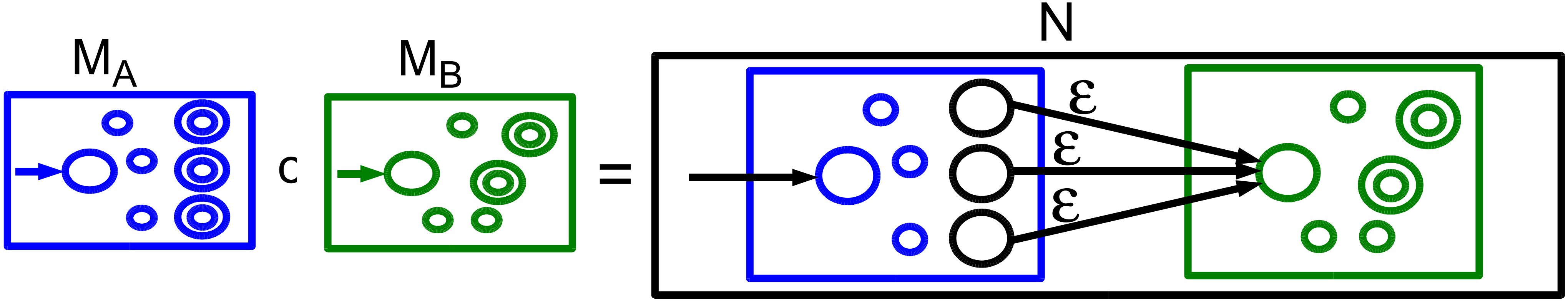

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- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
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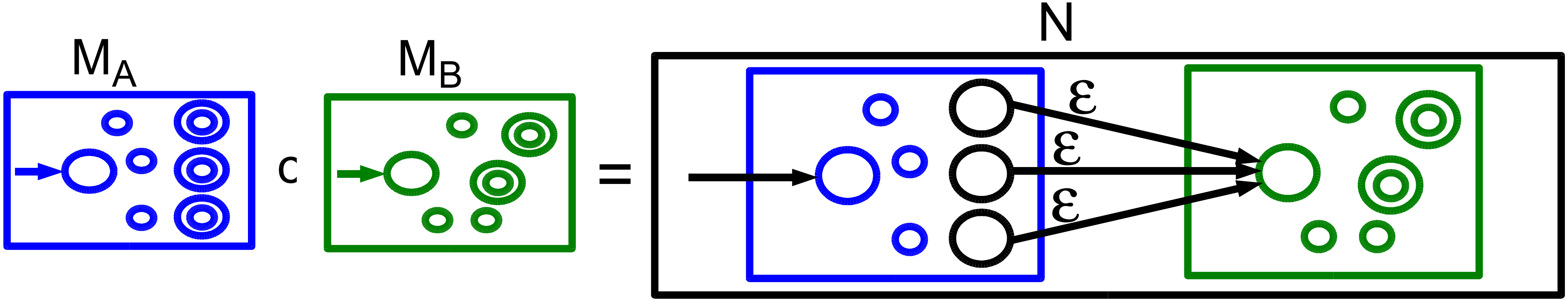

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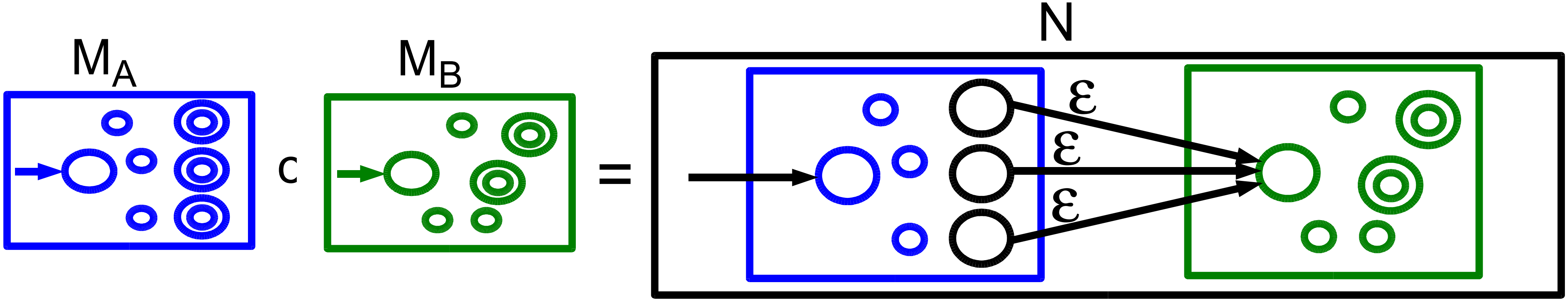

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- Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:
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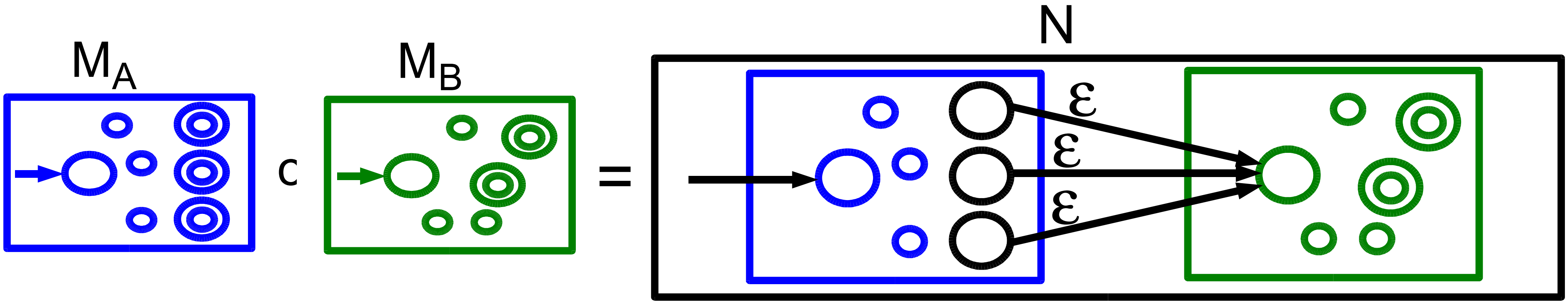

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- $\delta(r,\varepsilon) := \{ q_B \}$ if r in F_A
- $\delta(r,x) := \{ \delta_B(r,x) \}$ if r in Q_B and $x \neq \varepsilon$
- We have $L(N) = A \circ B$

Example

Is L = {w in {0,1}* : w contains a 1 after a 0}

regular?

Note: L = {01, 0001001, 111001, … }

Example

Is $L = \{w \text{ in } \{0,1\}^* : w \text{ contains a 1 after a 0}\}$

regular?

Let $L_0 = \{w : w \text{ contains a } 0\}$

$L_1 = \{w : w \text{ contains a } 1\}$. Then $L = L_0 \circ L_1$.

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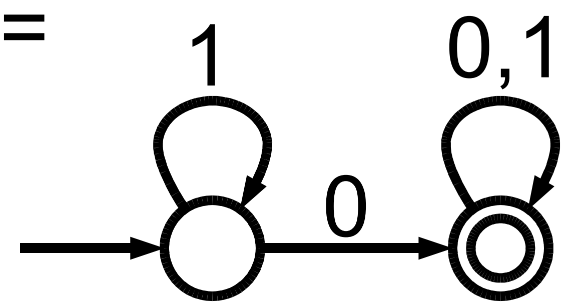
$M_0 =$



$L(M_0) = L_0$

Example

Is $L = \{w \text{ in } \{0,1\}^* : w \text{ contains a 1 after a 0}\}$

regular?

Let $L_0 = \{w : w \text{ contains a } 0\}$

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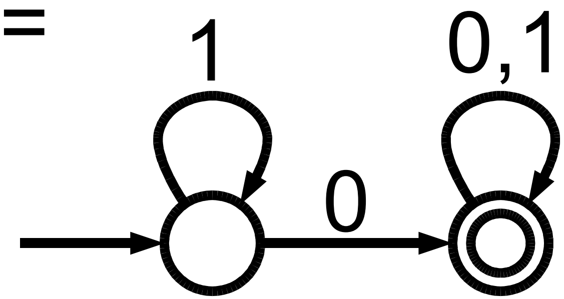
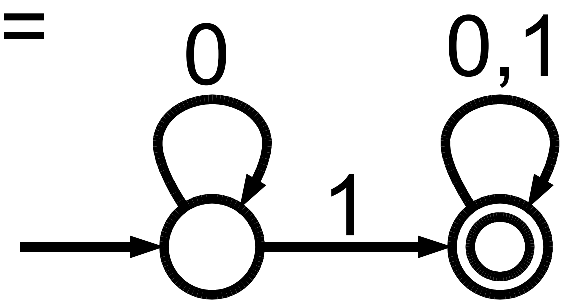


$L(M_0) = L_0$

$M_1 =$



$L(M_1) = L_1$

Example

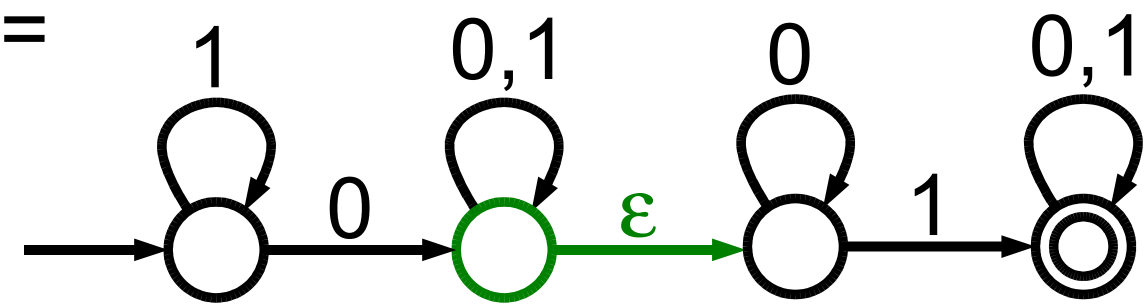
Is $L = \{w \text{ in } \{0,1\}^* : w \text{ contains a 1 after a 0}\}$

regular?

Let $L_0 = \{w : w \text{ contains a } 0\}$

$L_1 = \{w : w \text{ contains a } 1\}$. Then $L = L_0 \circ L_1$.

M =



$$L(M) = L(M_0) \circ L(M_1) = L_0 \circ L_1 = L$$

⇒ L is regular.

We now return to the question:

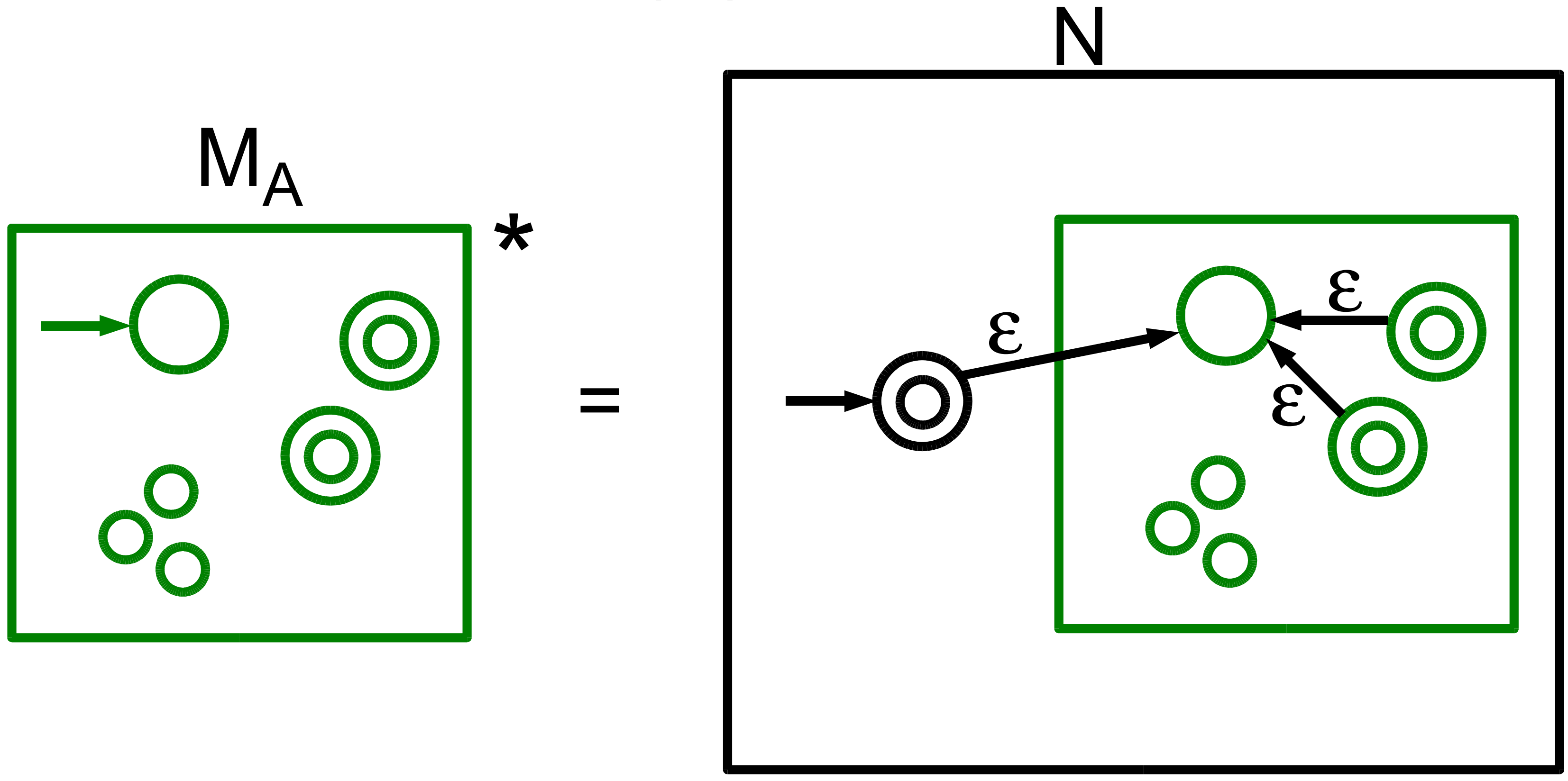
- Suppose A, B are regular languages, then
- not A := { w : w is not in A } REGULAR
- A U B := { w : w in A or w in B } REGULAR
- A o B := { w_1 w_2 : $w_1 \in$ A and $w_2 \in$ B } REGULAR
- A* := { w_1 w_2 … w_k : $k \geq 0$, w_i in A for every i }

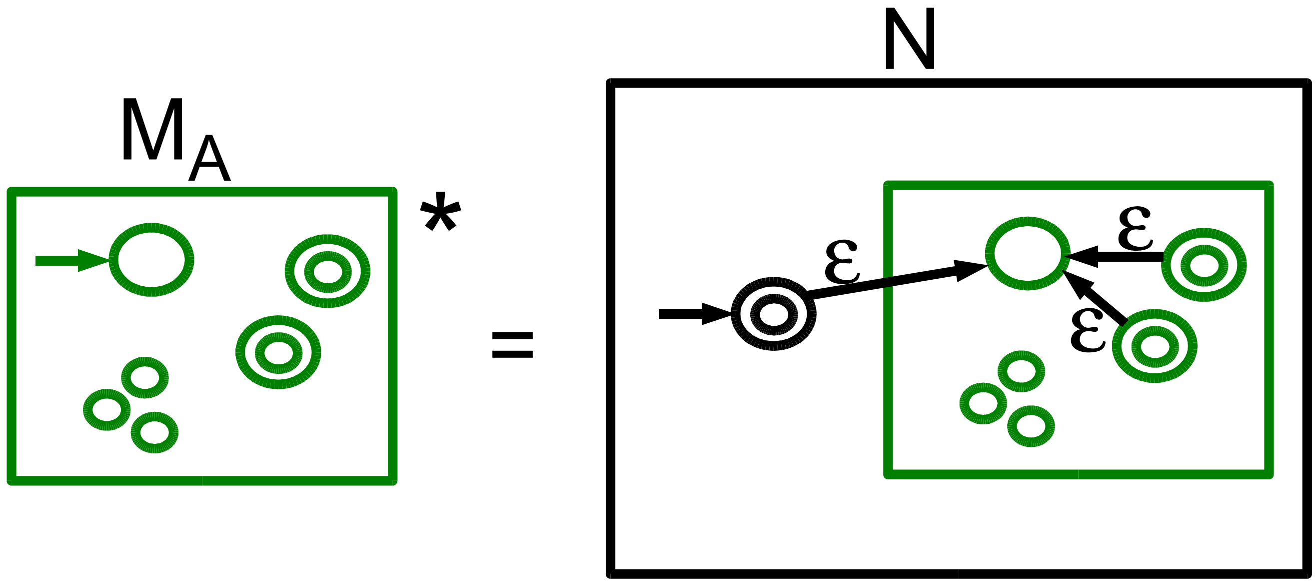
Theorem: If A is a regular language, then so is

$A^* := \{ w : w = w_1 \ldots w_k,\ w_i \text{ in } A \text{ for } i=1,\ldots,k \}$

- Proof idea: Given DFA $M_A : L(M_A) = A$,

Construct NFA $N : L(N) = A^*$



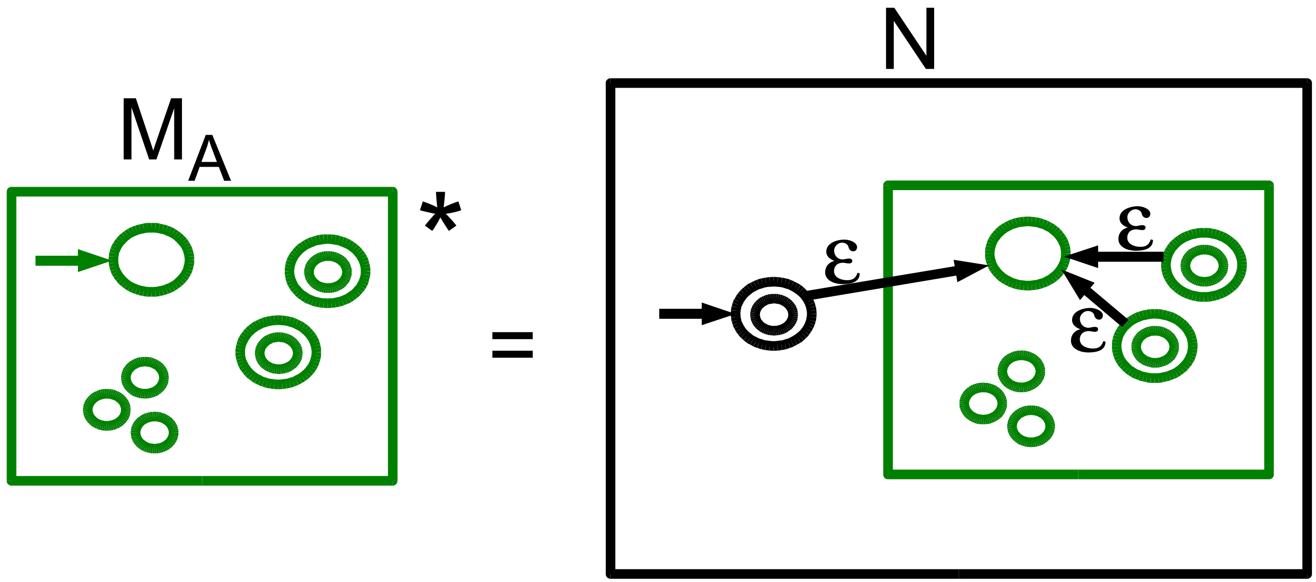


Construction:

• Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$: $L(M_A) = A$,

Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

• $Q :=$?

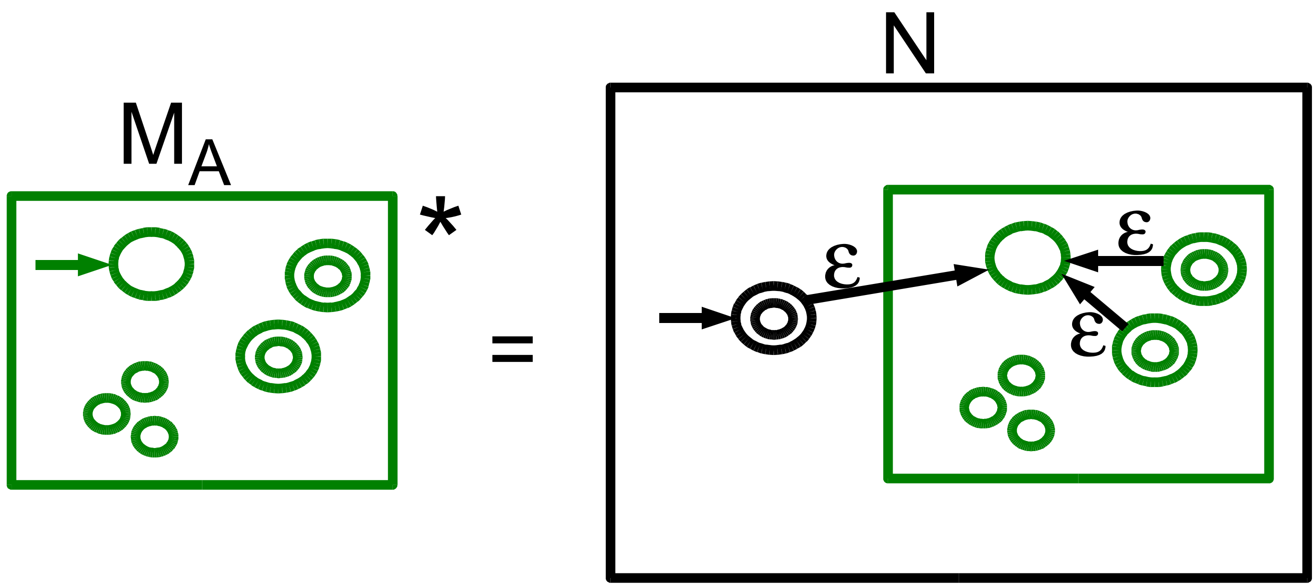


Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

- $Q := \{q\} \cup Q_A$, $F :=$?

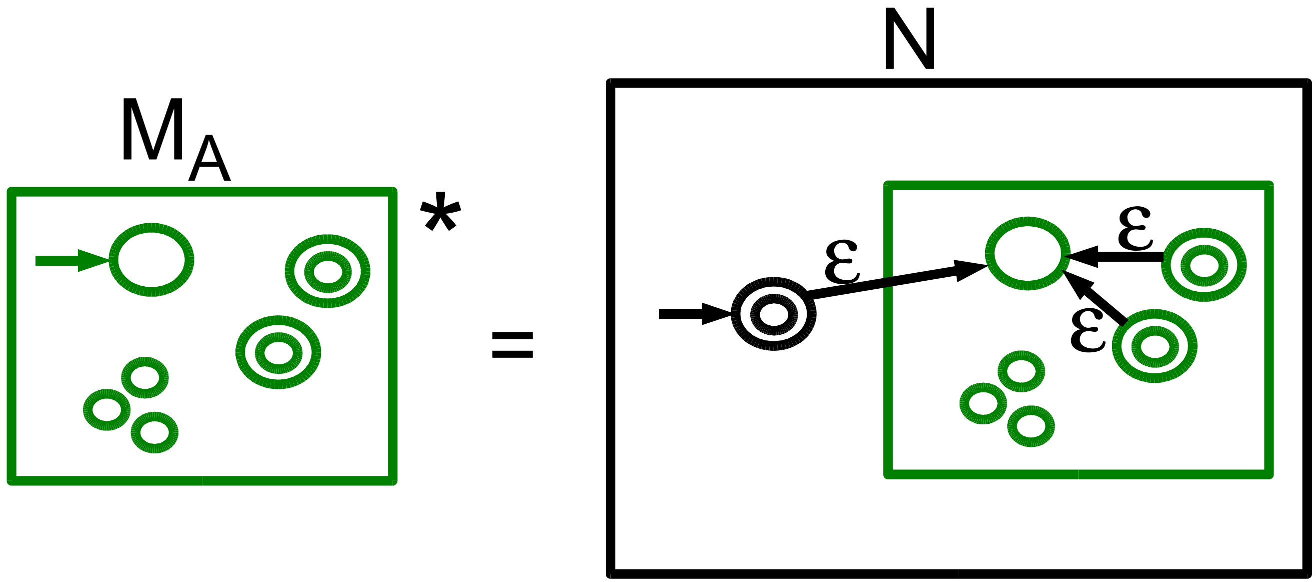


Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A) : L(M_A) = A$,

Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

- $Q := \{q\} \cup Q_A$, $F := \{q\} \cup F_A$
- $\delta(r,x) :=$? if r in Q_A and $x \neq \varepsilon$

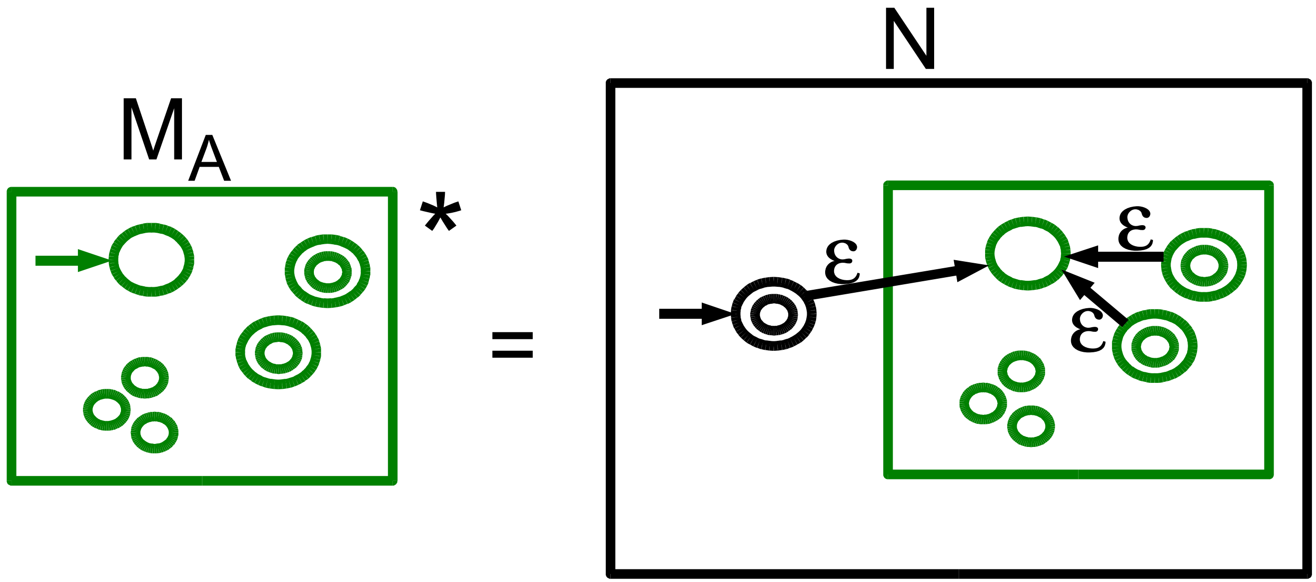


Construction:

- Given DFA $M_A = (Q_A, \Sigma, \delta_A, q_A, F_A)$: $L(M_A) = A$,

Construct NFA $N = (Q, \Sigma, \delta, q, F)$ where:

- $Q := \{q\} \cup Q_A$, $F := \{q\} \cup F_A$
- $\delta(r,x) := \{ \delta_A(r,x) \}$ if r in Q_A and $x \neq \varepsilon$
- $\delta(r,\varepsilon) :=$? if r in $\{q\} \cup F_A$



Construction:

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- $\delta(r,\varepsilon) := \{ q_A \}$ if r in $\{q\} \cup F_A$
- We have $L(N) = A^*$

Example

Is $L = \{w \text{ in } \{0,1\}^* : w \text{ has even length}\}$

regular?

Example

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regular?

Let $L_0 = \{w : w \text{ has length} = 2\}$. Then $L = L_0{}^*$.

Example

Is $L = \{w \text{ in } \{0,1\}^* : w \text{ has even length}\}$

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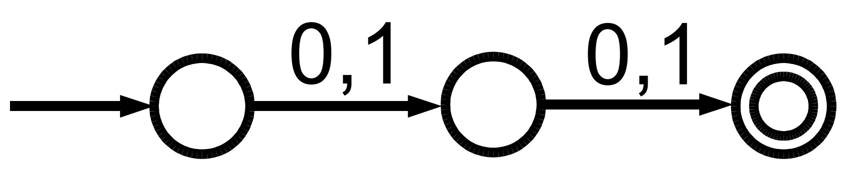
Let $L_0 = \{w : w \text{ has length} = 2\}$. Then $L = L_0^*$.

$M_0 =$



$L(M_0) = L_0$

Example

Is $L = \{w \text{ in } \{0,1\}^* : w \text{ has even length}\}$

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Let $L_0 = \{w : w \text{ has length} = 2\}$. Then $L = L_0{}^*$.

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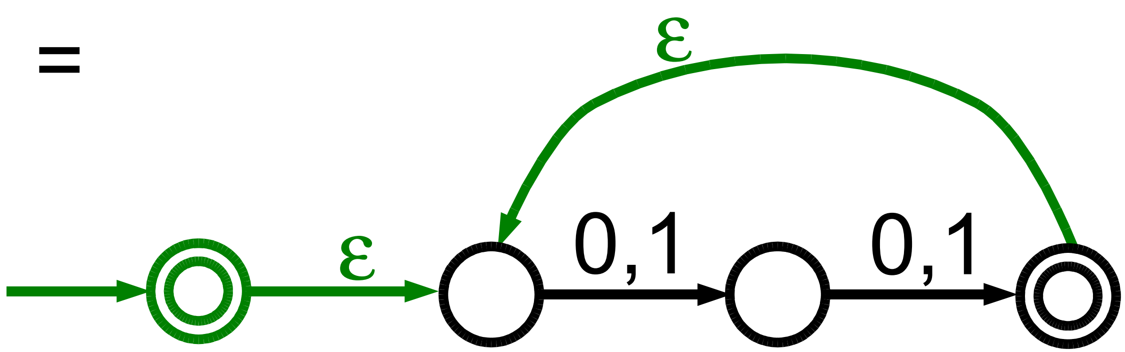

$$L(M) = L(M_0)^* = L_0{}^* = L$$

$\Rightarrow$ L is regular.

We now return to the question:

- Suppose A, B are regular languages, then
- not A := { w : w is not in A }
- A U B := { w : w in A or w in B }
- A o B := { w_1 w_2 : w_1 in A and w_2 in B }
- A* := { w_1 w_2 … w_k : $k \geq 0$, w_i in A for every i }

are all regular!

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What about $A \cap B$:= { w : w in A and w in B } ?

We now return to the question:

- Suppose A, B are regular languages, then
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- A* := { w_1 w_2 … w_k : $k \geq 0$, w_i in A for every i }

De Morgan's laws: A ∩ B = not ((not A) U (not B))

By above, (not A) is regular, (not B) is regular,

(not A) U (not B) is regular,

not ((not A) U (not B)) = A ∩ B regular

We now return to the question:

- Suppose A, B are regular languages, then
- not A := { w : w is not in A }
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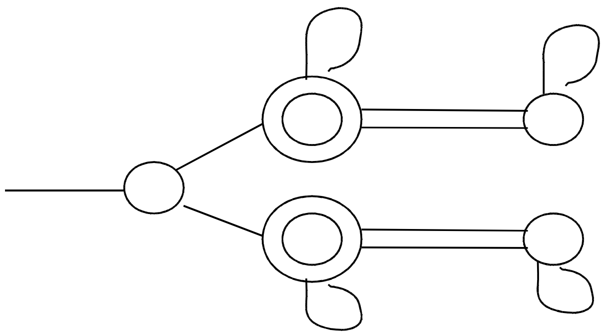
- A ∩ B := { w : w in A and w in B }

are all regular

How to specify a regular language?

Write a picture → complicated



Write down formal definition → complicated

$\delta(q_0, 0) = q_0, \ldots$

Use symbols from Σ and operations *, o, U → good

({0} * U {1}) o {001}

Regular expressions: anything you can write with

$\emptyset$, ε , symbols from Σ, and operations *, o, U

Conventions:

- Write a instead of {a}
- Write AB for A o B
- Write ∑ for $U_{a \in \sum}$ a So if ∑ = {a,b} then ∑ = a U b
- Operation * has precedence over o, and o over U

so 1 U 01* means 1U(0(1)*)

Example: 110, 0*, Σ*, Σ*001Σ*, (ΣΣ)*, 01 U 10

Definition Regular expressions RE over Σ are:

$\varnothing$	
ε	
a	if a in Σ
R R'	if R, R' are RE
R U R'	if R, R' are RE
R*	if R is RE

Definition The language described by RE:

$L(\emptyset) = \emptyset$

$L(\varepsilon) = \{\varepsilon\}$

$L(a) = \{a\}$ if a in Σ

$L(R\ R') = L(R) \circ L(R')$

$L(R \cup R') = L(R) \cup L(R')$

$L(R^*) = L(R)^*$

Example $\Sigma = \{ a, b\}$

RE	Language
• ab U ba	?
• a^*	
• $(a \cup b)^*$	
• a^*ba^*	
• $\Sigma^*b\Sigma^*$	
• $\Sigma^*aab\Sigma^*$	
• $(\Sigma\Sigma)^*$	
• $(a^*ba^*ba^*)^*$	
• $a^*baba^*a\emptyset$	

Example $\Sigma = \{ a, b\}$

RE	Language
• ab U ba	{ab, ba}
• a*	
• (a U b)*	
• a*ba*	
• $\Sigma^*b\Sigma^*$	
• $\Sigma^*aab\Sigma^*$	
• $(\Sigma\Sigma)^*$	
• (a*ba*ba*)*	
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Example $\Sigma = \{ a, b\}$

RE Language

- ab U ba $\{ab, ba\}$
- a* $\{\varepsilon, a, aa, \dots \} = \{ w : w \text{ has only } a\}$
- (a U b)*
- a*ba*
- $\Sigma^*b\Sigma^*$
- $\Sigma^*aab\Sigma^*$
- $(\Sigma\Sigma)^*$
- (a*ba*ba*)*
- a*baba*a$\emptyset$

Example $\Sigma = \{ a, b\}$

RE	Language
• ab U ba	{ab, ba}
• a*	$\{\varepsilon$, a, aa, … } = { w : w has only a}
• (a U b)*	all strings
• a*ba*	
• $\Sigma^*b\Sigma^*$	
• $\Sigma^*aab\Sigma^*$	
• $(\Sigma\Sigma)^*$	
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Example $\Sigma = \{ a, b\}$

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• ab U ba	{ab, ba}
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• (a U b)*	all strings
• a*ba*	{w : w has exactly one b}
• $\Sigma^*b\Sigma^*$	
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Example $\Sigma = \{ a, b\}$

RE	Language
• ab U ba	{ab, ba}
• a^*	$\{\varepsilon, a, aa, \ldots \} = \{ w : w$ has only $a\}$
• $(a \cup b)^*$	all strings
• a^*ba^*	$\{w : w$ has exactly one $b\}$
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Example $\Sigma = \{ a, b\}$

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• a*	$\{\varepsilon$, a, aa, … } = { w : w has only a}
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• $\Sigma^*b\Sigma^*$	{w : w has at least one b}
• $\Sigma^*aab\Sigma^*$	{w : w contains the string aab}
• $(\Sigma\Sigma)^*$	
• (a*ba*ba*)*	
• a*baba*a$\emptyset$	

Example $\Sigma = \{ a, b\}$

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Example $\Sigma = \{ a, b\}$

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Example $\Sigma = \{ a, b\}$

RE	Language
$ab \cup ba$	$\{ab, ba\}$
a^*	$\{\varepsilon, a, aa, \ldots \} = \{ w : w \text{ has only } a\}$
$(a \cup b)^*$	all strings
a^*ba^*	$\{w : w$ has exactly one $b\}$
$\Sigma^*b\Sigma^*$	$\{w : w$ has at least one $b\}$
$\Sigma^*aab\Sigma^*$	$\{w : w$ contains the string $aab\}$
$(\Sigma\Sigma)^*$	$\{w : w$ has even length$\}$
$(a^*ba^*ba^*)^*$	$\{w : w$ contains even number of $b\}$
$a^*baba^*a\emptyset$	$\emptyset$ (anything $\circ\, \emptyset = \emptyset$)

Theorem: For every RE R there is NFA M: L(M) = L(R)

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Construction:

- R = $\emptyset$ M := ?

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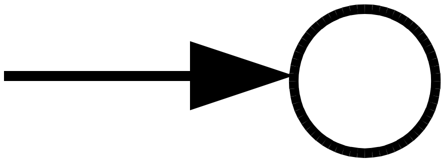
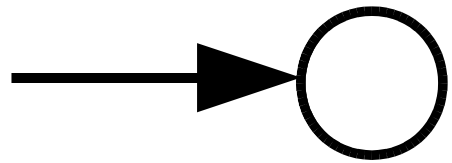
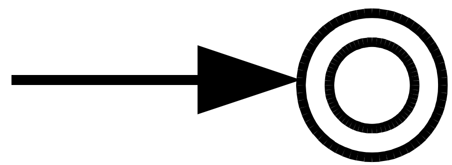
- R = $\emptyset$ M := 

- R = ε M := ?

Theorem: For every RE R there is NFA M: L(M) = L(R)

Construction:

- R = $\varnothing$ M := 

- R = ε M := 

- R = a M := ?

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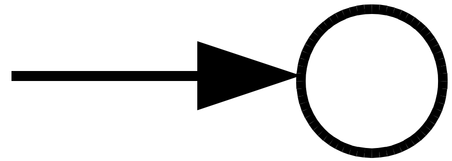
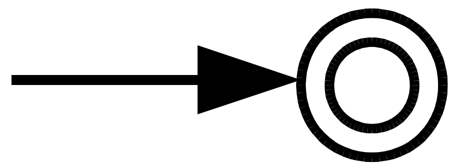
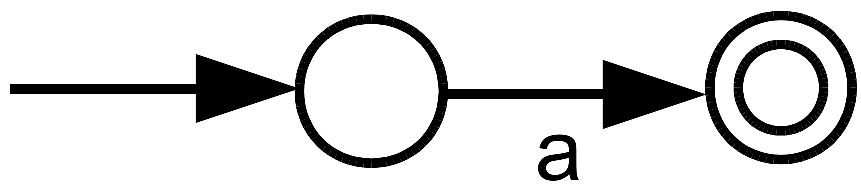
Construction:

- R = ∅ M := 

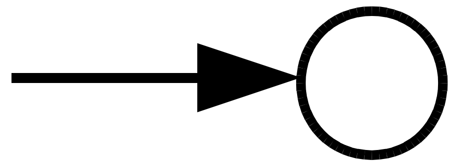
- R = ε M := 

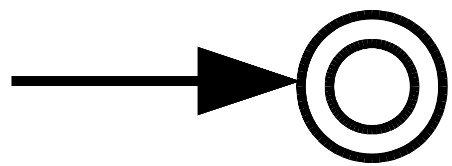
- R = a M := 

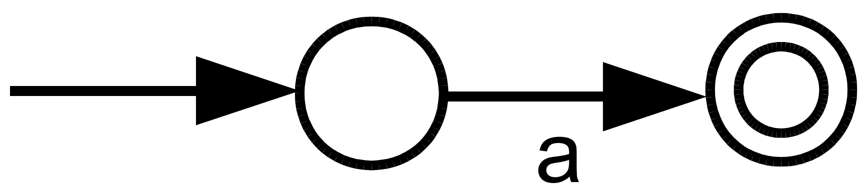
- R = R U R' ?

Theorem: For every RE R there is NFA M: L(M) = L(R)

Construction:

- R = ∅ M := 

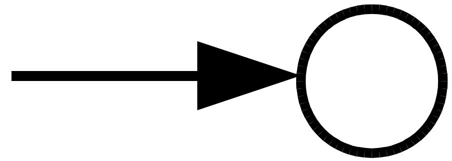
- R = ε M := 

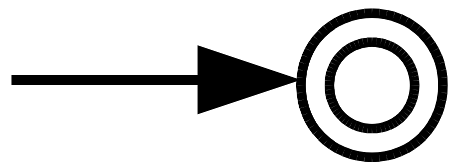
- R = a M := 

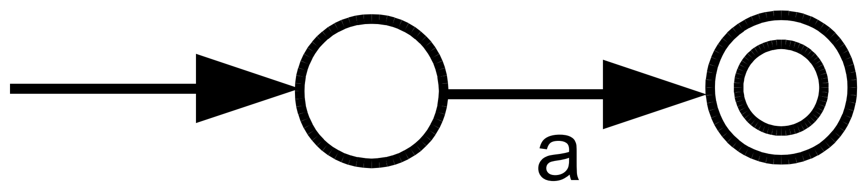
- R = R U R' use construction for A U B seen earlier
- R = R o R' ?

Theorem: For every RE R there is NFA M: L(M) = L(R)

Construction:

- R = ∅ M := 

- R = ε M := 

- R = a M := 

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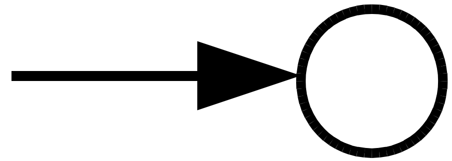
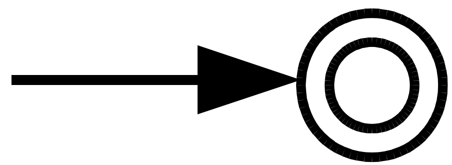
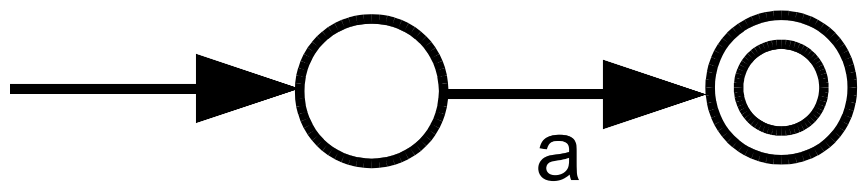
- R = ∅ M := 

- R = ε M := 

- R = a M := 

- R = R U R' use construction for A U B seen earlier
- R = R o R' use construction for A o B seen earlier
- R = R* use construction for A* seen earlier

Example: RE → NFA

$$RE = (ab \cup a)^*$$

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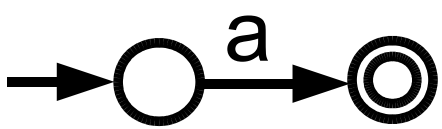
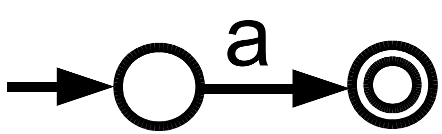
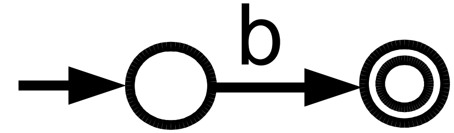
M_a = 

$$L(M_a)=L(a)$$

Example: RE → NFA

$$RE = (ab \cup a)^*$$

$M_a =$ 

$M_b =$ 

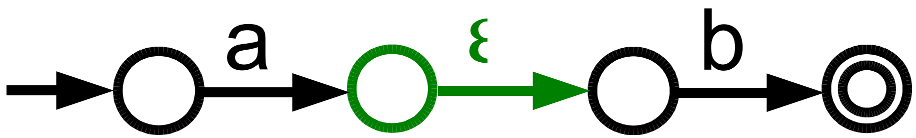
$L(M_a)=L(a)$

$L(M_b)=L(b)$

Example: RE → NFA

$$RE = (ab \cup a)^*$$

M_{ab} =

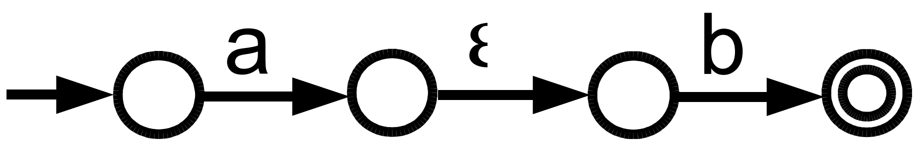


$L(M_{ab})=L(ab)$

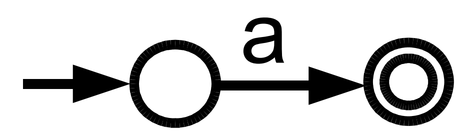
Example: RE → NFA

$$RE = (ab \cup a)^*$$

M_{ab} =



M_a =



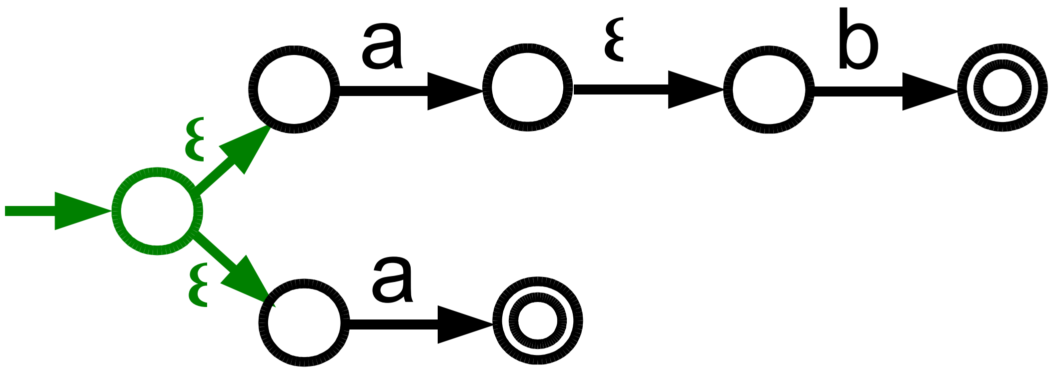
$L(M_{ab}) = L(ab)$

$L(M_a) = L(a)$

Example: RE → NFA

$$RE = (ab \cup a)^*$$

$M_{ab \cup a}$ =

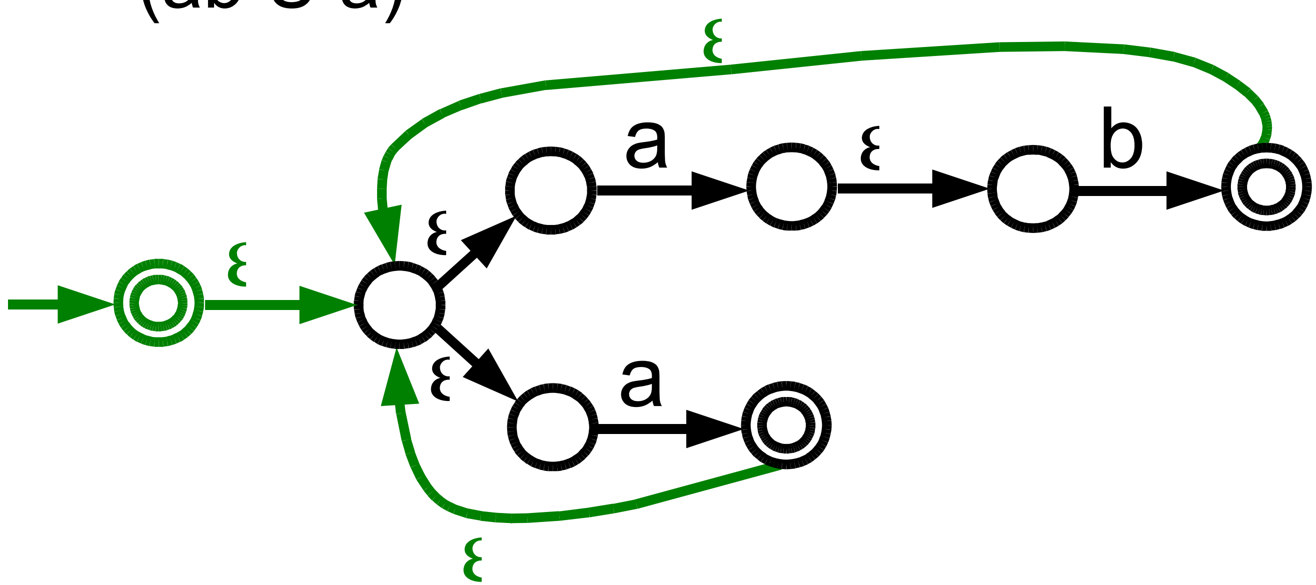


$$L(M_{ab \cup a}) = L(ab \cup a)$$

Example: RE → NFA

RE = (ab U a)*

$M_{(ab\ U\ a)^*}$ =



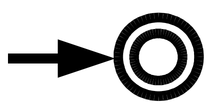
$L(M_{(ab\ U\ a)^*})$=L((ab U a)*)=L(RE)

ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

ANOTHER Example: RE → NFA

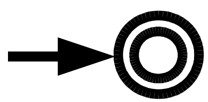
$$RE = (\varepsilon \cup a)ba^*$$

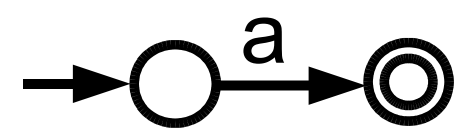
$M_\varepsilon =$ 

$$L(M_\varepsilon) = L(\varepsilon)$$

ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

$M_\varepsilon =$ 

$M_a =$ 

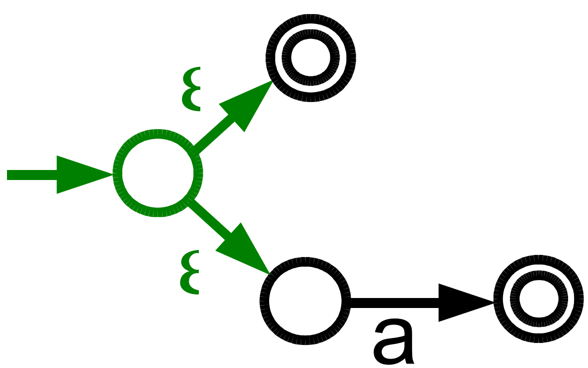
$L(M_\varepsilon) = L(\varepsilon)$

$L(M_a) = L(a)$

ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

$M_{\varepsilon \cup a} =$

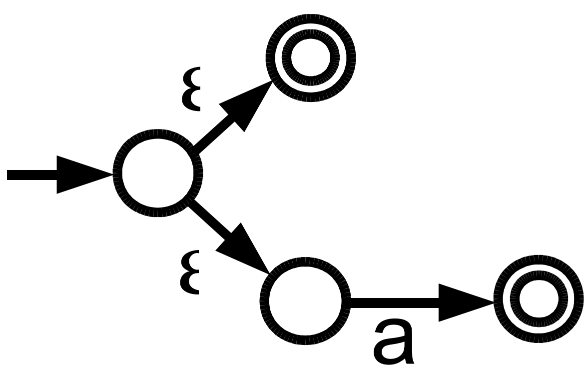


$$L(M_{\varepsilon \cup a}) = L(\varepsilon \cup a)$$

ANOTHER Example: RE → NFA

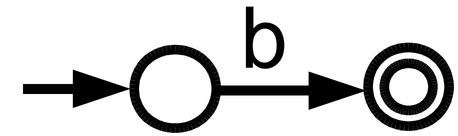
$$RE = (\varepsilon \cup a)ba^*$$

$M_{\varepsilon \cup a} =$



$L(M_{\varepsilon \cup a}) = L(\varepsilon \cup a)$

$M_b =$

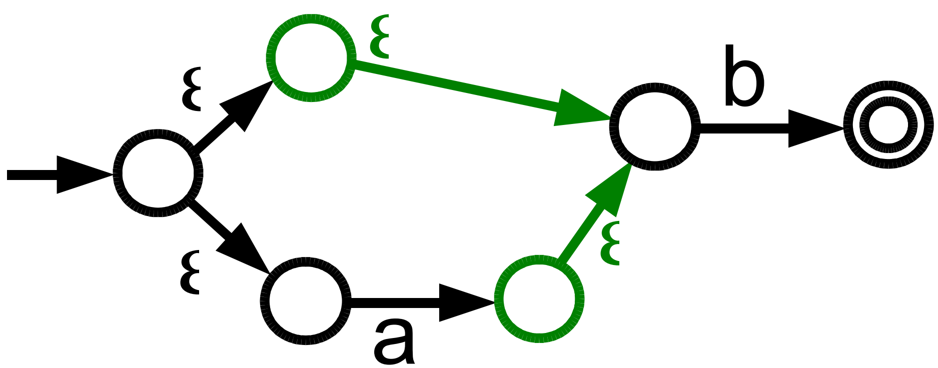


$L(M_b) = L(b)$

ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

$$M_{(\varepsilon \cup a)b} =$$

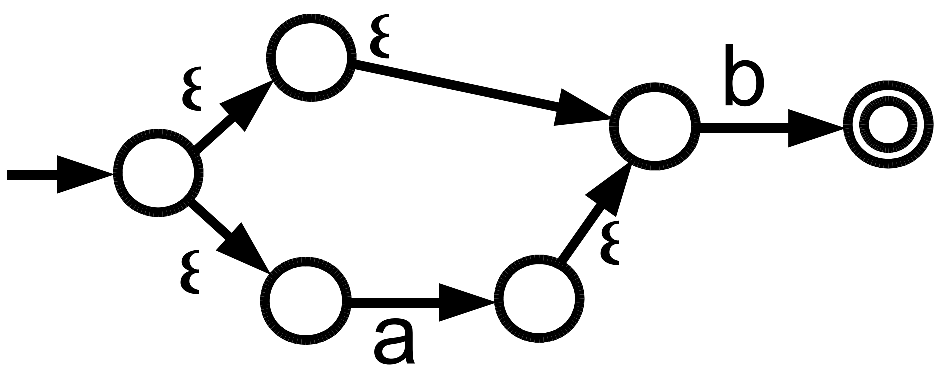


$$L(M_{(\varepsilon \cup a)b}) = L((\varepsilon \cup a)b)$$

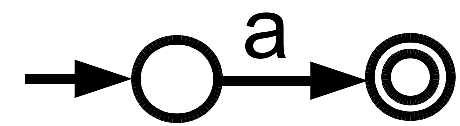
ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

$M_{(\varepsilon \cup a)b} =$



$M_a =$



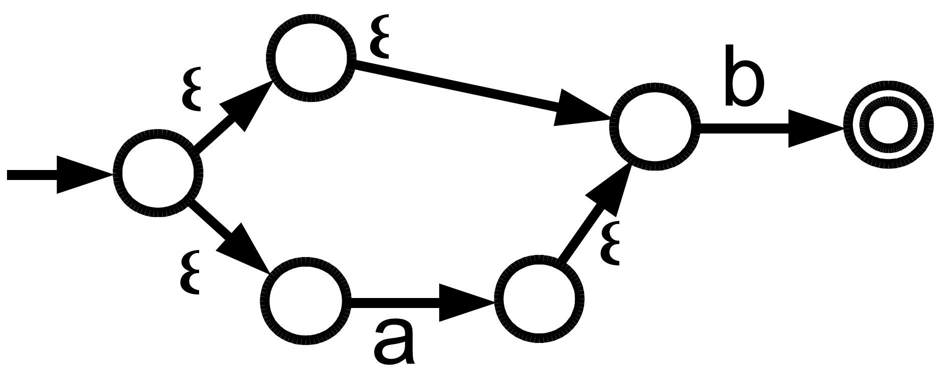
$L(M_a) = L(a)$

$L(M_{(\varepsilon \cup a)b}) = L((\varepsilon \cup a)b)$

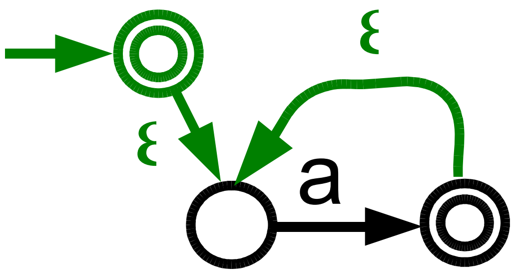
ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

$M_{(\varepsilon \cup a)b} =$



$M_{a^*} =$



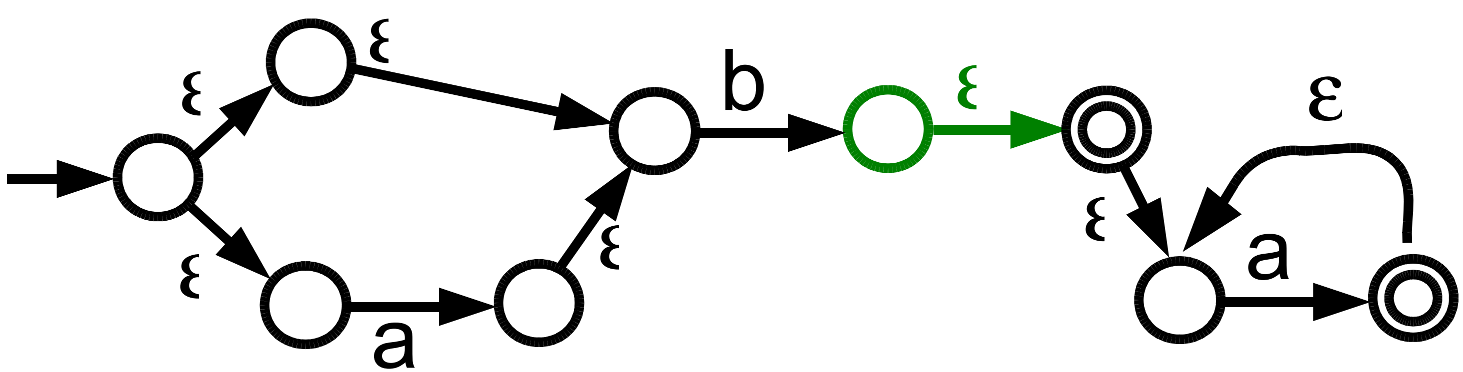
$L(M_{a^*}) = L(a^*)$

$L(M_{(\varepsilon \cup a)b}) = L((\varepsilon \cup a)b)$

ANOTHER Example: RE → NFA

$$RE = (\varepsilon \cup a)ba^*$$

$$M_{(\varepsilon \cup a)ba^*} =$$



$$L(M_{(\varepsilon \cup a)ba^*}) = L((\varepsilon \cup a)ba^*) = L(RE)$$

Recap:

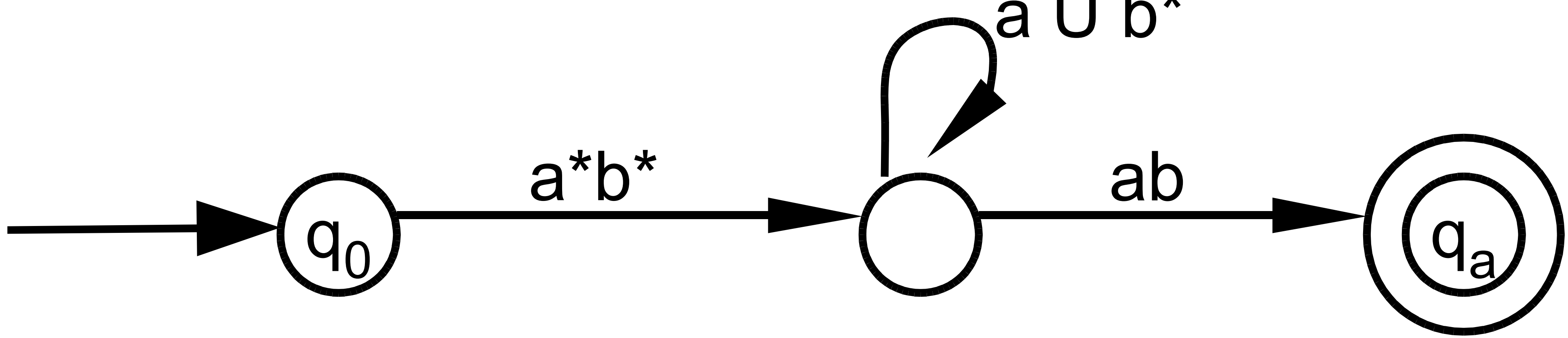
Here "$\Rightarrow$" means "can be converted to"

We have seen: RE $\Rightarrow$ NFA $\Leftrightarrow$ DFA

Next we see: DFA $\Rightarrow$ RE

In two steps: DFA $\Rightarrow$ Generalized NFA $\Rightarrow$ RE

Generalized NFA (GNFA)

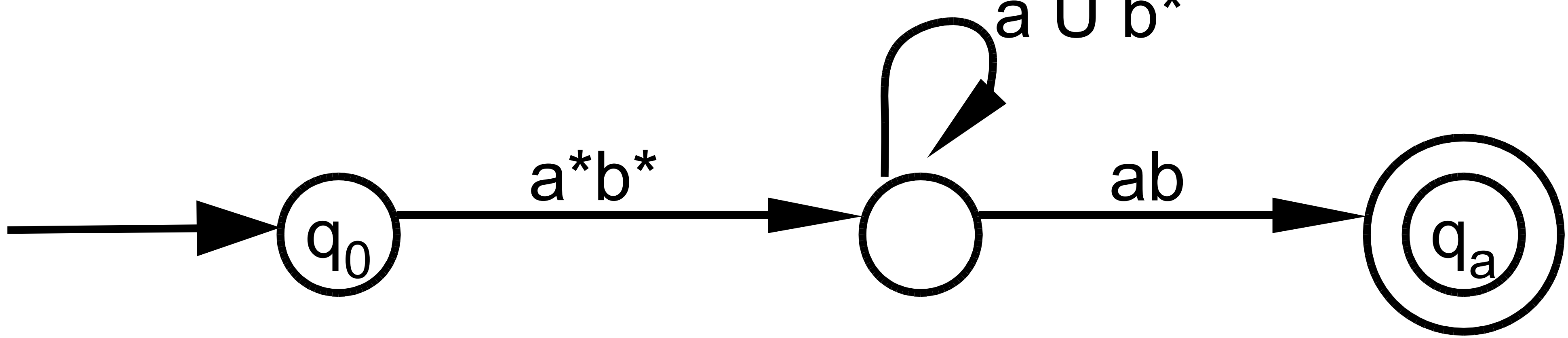


Nondeterministic

Transitions labelled by RE

Read blocks of input symbols at a time

Generalized NFA (GNFA)



Convention:

Unique final state

Exactly one transition between each pair of states

except nothing going into start state

nothing going out of final state

If arrow not shown in picture, label = ∅

- Definition: A generalized finite automaton (GNFA)
- is a 5-tuple $(Q, \Sigma, \delta, q_0, q_a)$ where
- Q is a finite set of states
- Σ is the input alphabet
- $\delta : (Q - \{q_a\}) \times (Q - \{q_0\}) \rightarrow$ Regular Expressions
- q_0 in Q is the start state
- q_a in Q is the accept state

- Definition: GNFA $(Q, \Sigma, \delta, q_0, q_a)$ accepts a string w if
- $\exists$ integer k, $\exists$ k strings $w_1, w_2, \ldots, w_k \in \Sigma^*$

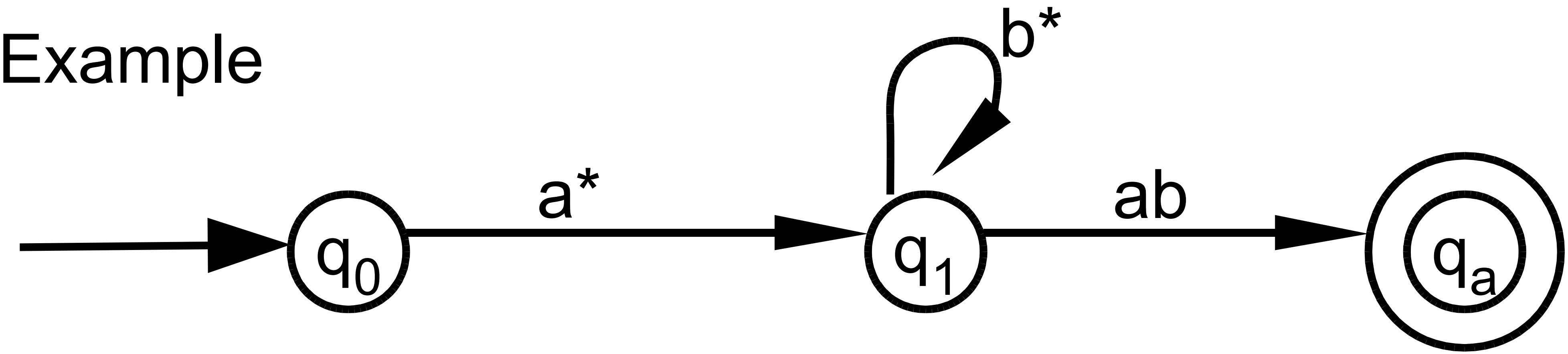
  such that $w = w_1 w_2 \ldots w_k$

  (divide w in k strings)

- $\exists$ sequence of k+1 states $r_0, r_1, .., r_k$ in Q such that:
- $r_0 = q_0$
- $w_{i+1} \in L(\delta(r_i, r_{i+1})) \; \forall\, 0 \le i < k$
- $r_k = q_a$

- Differences with NFA are in green

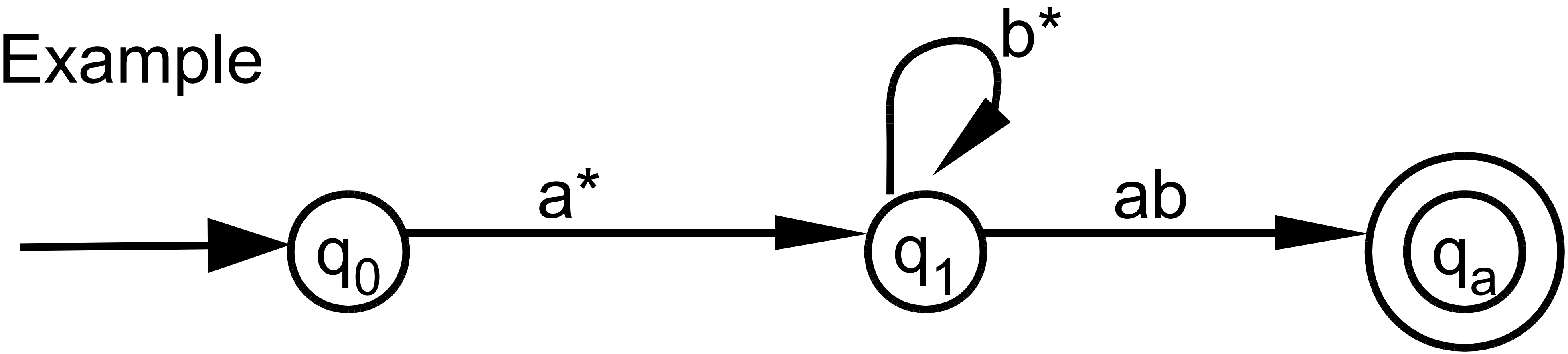
Example



Accepts w = aaabbab

w_1=?

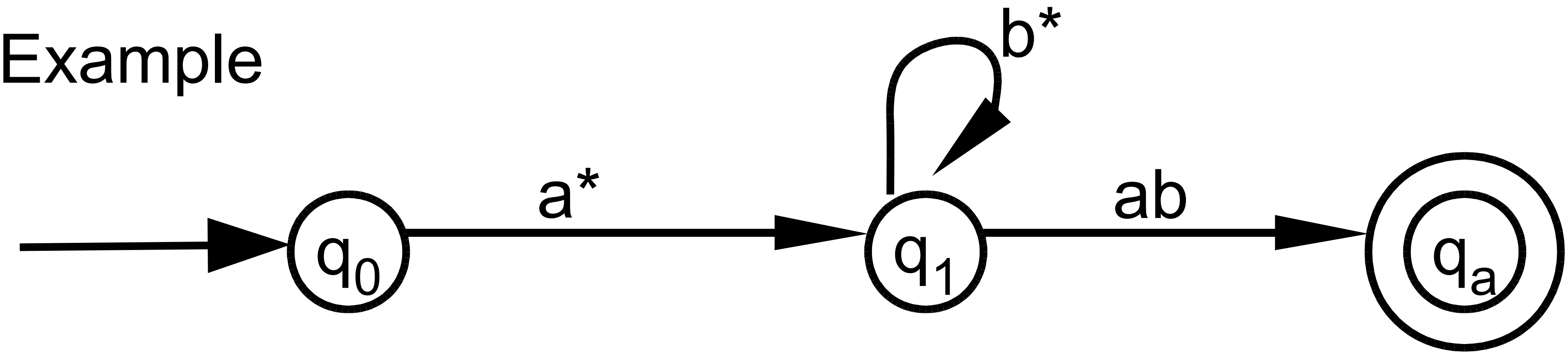
Example



Accepts w = aaabbab

w_1=aaa w_2=?

Example

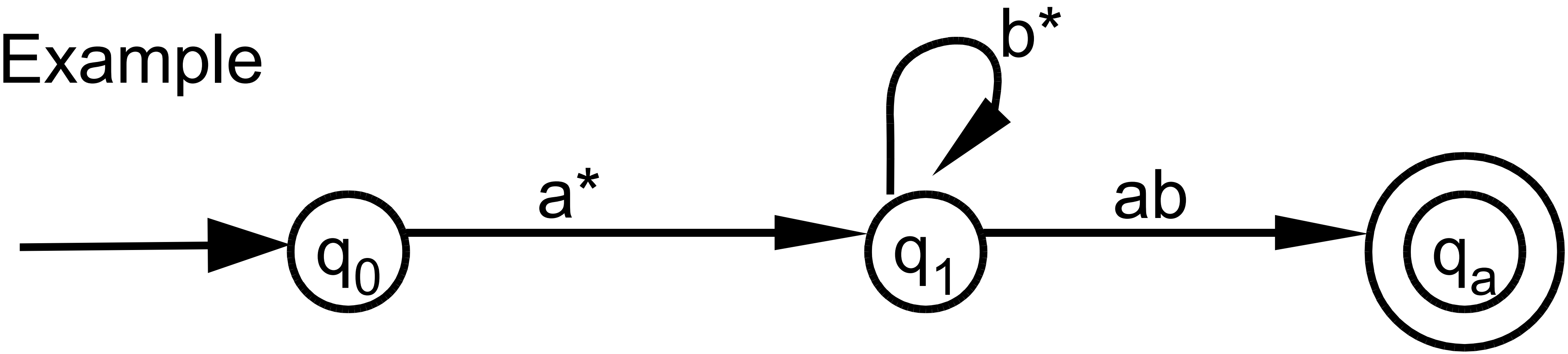


Accepts w = aaabbab

w_1=aaa w_2=bb w_3=ab

$r_0=q_0$ r_1=?

Example



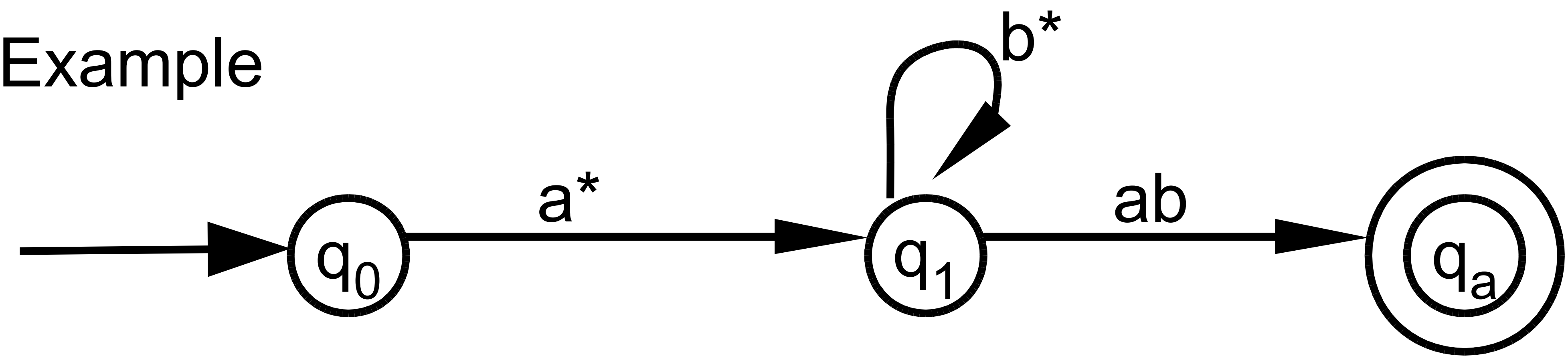
Accepts w = aaabbab

w_1=aaa w_2=bb w_3=ab

$r_0=q_0$ $r_1=q_1$ r_2=?

$w_1 = aaa \in L(\delta(r_0,r_1)) = L(\delta(q_0,q_1)) = L(a^*)$

Example



Accepts w = aaabbab

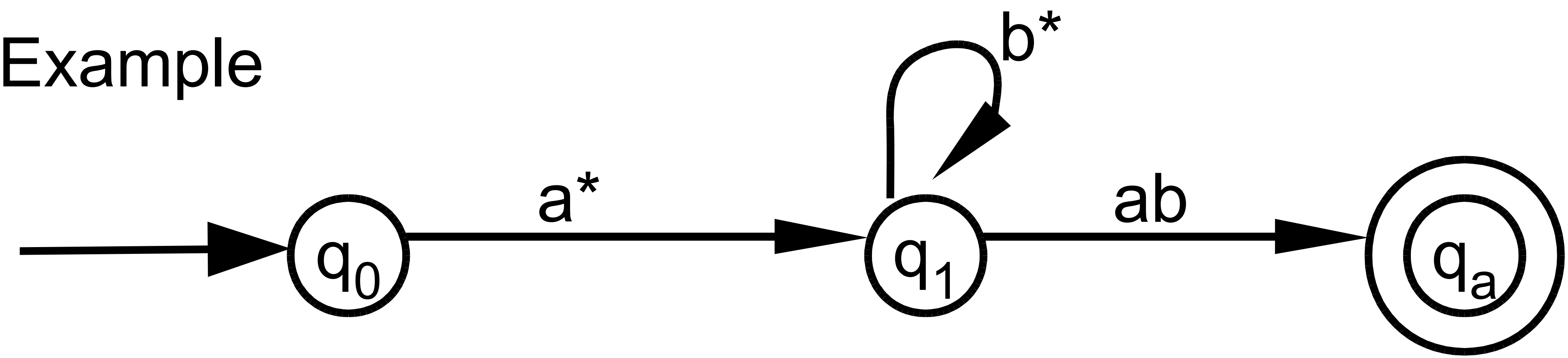
w_1=aaa w_2=bb w_3=ab

$r_0=q_0$ $r_1=q_1$ $r_2=q_1$ r_3 = ?

w_1 = aaa $\in L(\delta(r_0,r_1)) = L(\delta(q_0,q_1)) = L(a^*)$

w_2 = bb $\in L(\delta(r_1,r_2)) = L(\delta(q_1,q_1)) = L(b^*)$

Example



Accepts w = aaabbab

w_1=aaa w_2=bb w_3=ab

$r_0=q_0$ $r_1=q_1$ $r_2=q_1$ $r_3 = q_a$

$$w_1 = aaa \in L(\delta(r_0,r_1)) = L(\delta(q_0,q_1)) = L(a^*)$$

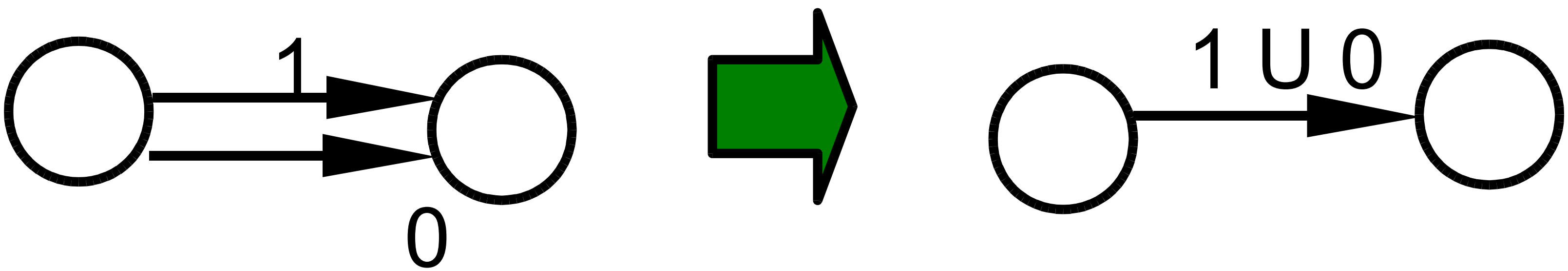
$$w_2 = bb \in L(\delta(r_1,r_2)) = L(\delta(q_1,q_1)) = L(b^*)$$

$$w_3 = ab \in L(\delta(r_2,r_3)) = L(\delta(q_1,q_a)) = L(ab)$$

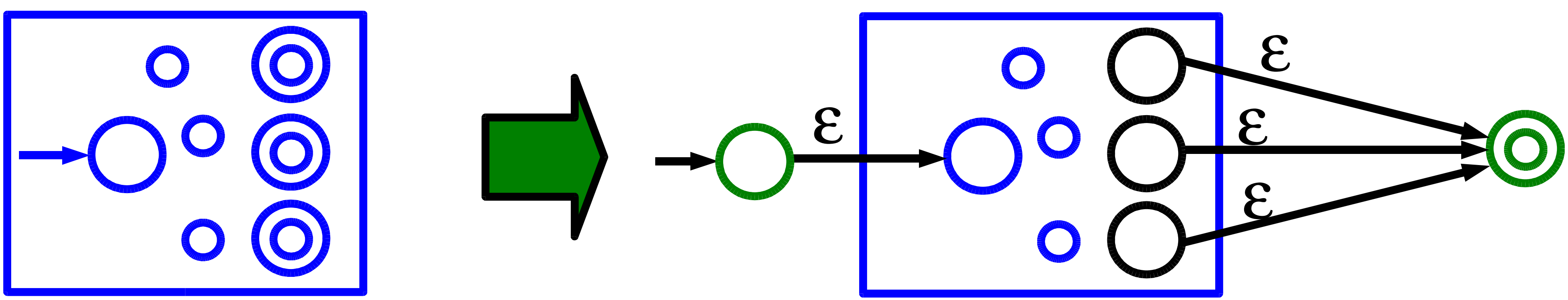
Theorem: ∀ DFA M ∃ GNFA N : L(N) = L(M)

Construction:

To ensure unique transition between each pair:



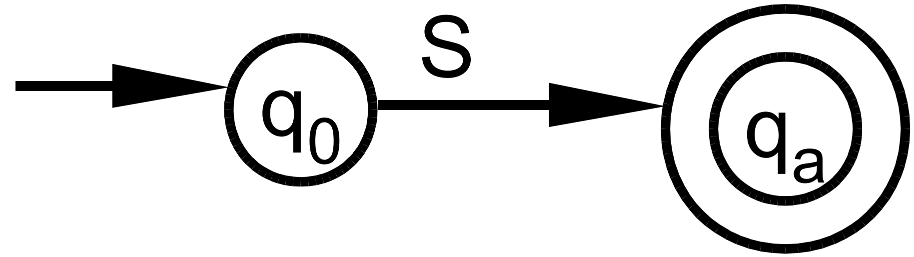
To ensure unique final state, no transitions ingoing start state, no transitions outgoing final state:



Theorem: $\forall$ GNFA N $\exists$ RE R : L(R) = L(N)

Construction:

If N has 2 states, then N =

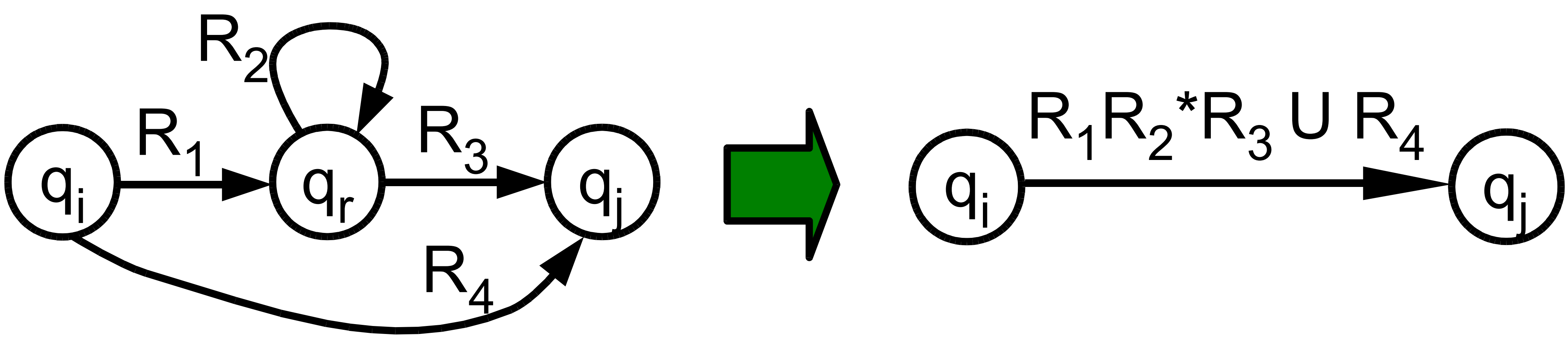


thus R := S

If N has > 2 states, eliminate some state $q_r \neq q_0, q_a$:

for every ordered pair q_i, q_j (possibly equal)

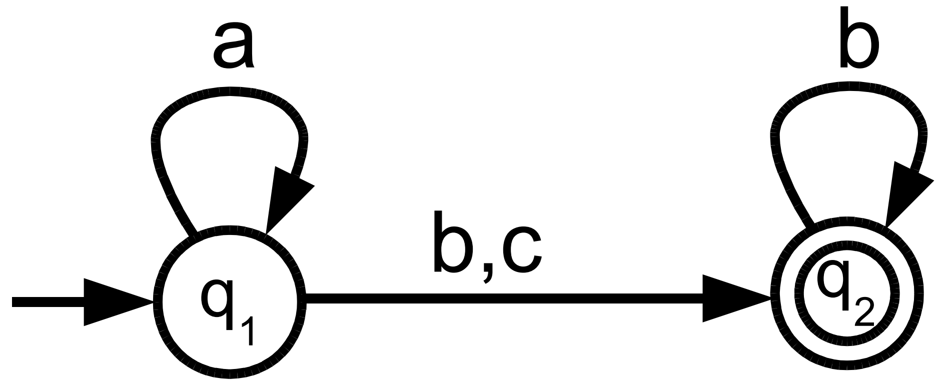
that are connected through q_r



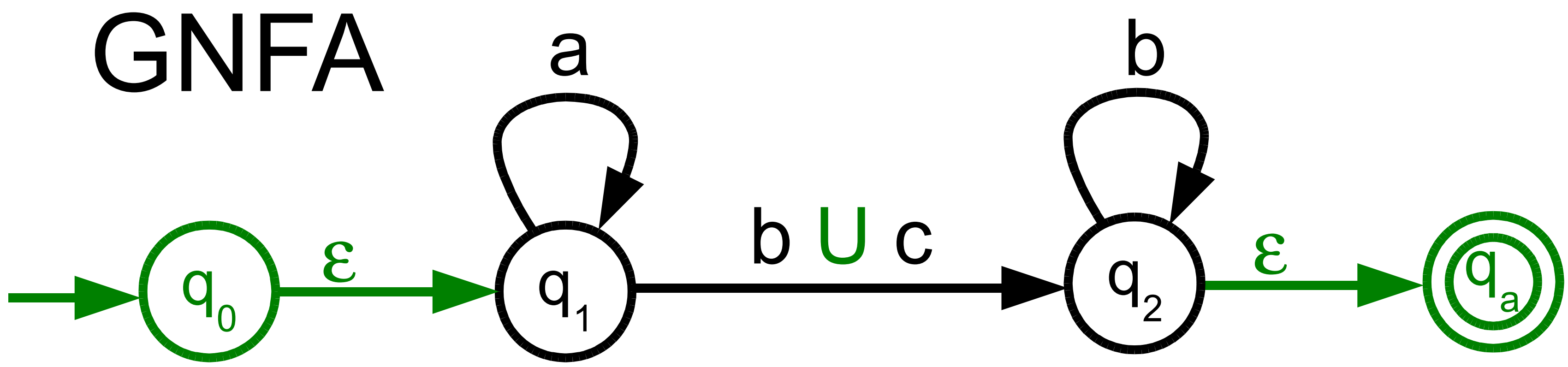
Repeat until 2 states remain

Example: DFA → GNFA → RE

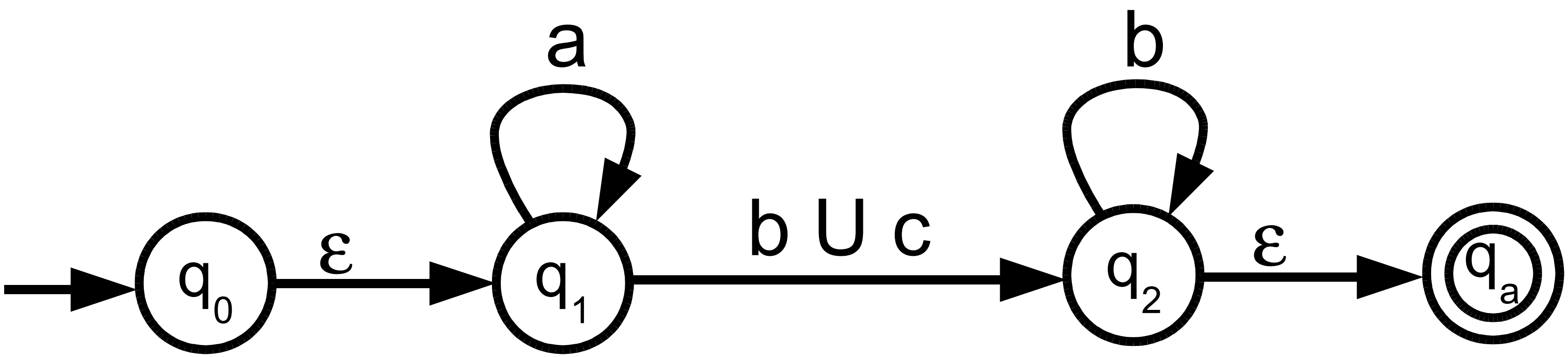
DFA



Example: DFA → GNFA → RE

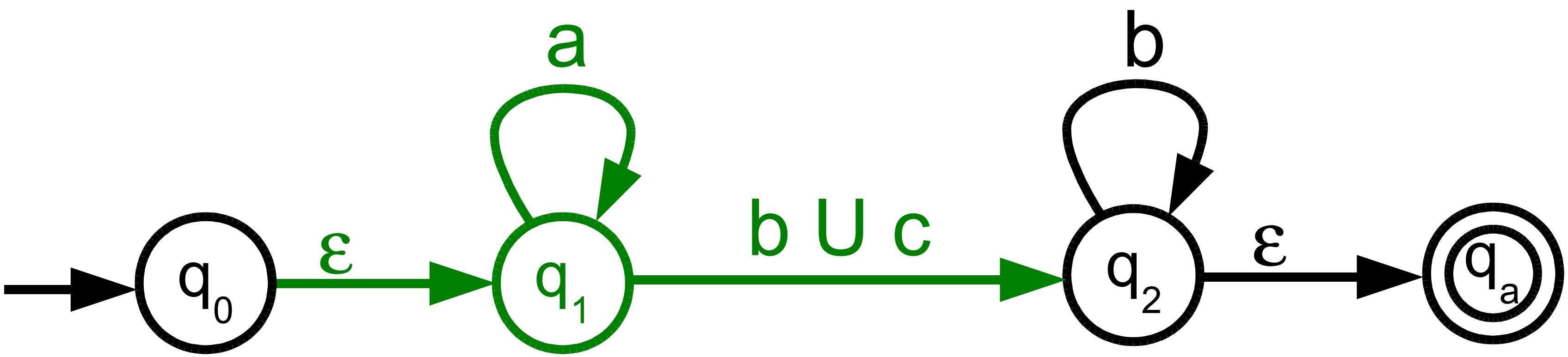


Example: DFA → GNFA → RE



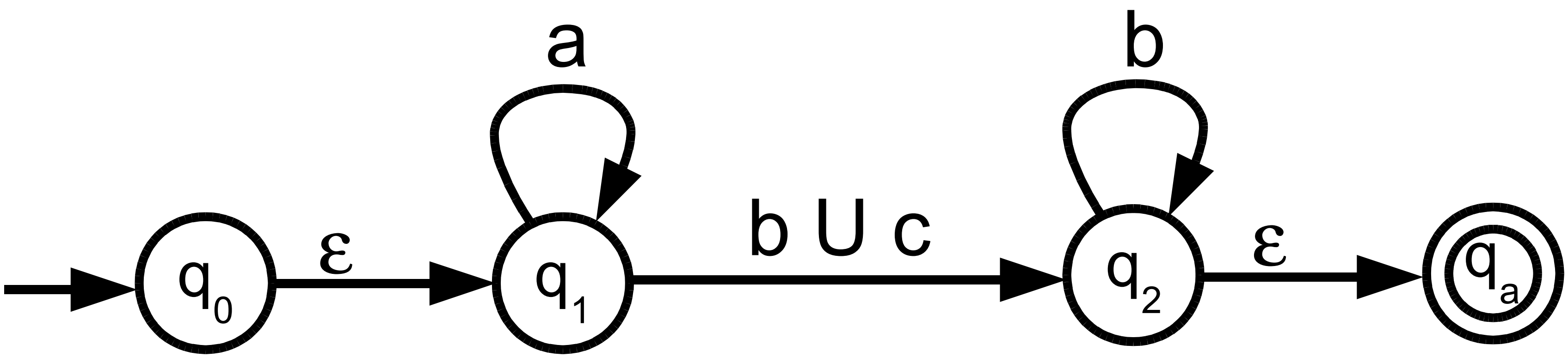
Eliminate q_1: re-draw GNFA with all other states

Example: DFA → GNFA → RE

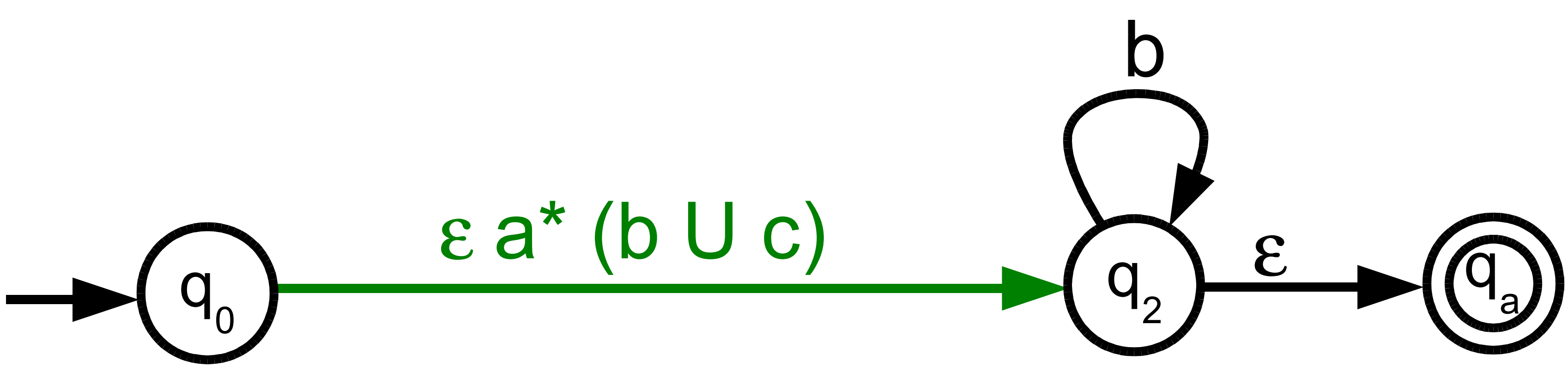


Eliminate q_1: find a path through q_1

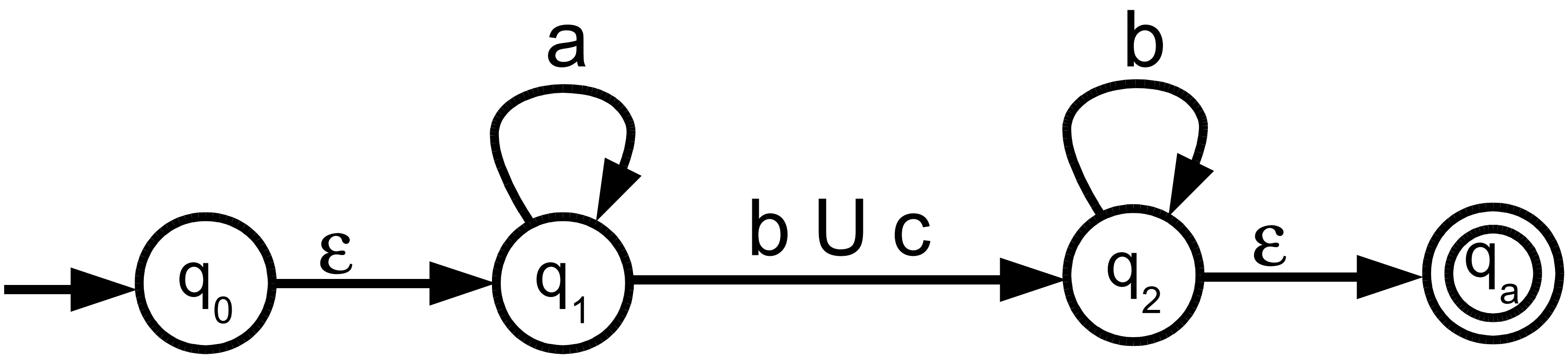
Example: DFA → GNFA → RE



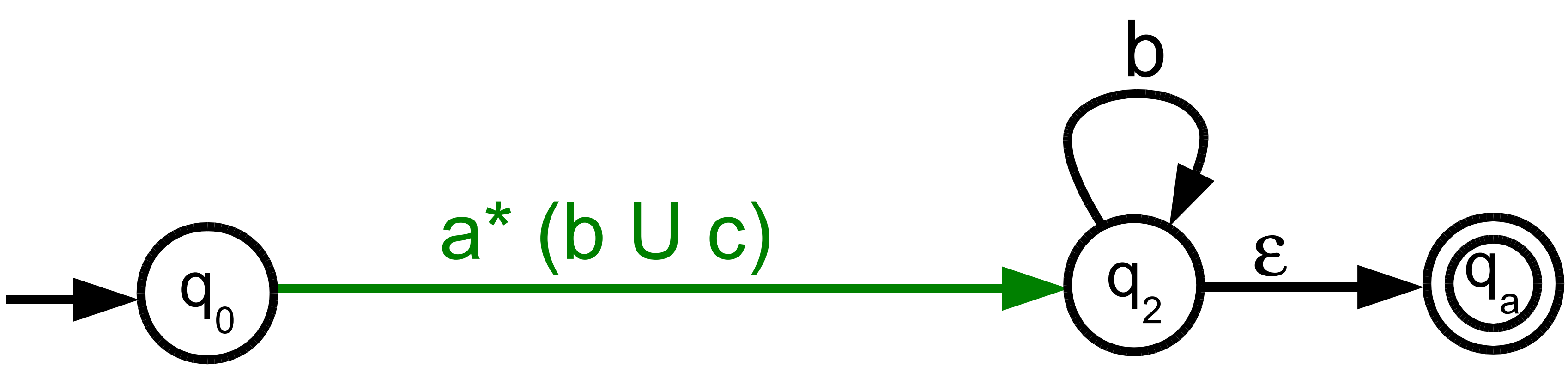
Eliminate q_1: add edge to new GNFA



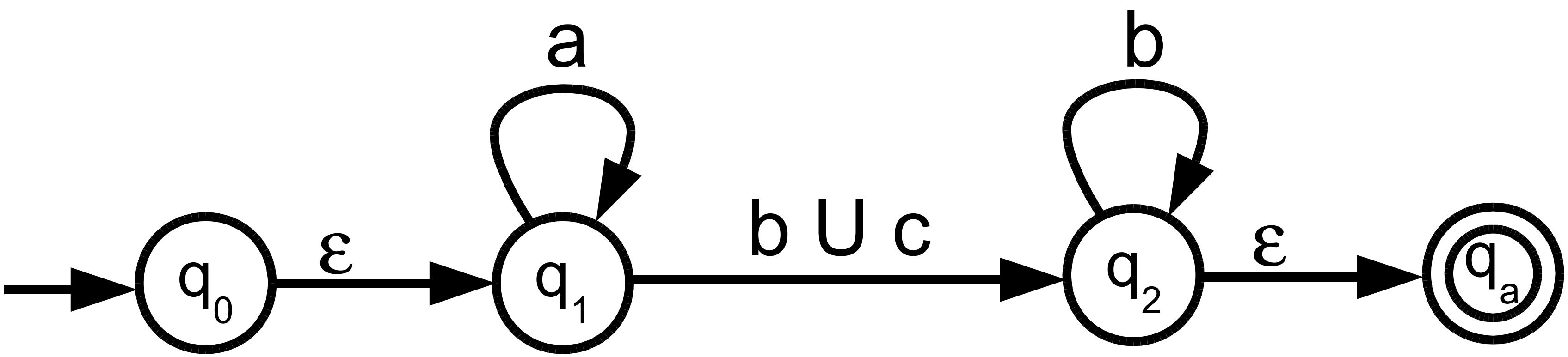
Example: DFA → GNFA → RE



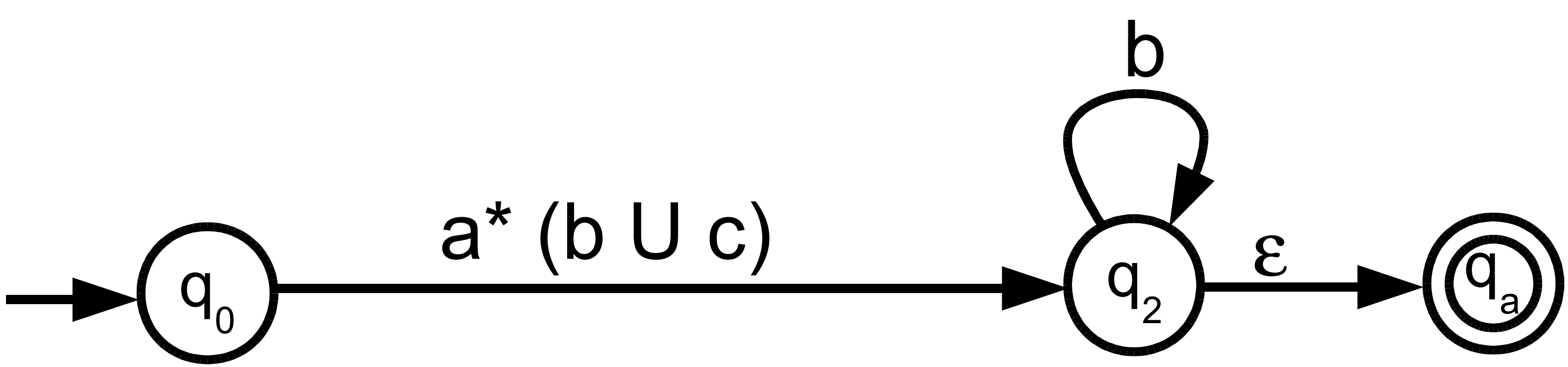
Eliminate q_1: simplify RE on new edge



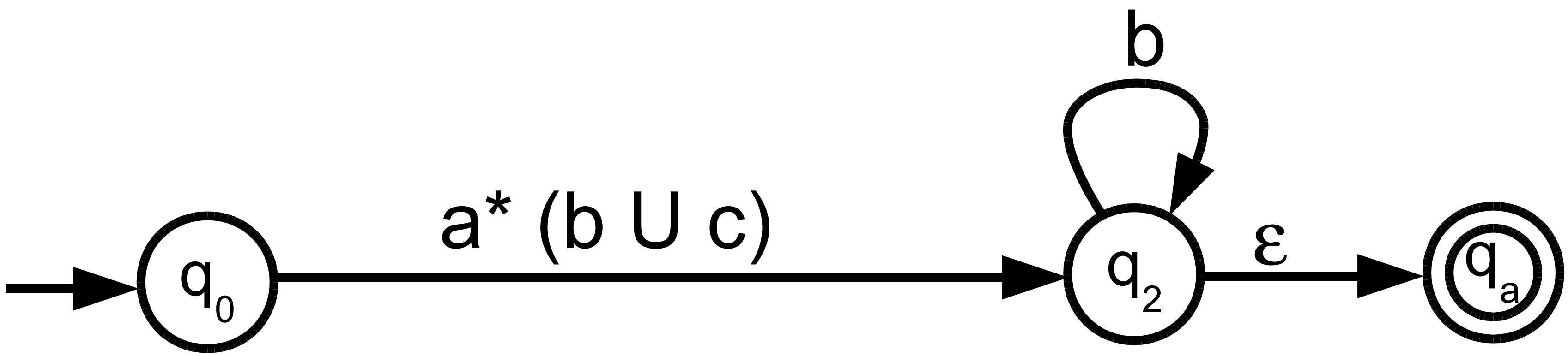
Example: DFA → GNFA → RE



Eliminate q_1: if no more paths through q_1, start over

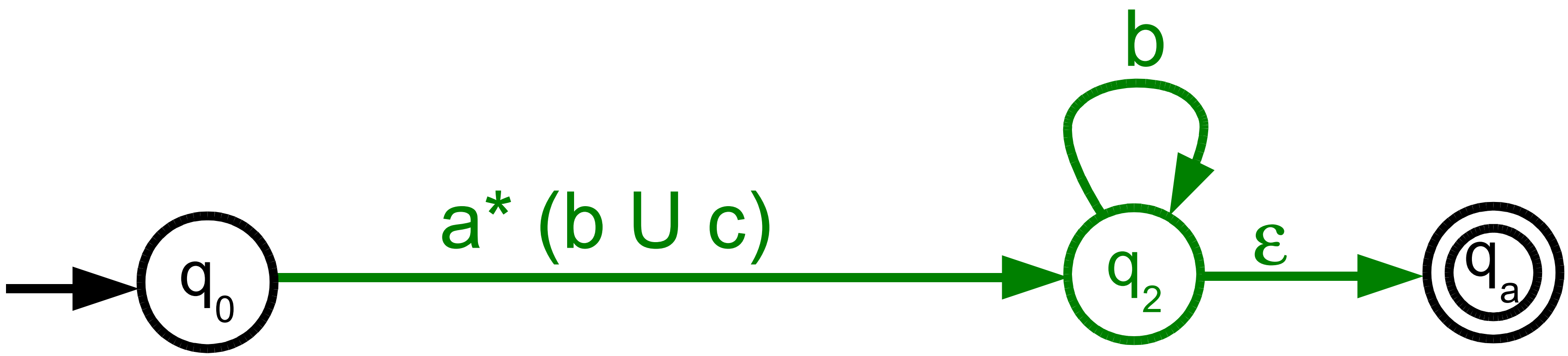


Example: DFA → GNFA → RE



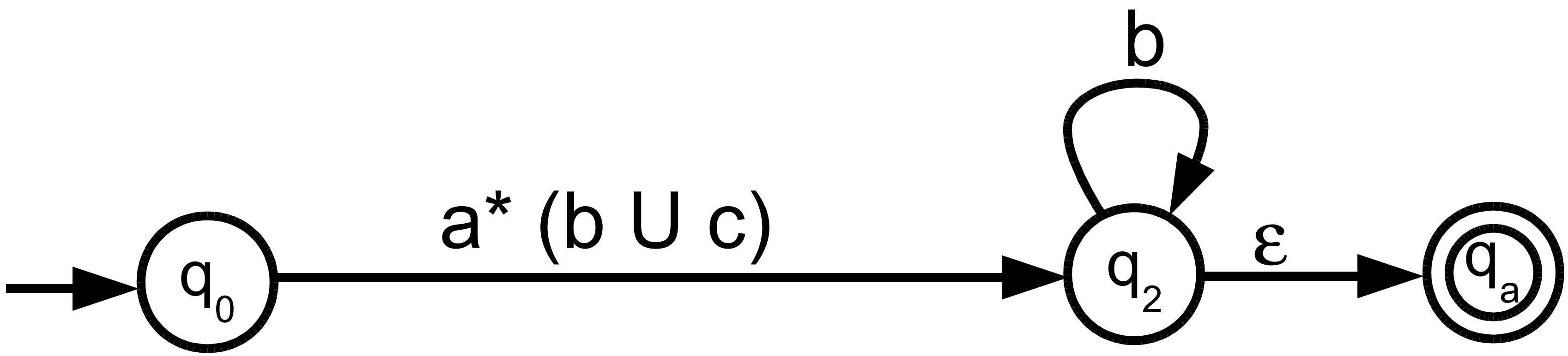
Eliminate q_2: re-draw GNFA with all other states

Example: DFA → GNFA → RE

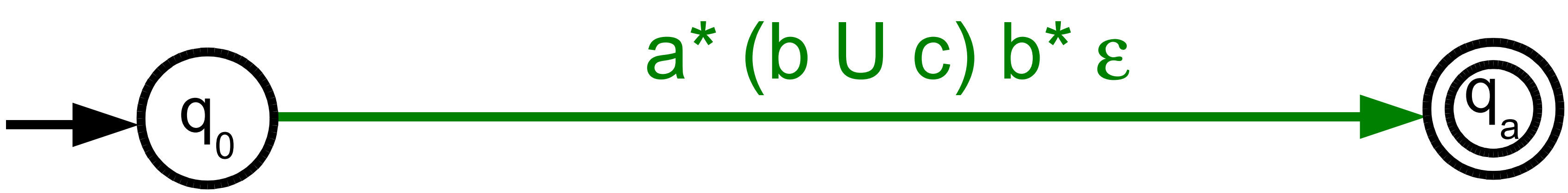


Eliminate q_2: find a path through q_2

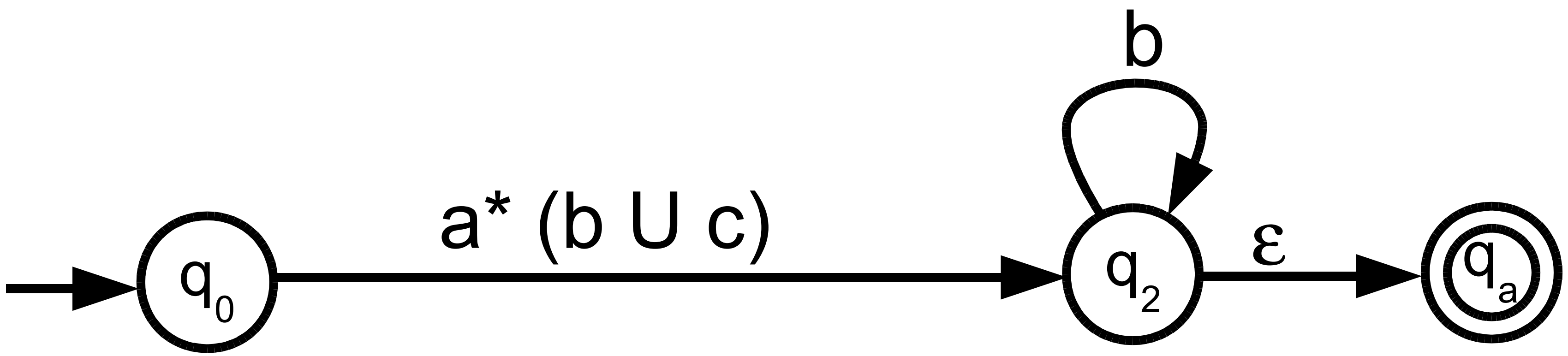
Example: DFA → GNFA → RE



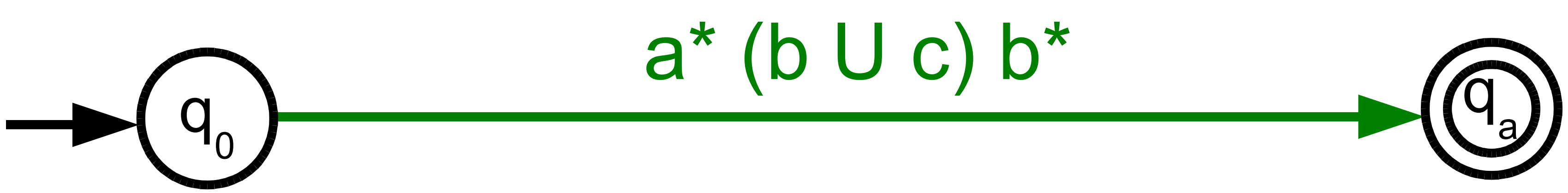
Eliminate q_2: add edge to new GNFA



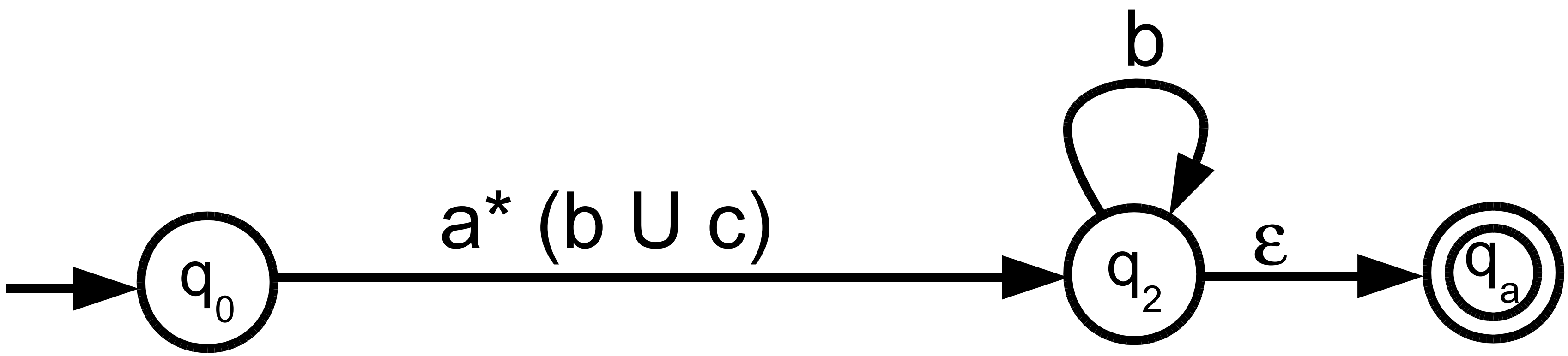
Example: DFA → GNFA → RE



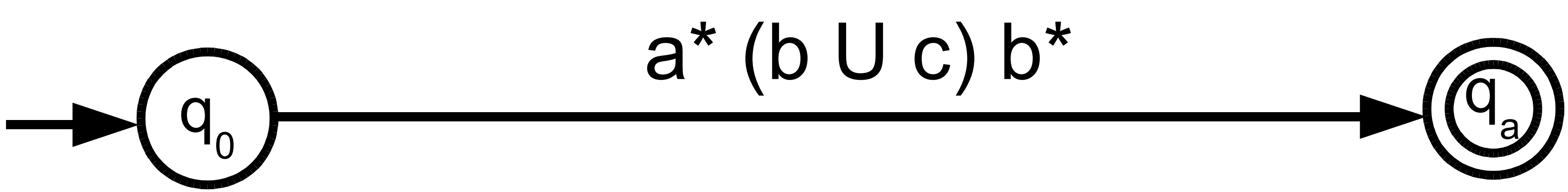
Eliminate q_2: simplify RE on new edge



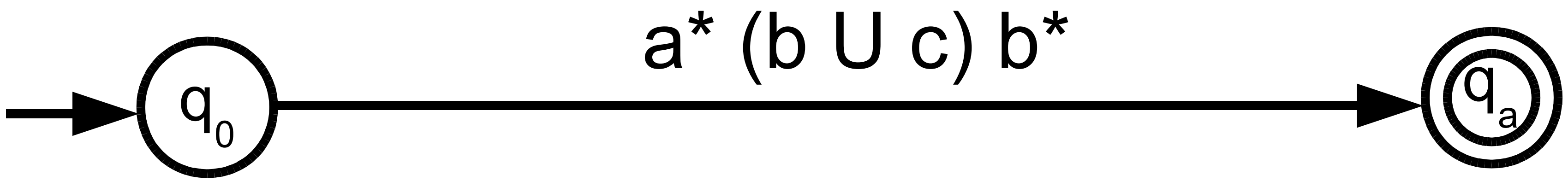
Example: DFA → GNFA → RE



Eliminate q_2: if no more paths through q_2, start over



Example: DFA → GNFA → RE

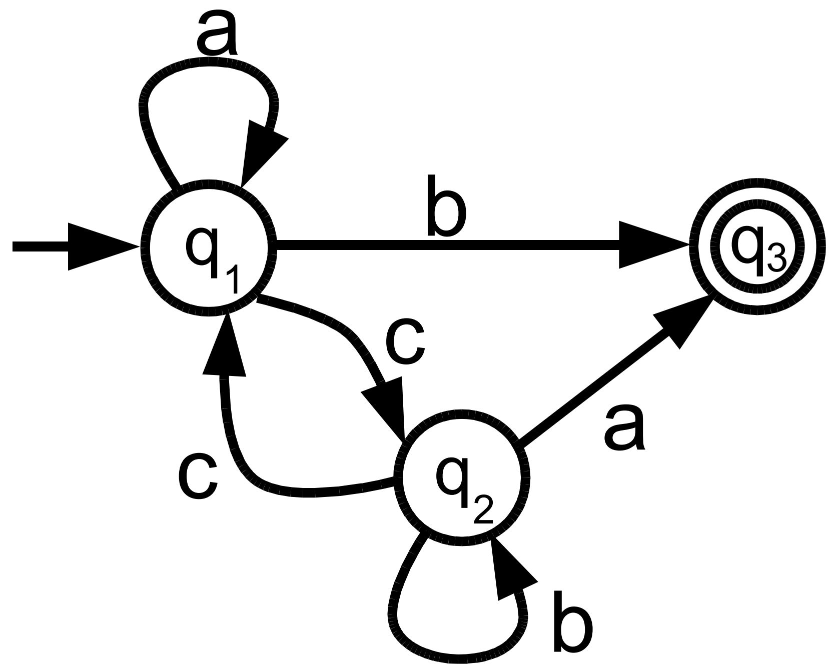


Only two states remain:

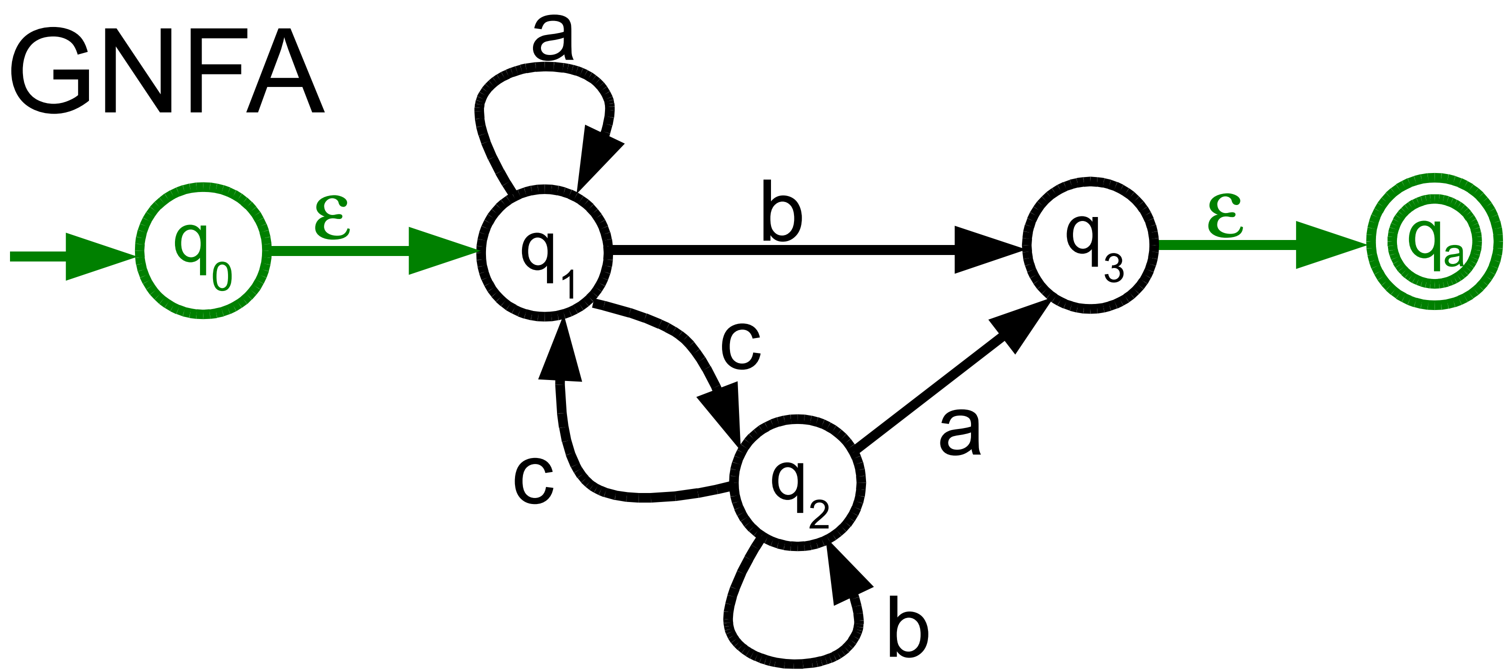
$$RE = a^* (b \cup c) b^*$$

ANOTHER Example: DFA → GNFA → RE

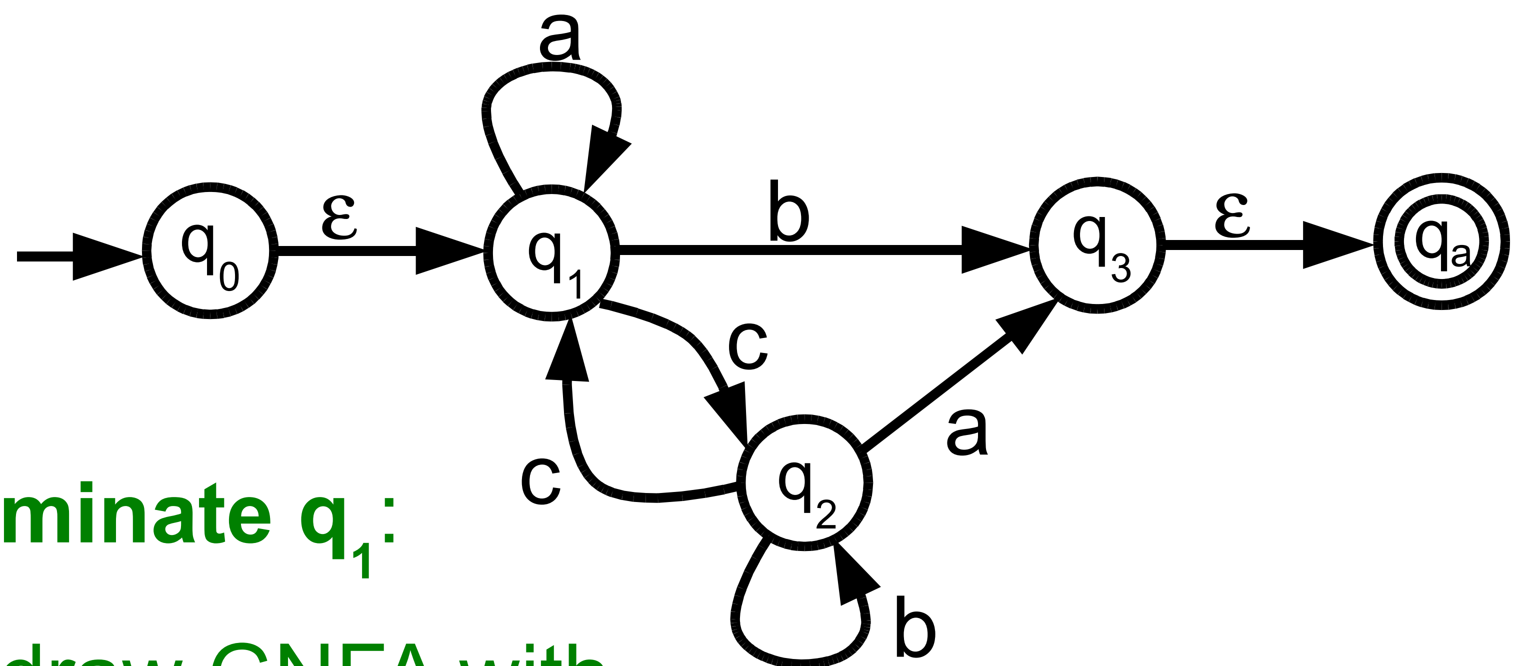
DFA



ANOTHER Example: DFA → GNFA → RE



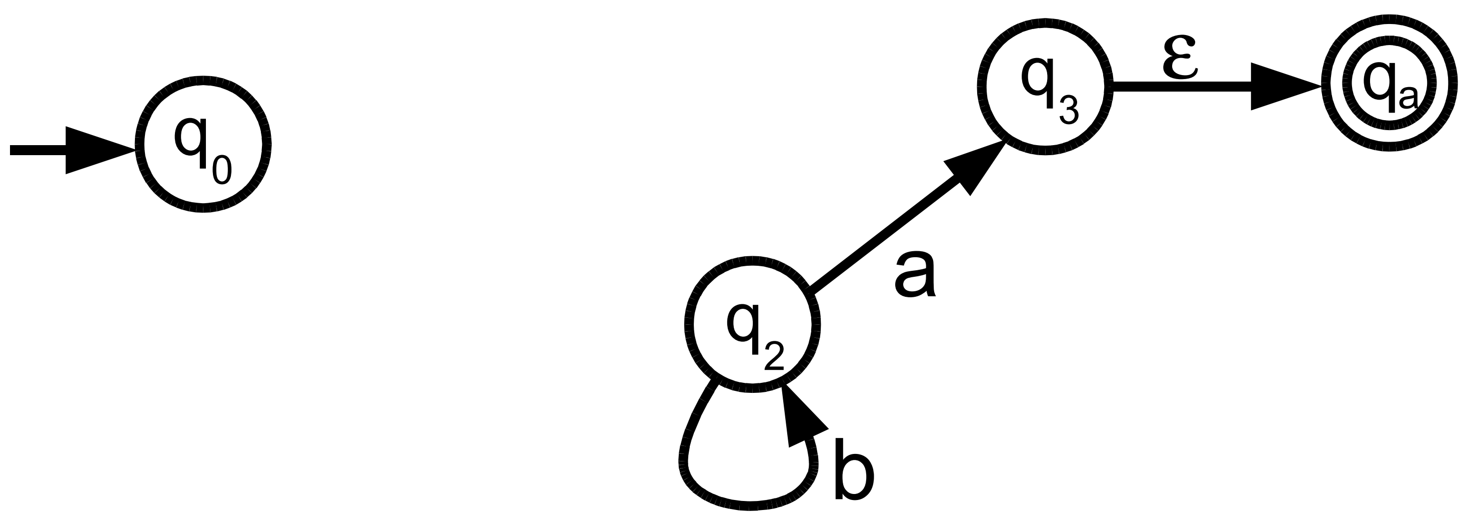
ANOTHER Example: DFA → GNFA → RE



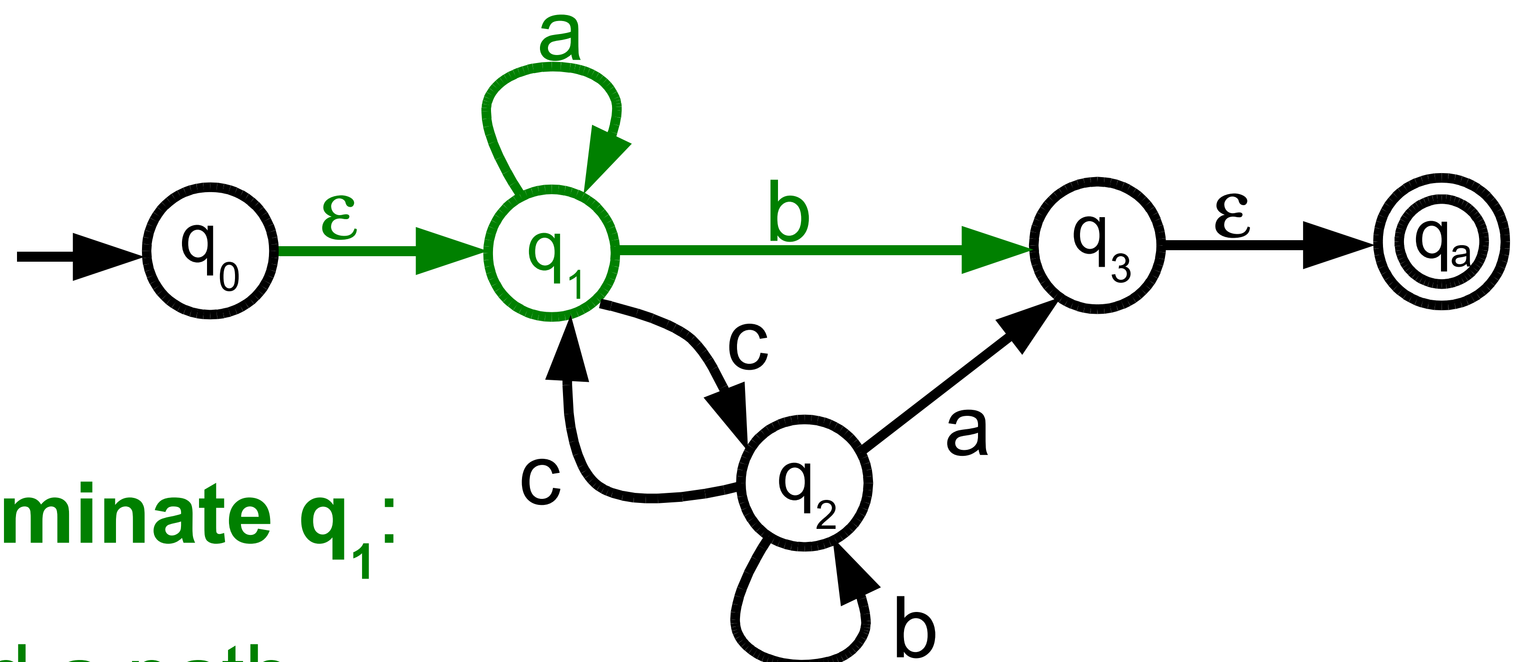
Eliminate q_1:

re-draw GNFA with

all other states

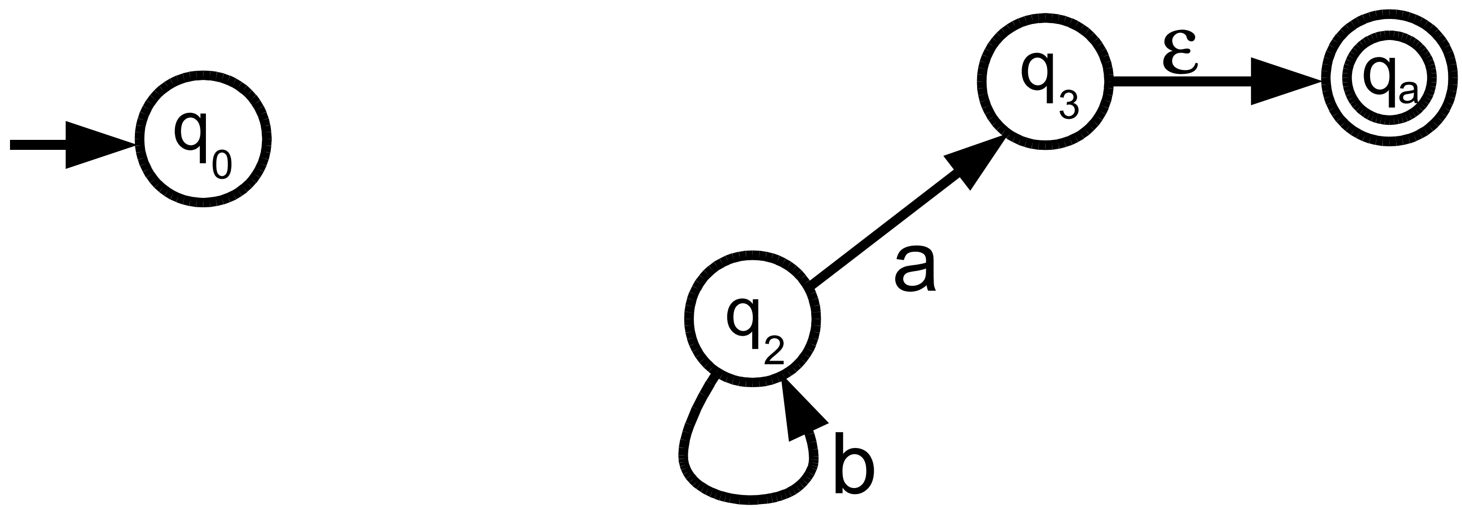


ANOTHER Example: DFA → GNFA → RE

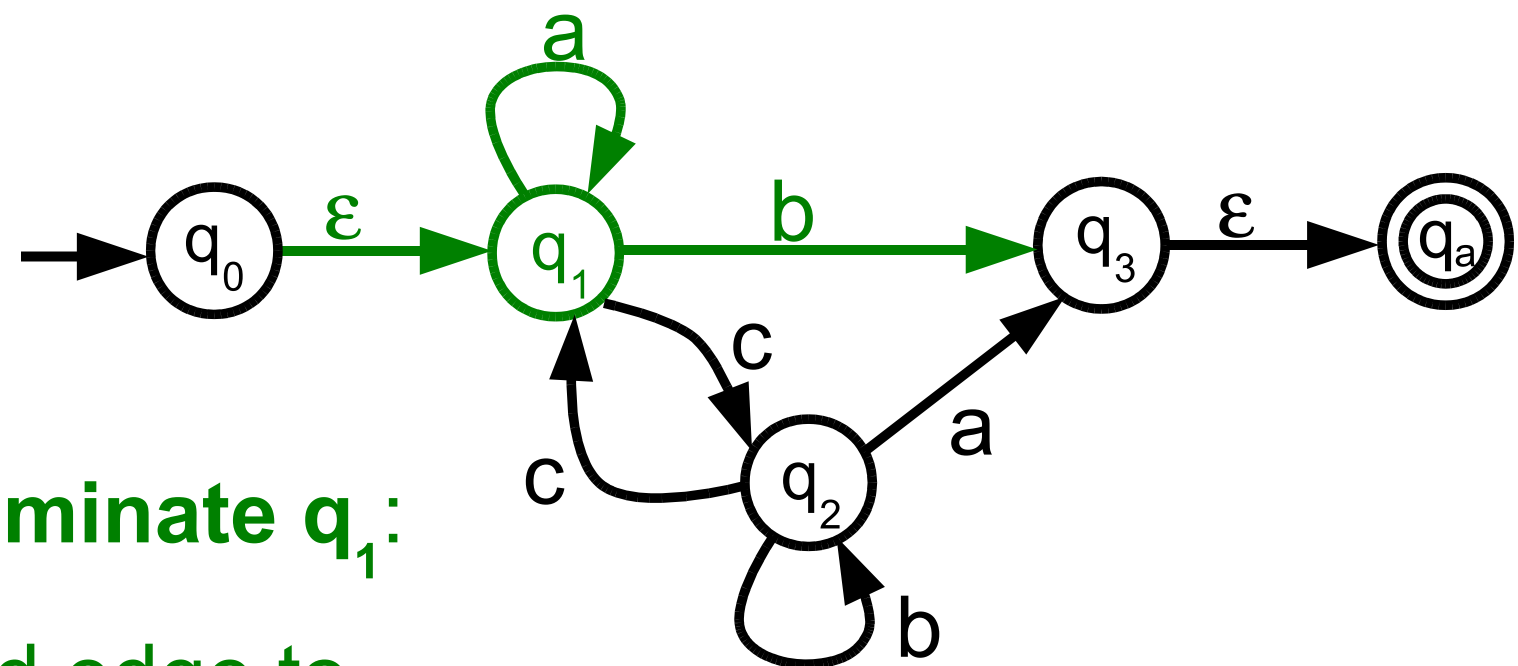


Eliminate q_1:

find a path through q_1



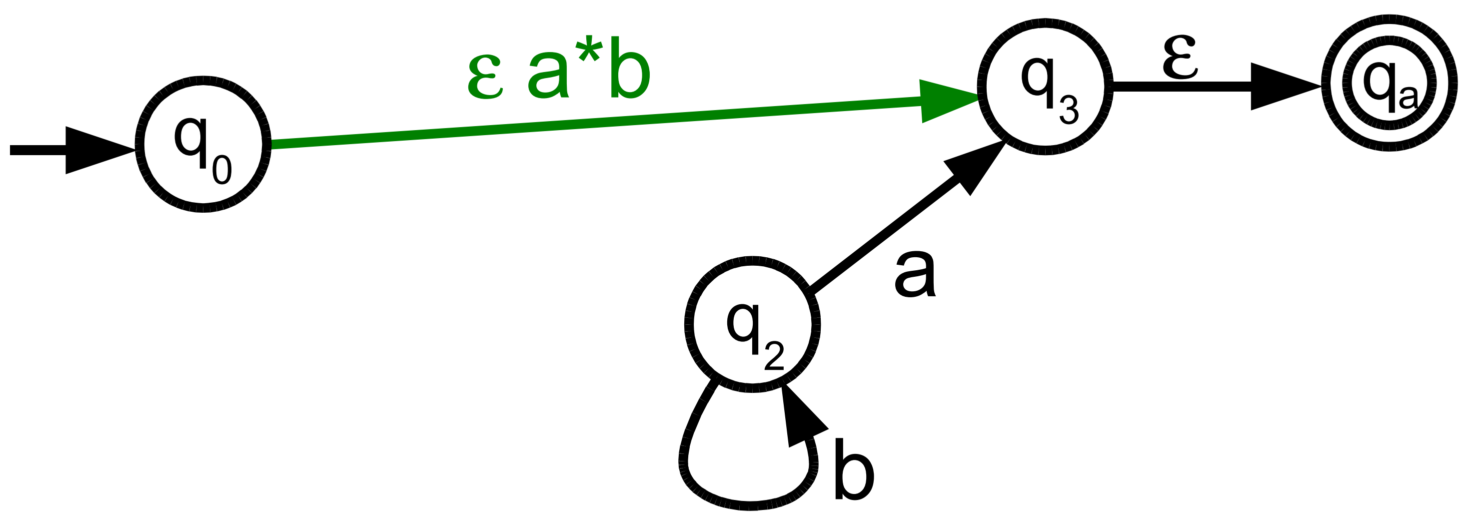
ANOTHER Example: DFA → GNFA → RE



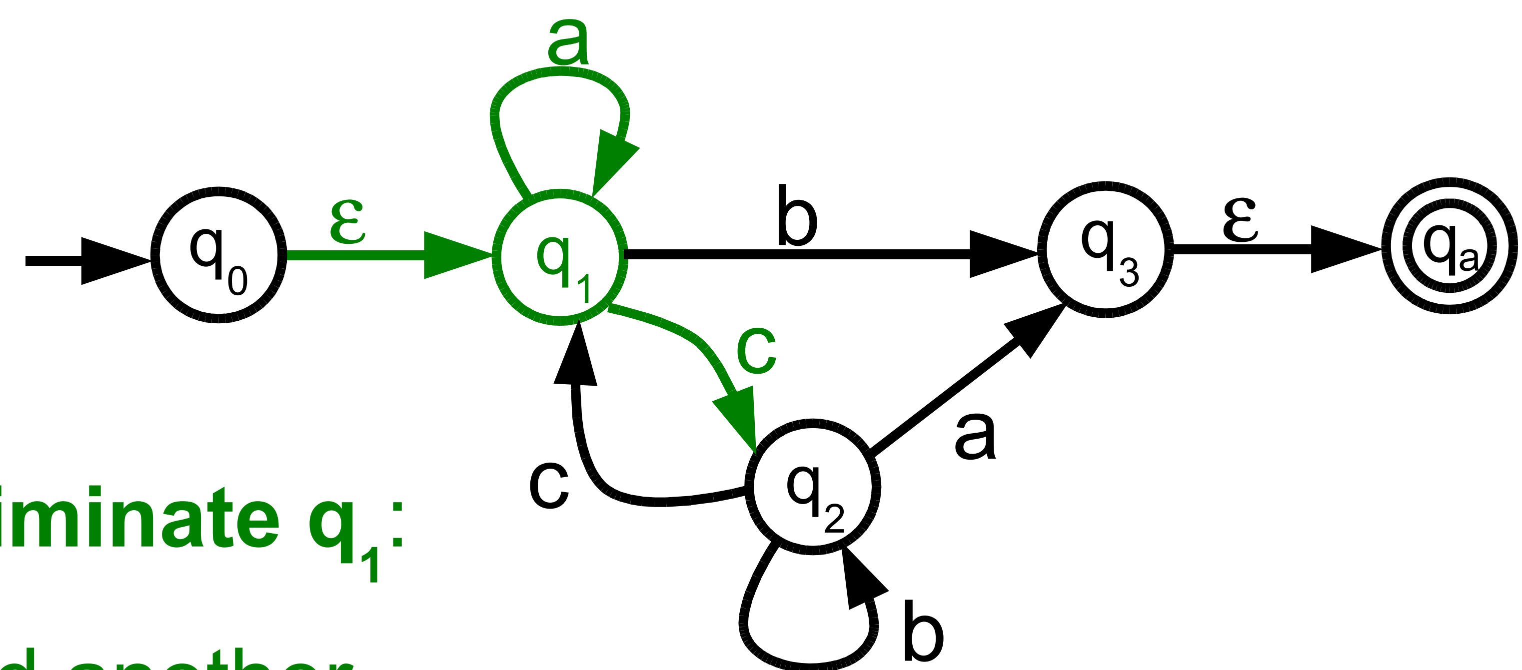
Eliminate q_1:

add edge to

new GNFA

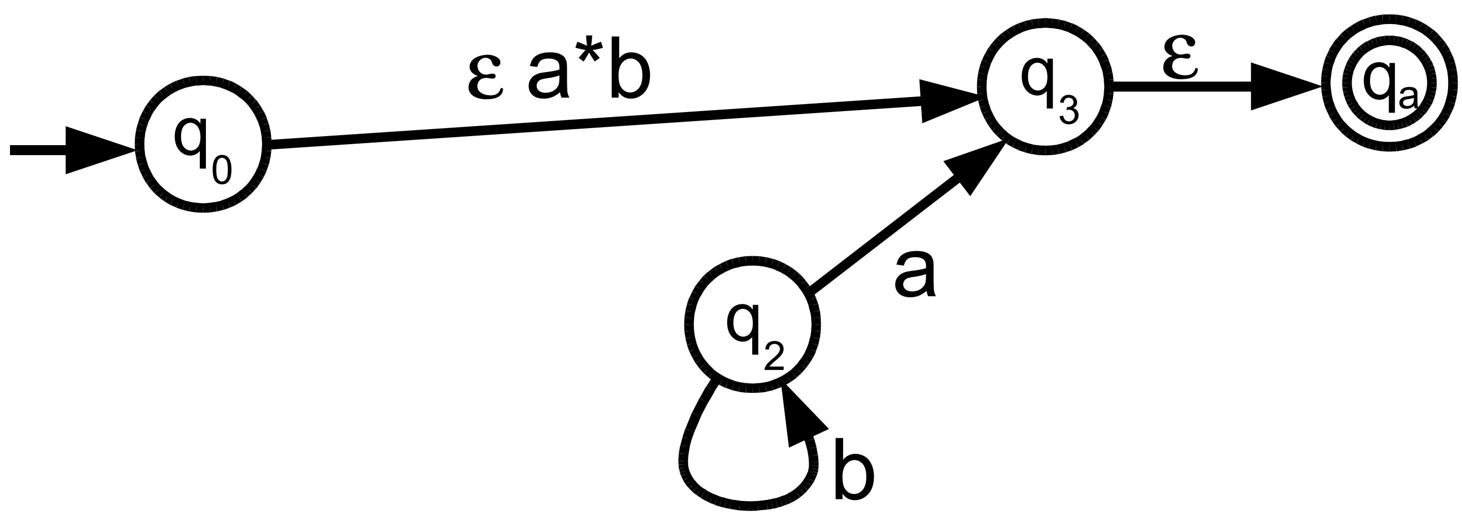


ANOTHER Example: DFA → GNFA → RE

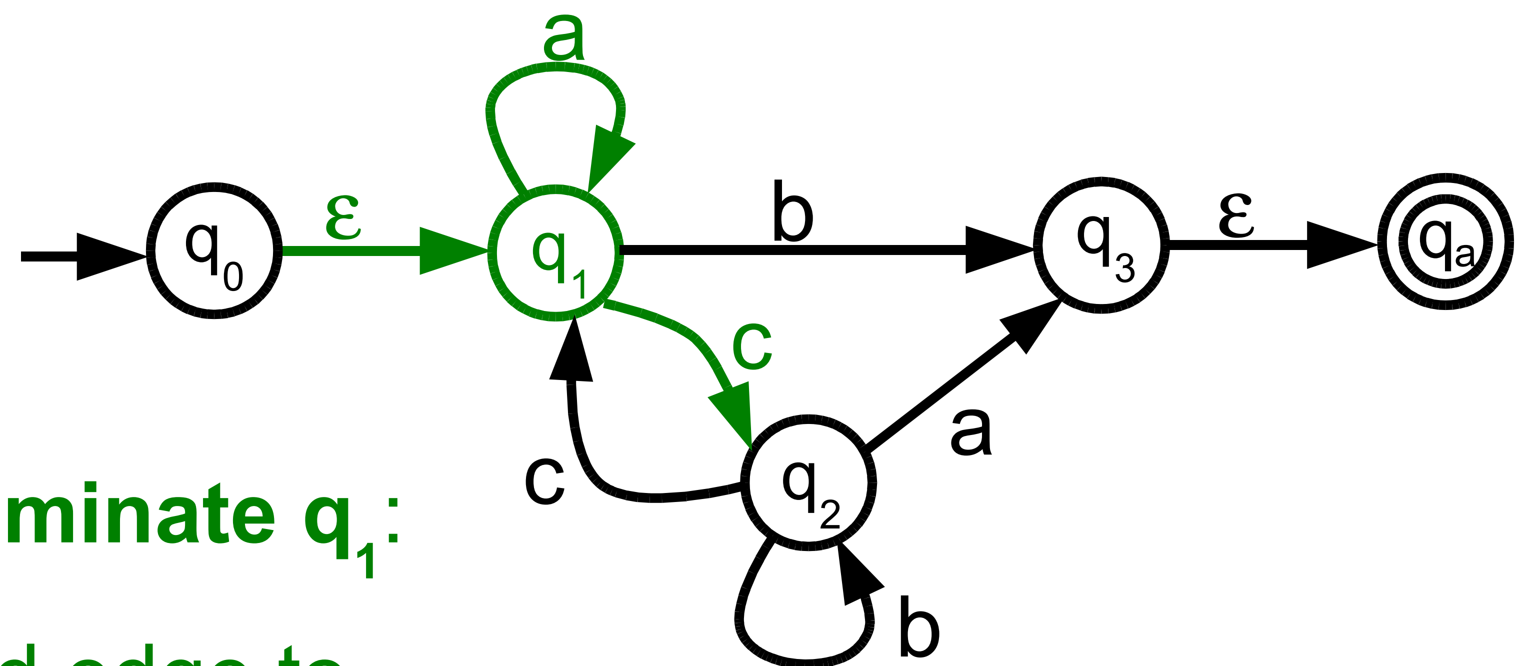


Eliminate q_1:

find another path through q_1



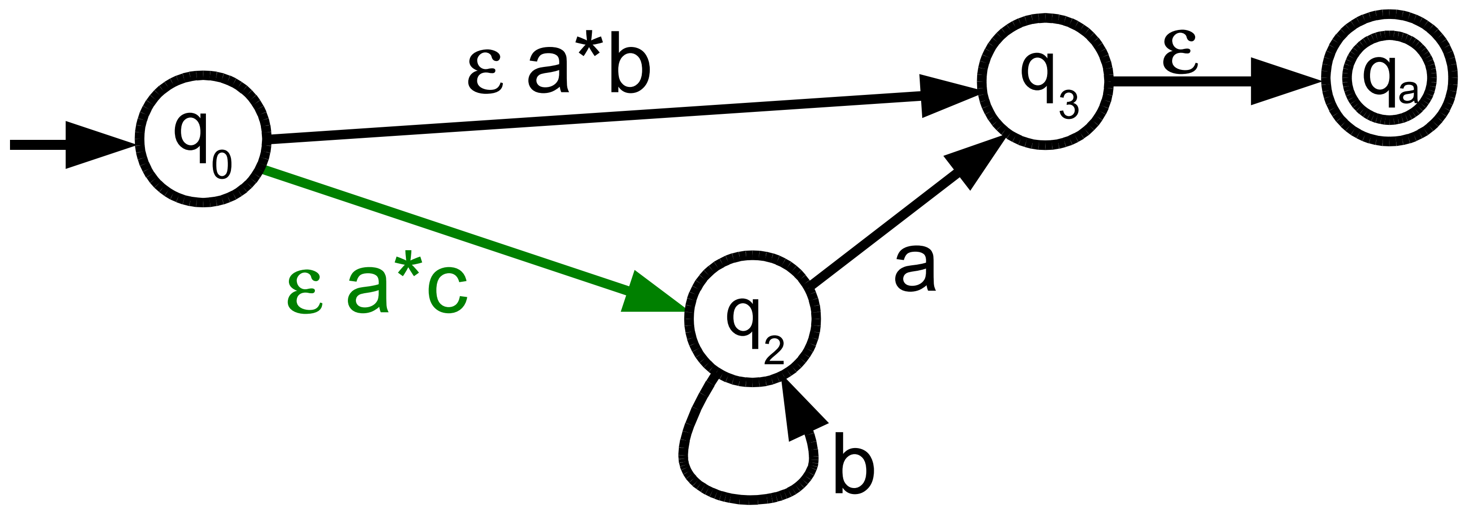
ANOTHER Example: DFA → GNFA → RE



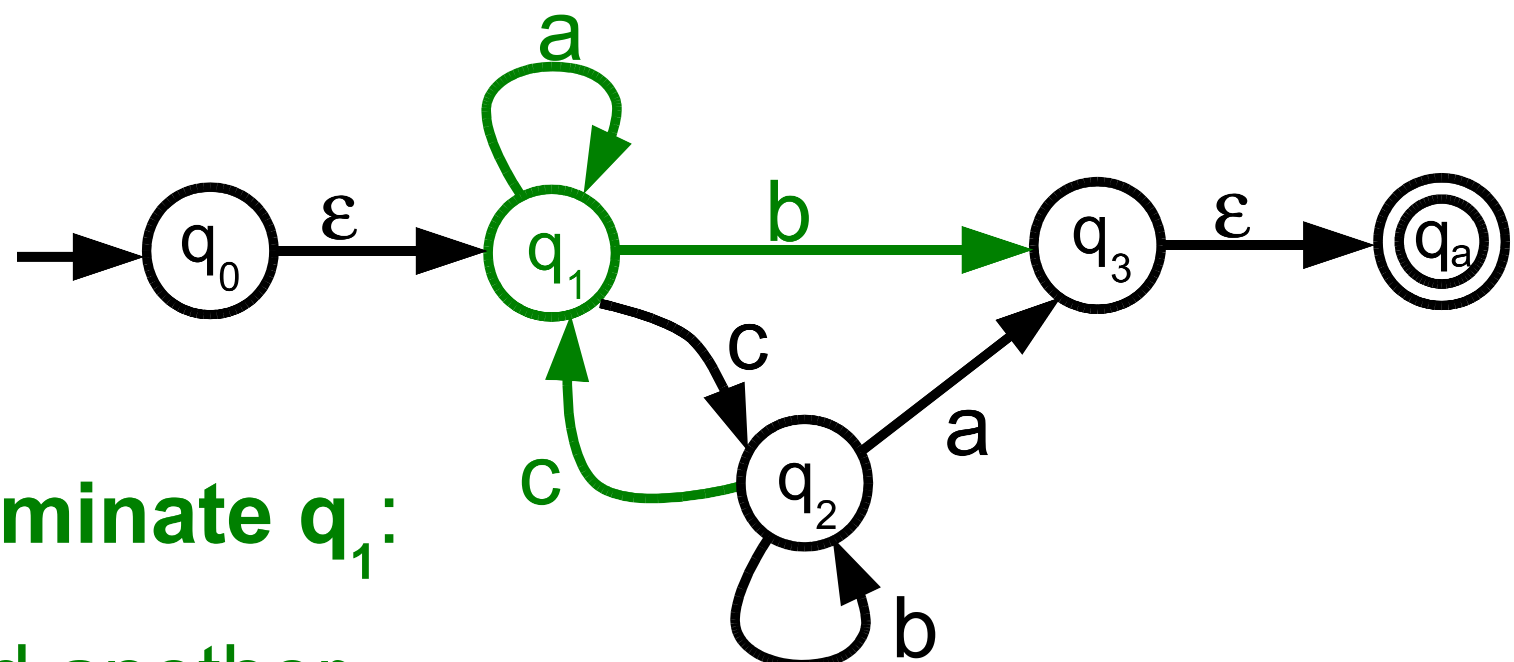
Eliminate q_1:

add edge to

new GNFA

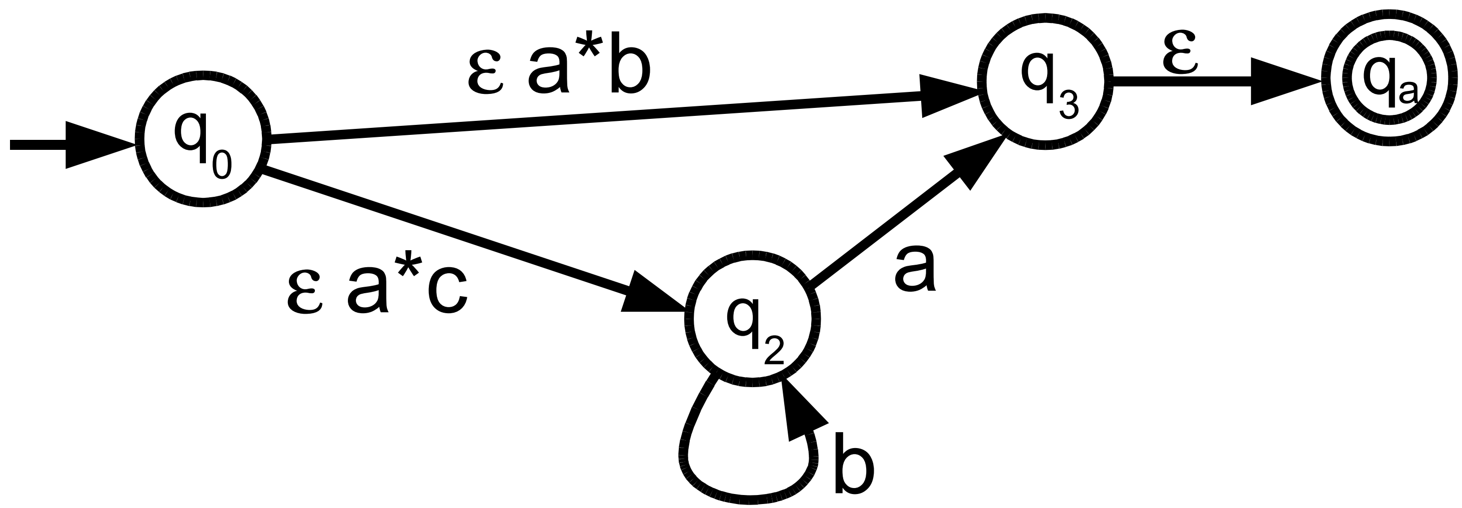


ANOTHER Example: DFA → GNFA → RE

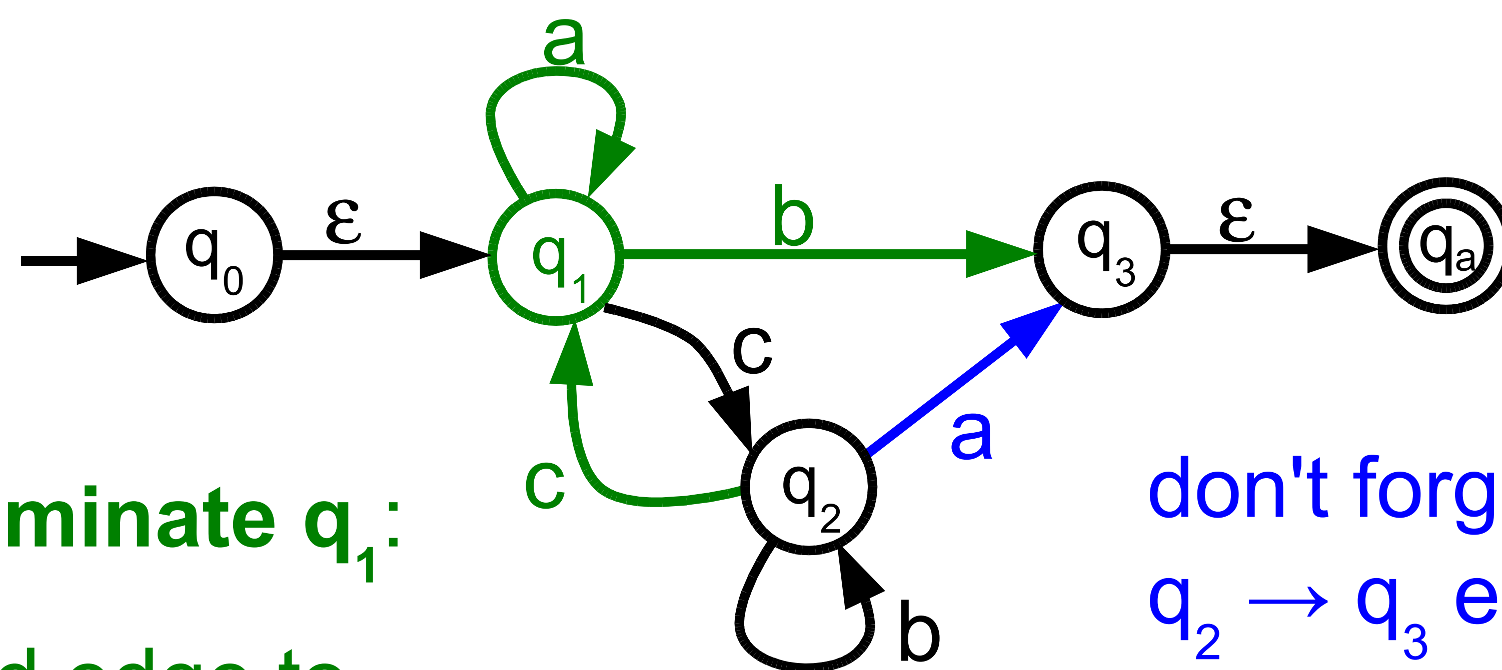


Eliminate q_1:

find another path through q_1



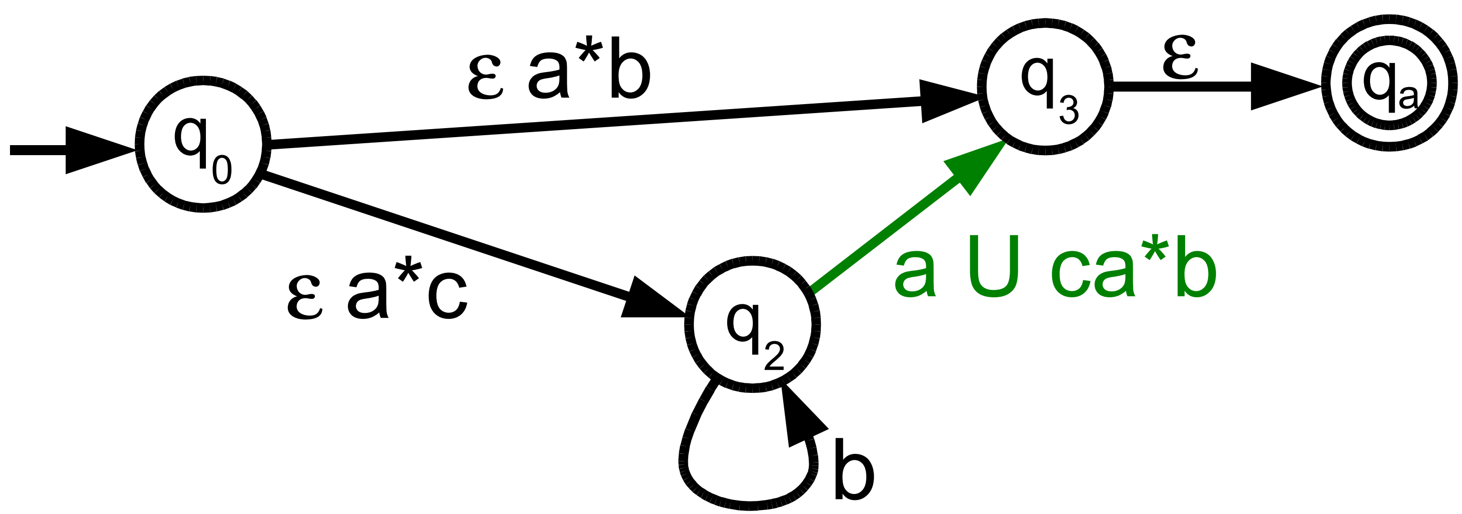
ANOTHER Example: DFA → GNFA → RE



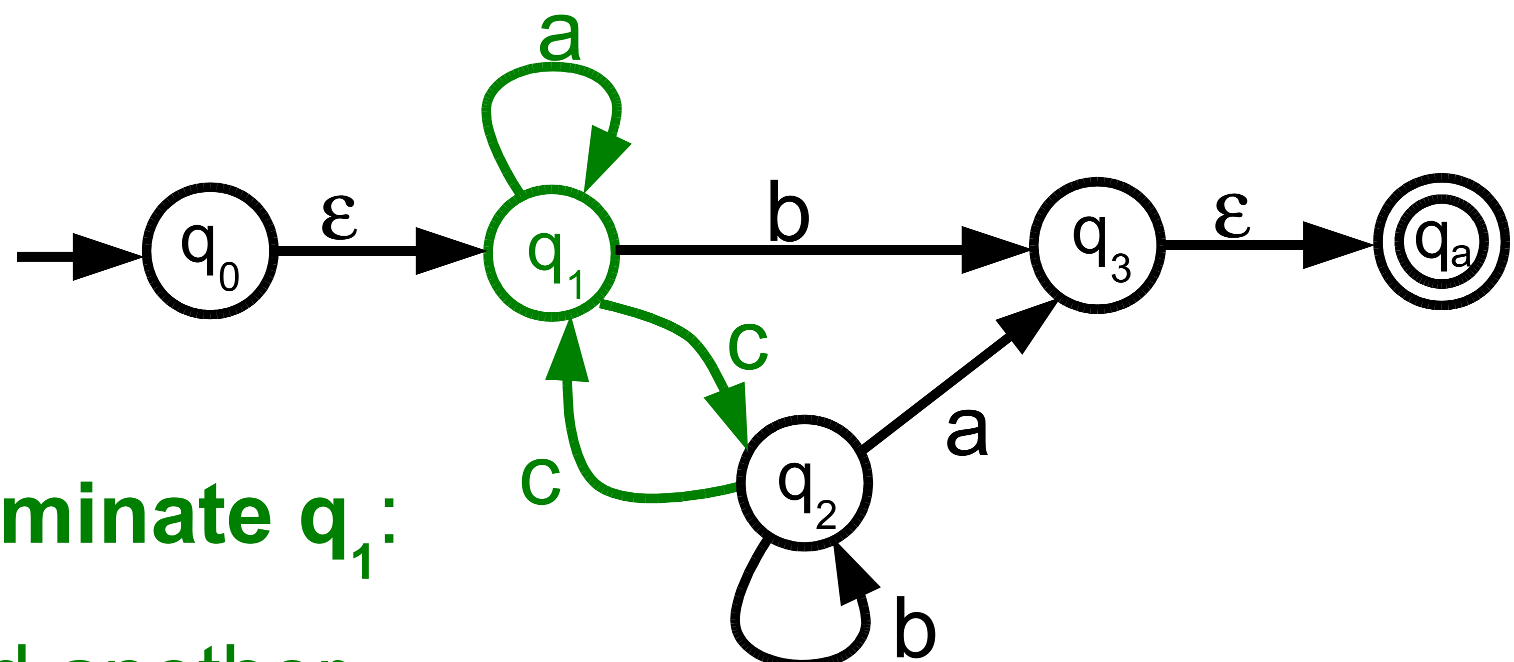
Eliminate q_1:

add edge to new GNFA

don't forget current $q_2 \to q_3$ edge!

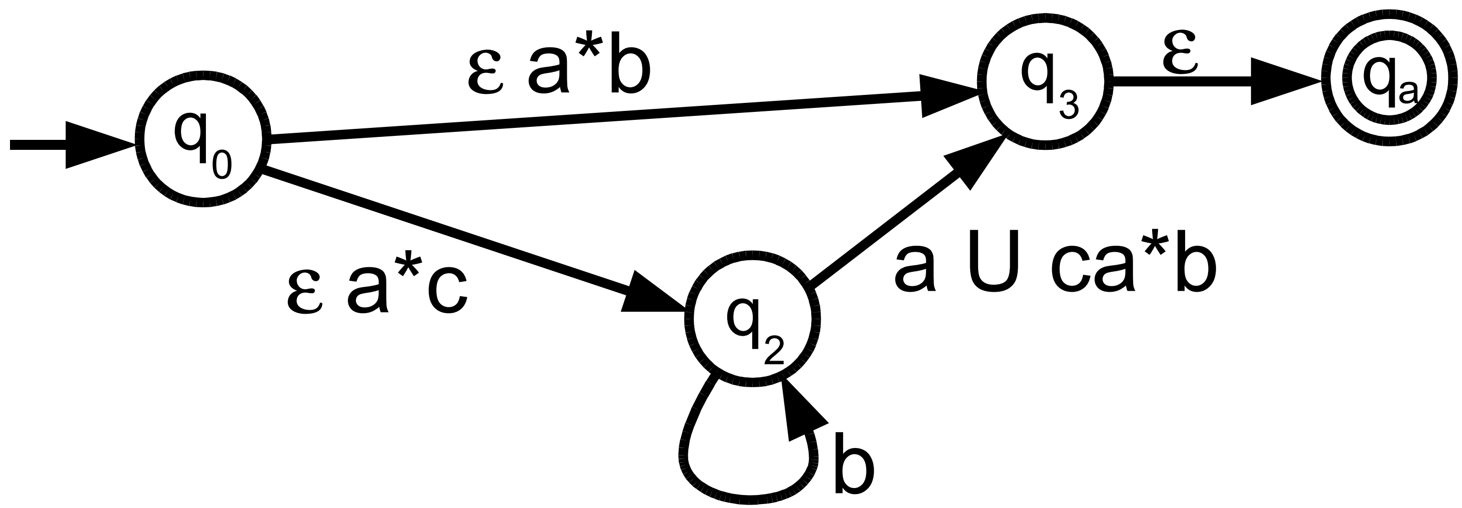


ANOTHER Example: DFA → GNFA → RE

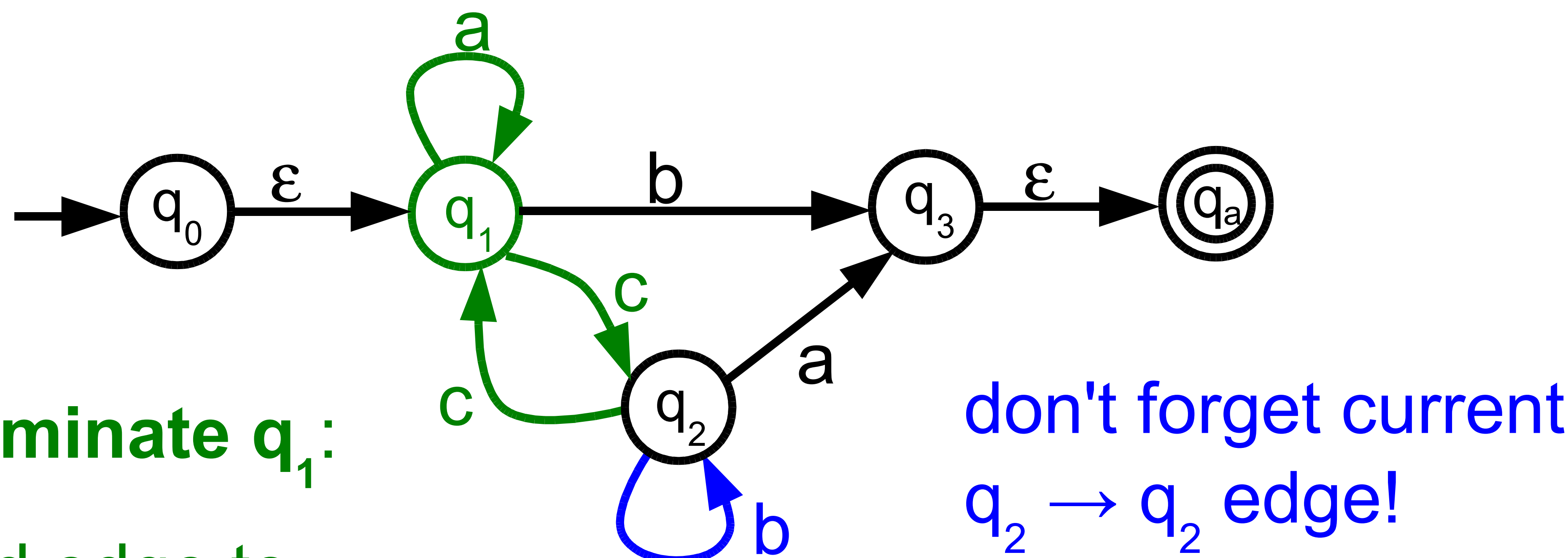


Eliminate q_1:

find another path through q_1



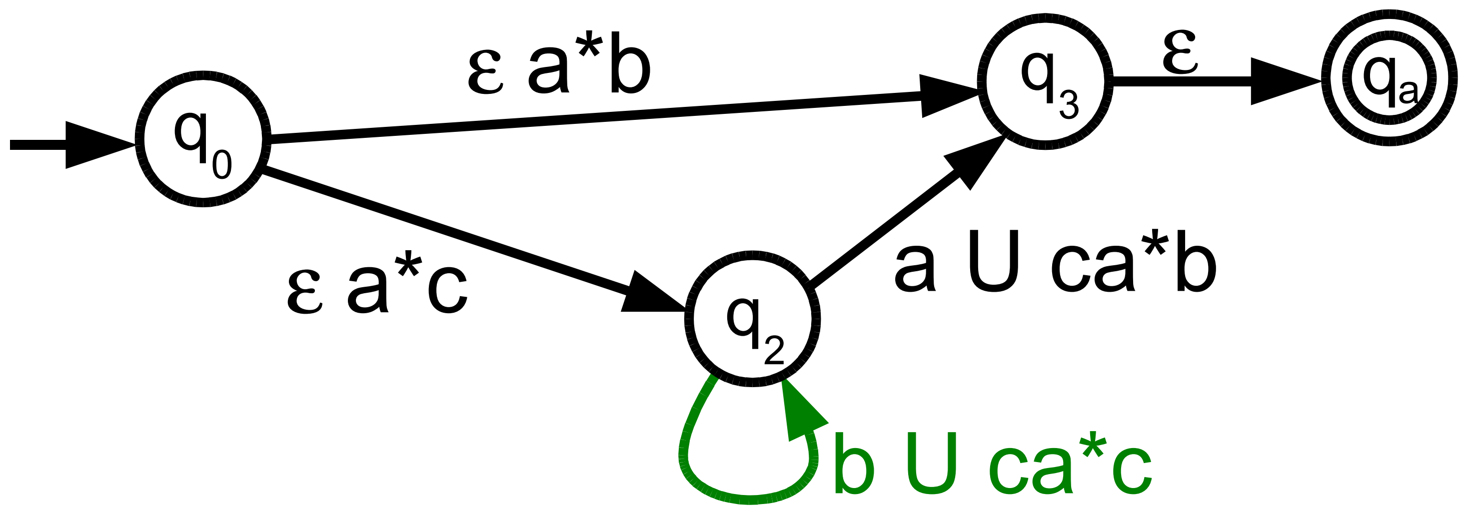
ANOTHER Example: DFA → GNFA → RE



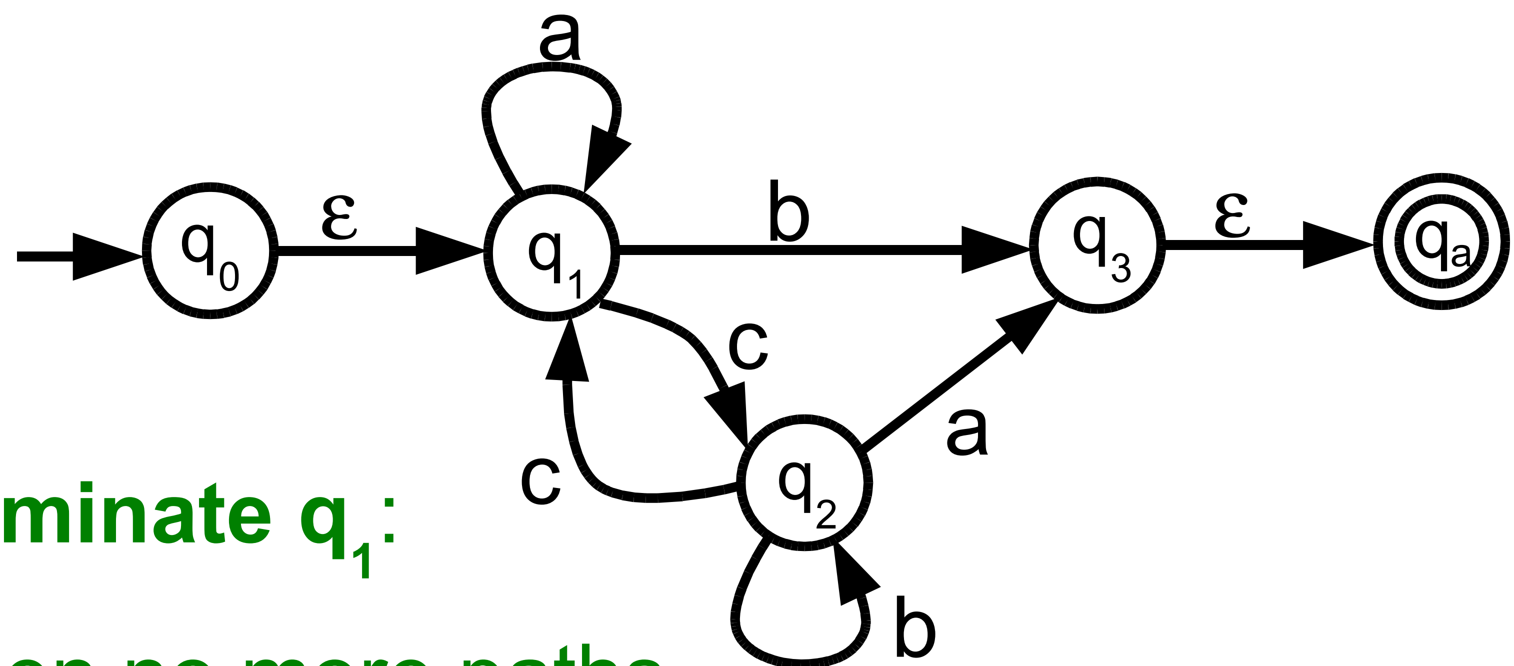
Eliminate q_1:

add edge to

new GNFA



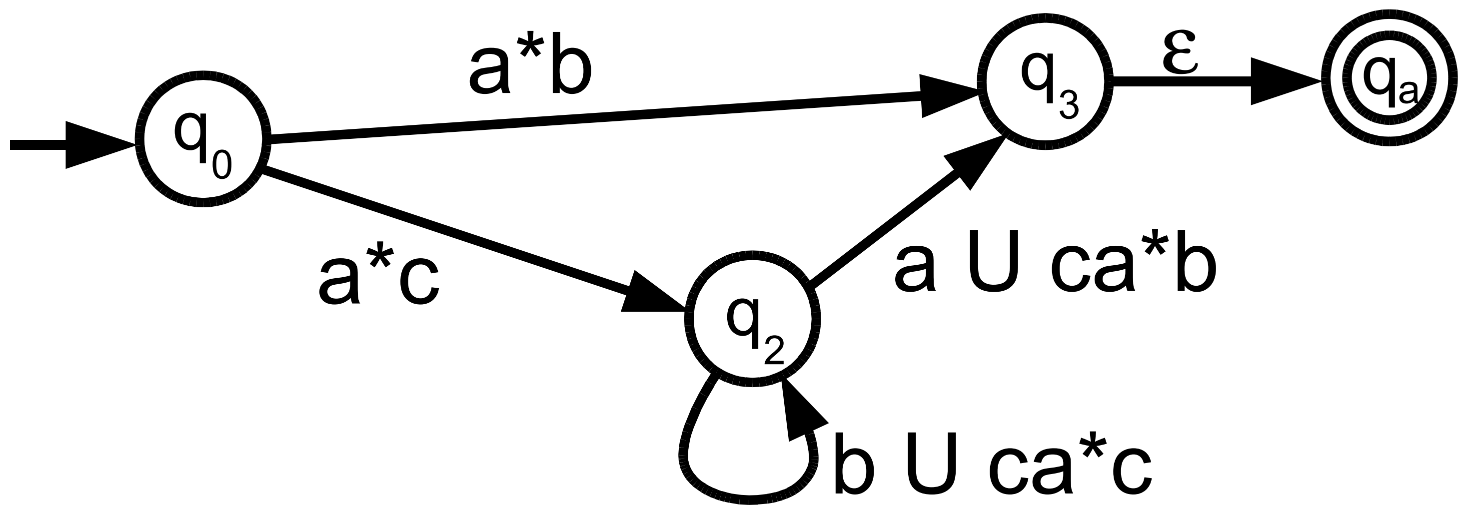
ANOTHER Example: DFA → GNFA → RE



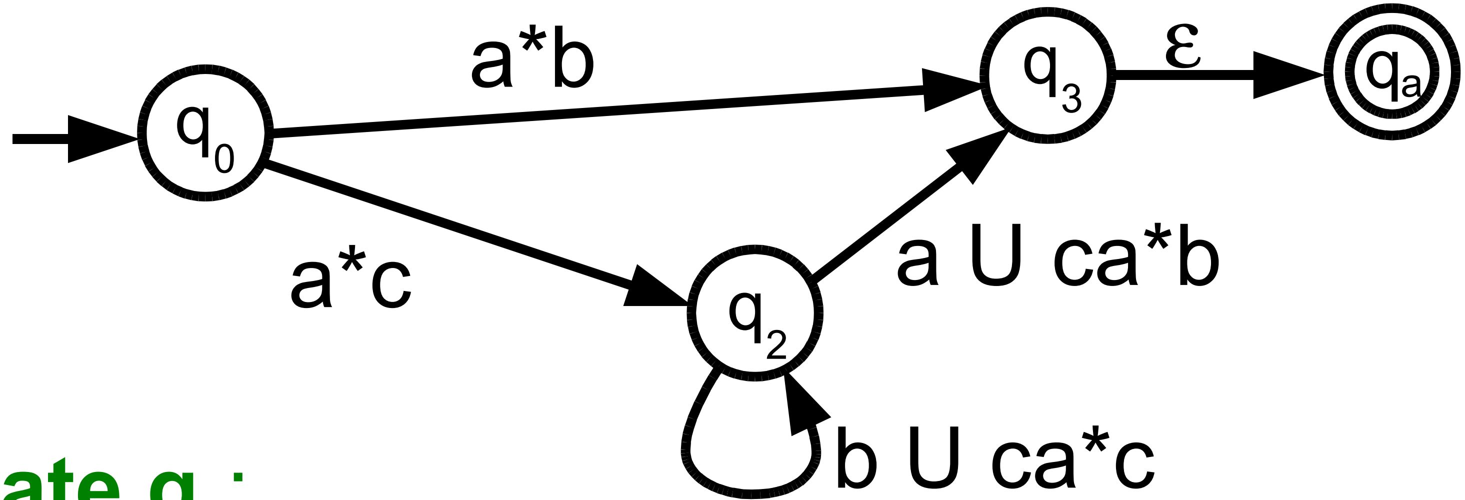
Eliminate q_1:

when no more paths through q_1, start over

(and simplify REs)



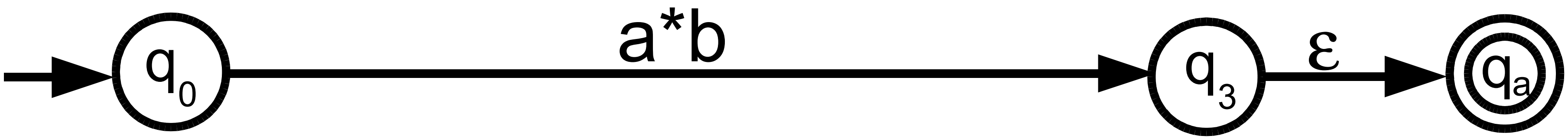
ANOTHER Example: DFA → GNFA → RE



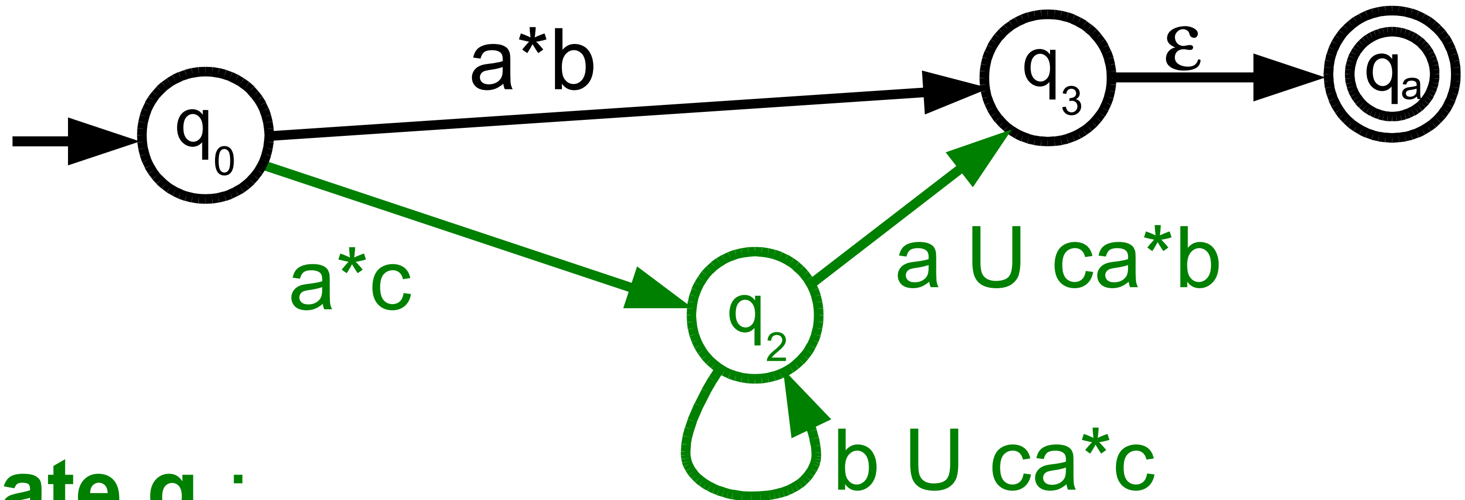
Eliminate q_2:

re-draw GNFA with

all other states

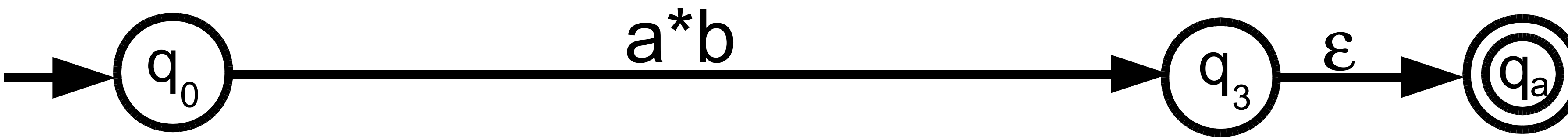


ANOTHER Example: DFA → GNFA → RE

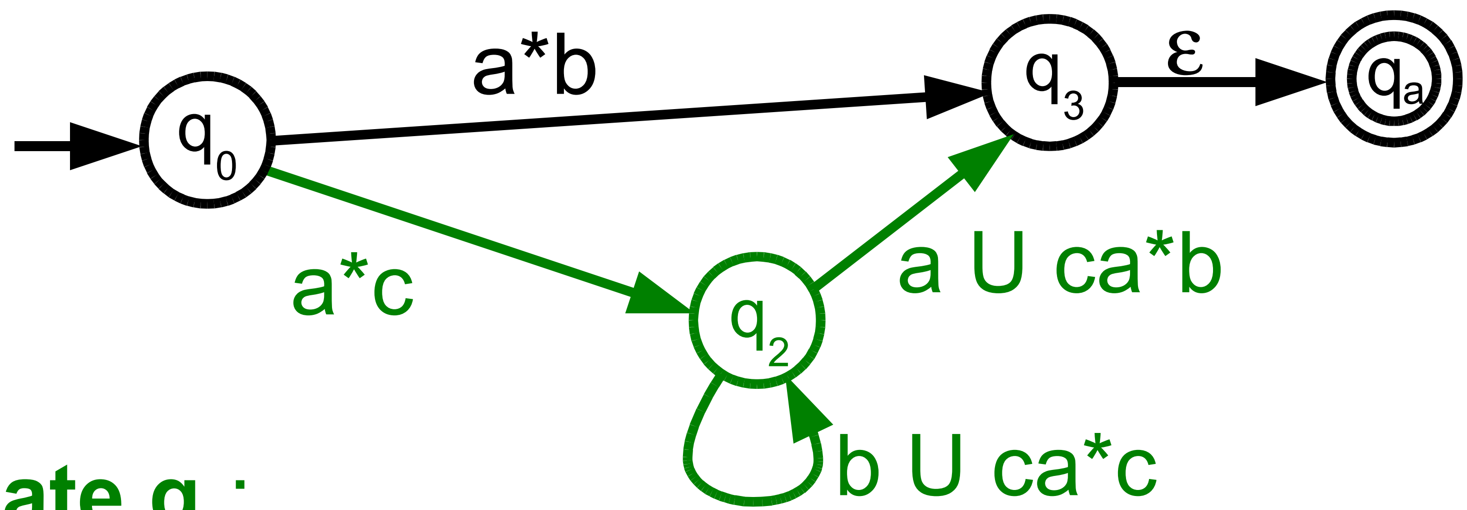


Eliminate q_2:

find a path through q_2

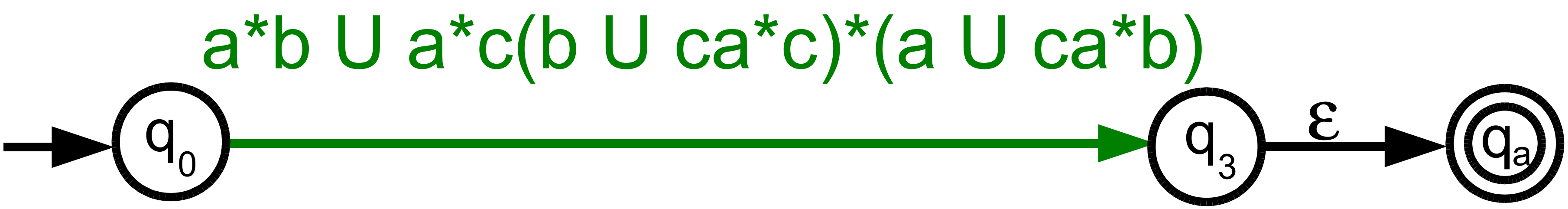


ANOTHER Example: DFA → GNFA → RE

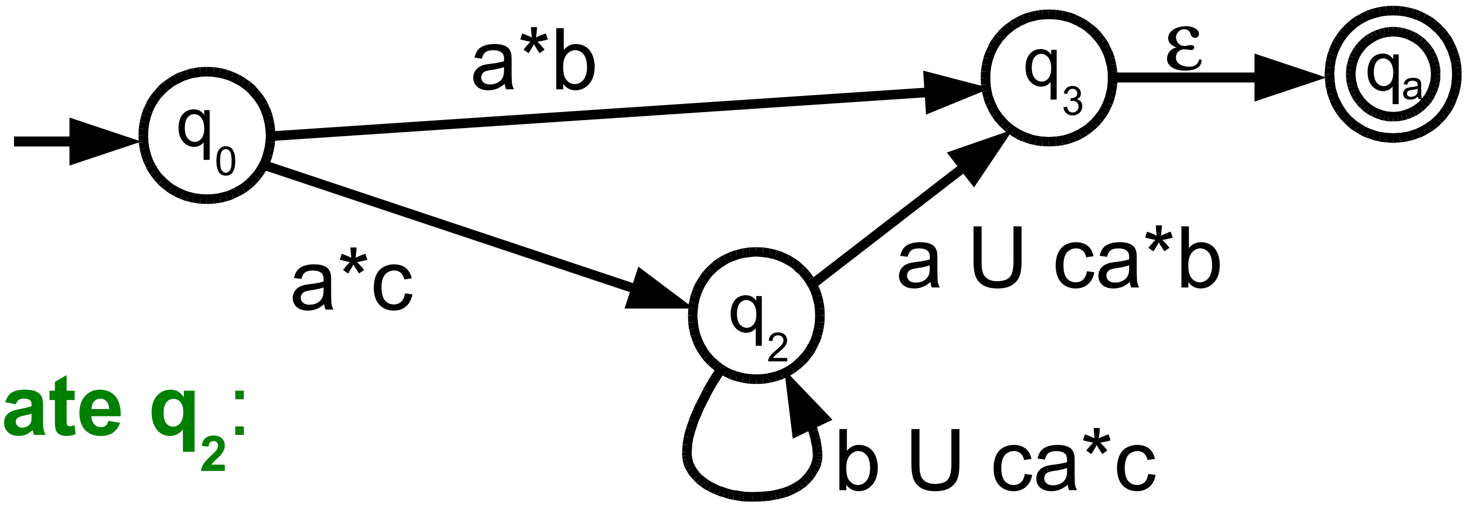


Eliminate q_2:

add edge to new GNFA

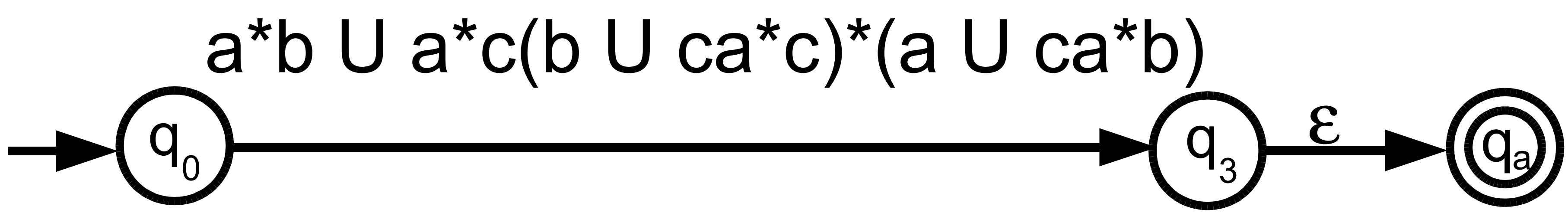


ANOTHER Example: DFA → GNFA → RE

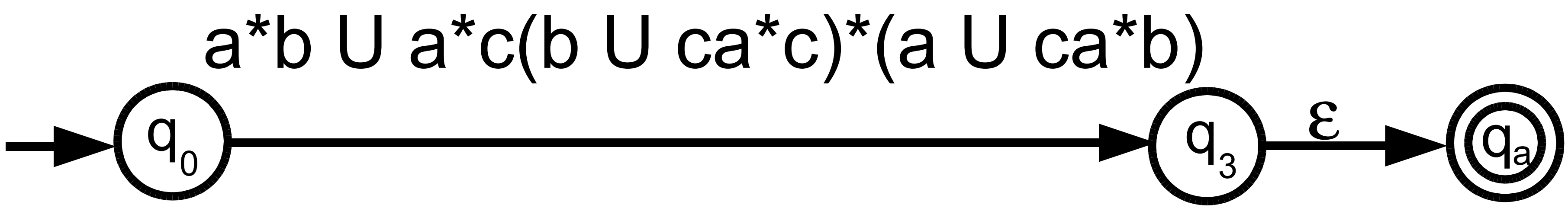


Eliminate q_2:

when no more paths through q_2, start over



ANOTHER Example: DFA → GNFA → RE

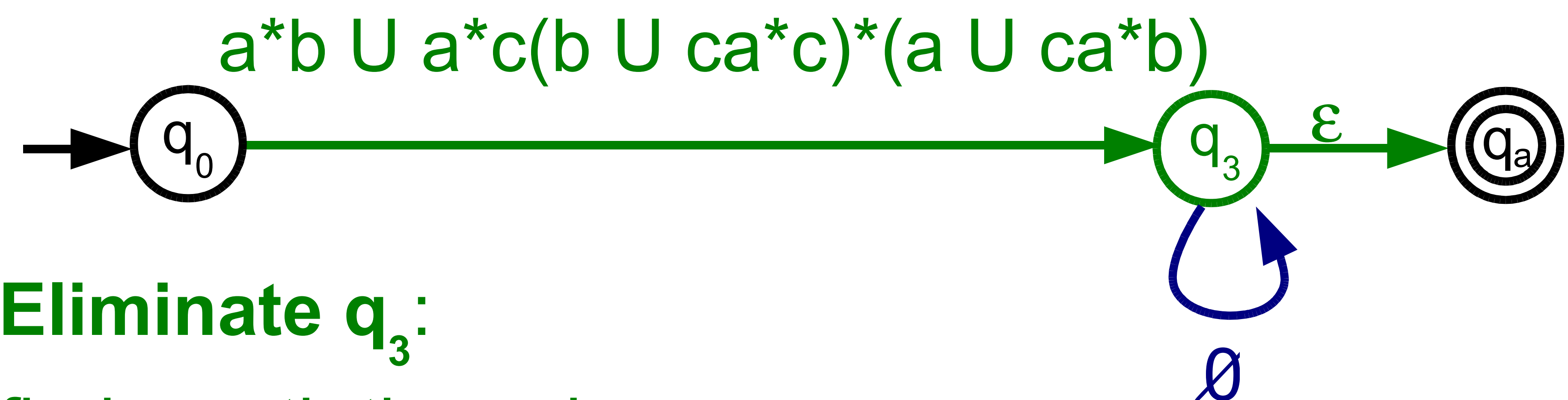


Eliminate q_3:

re-draw GNFA with

all other states

ANOTHER Example: DFA → GNFA → RE

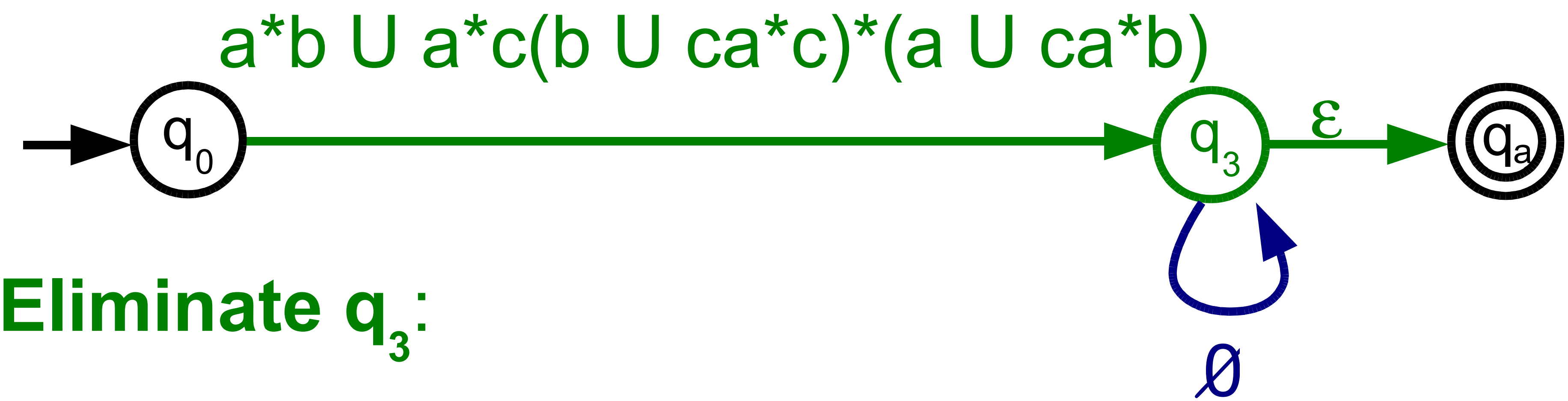


Eliminate q_3:

find a path through q_3

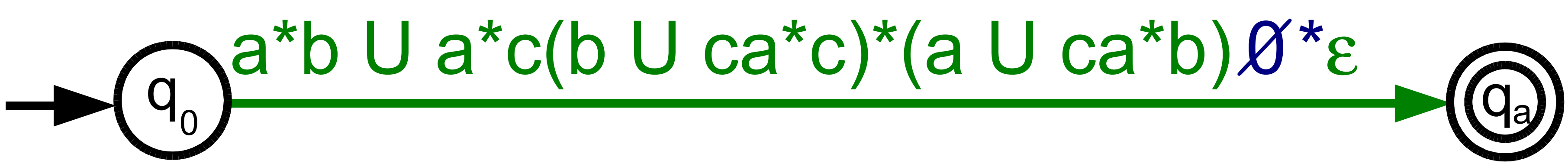
don't forget: no arrow means Ø

ANOTHER Example: DFA → GNFA → RE

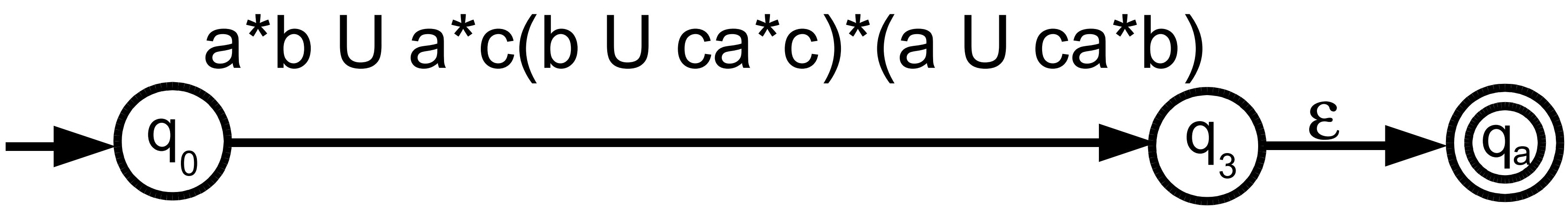


Eliminate q_3:

add edge to new GNFA



ANOTHER Example: DFA → GNFA → RE

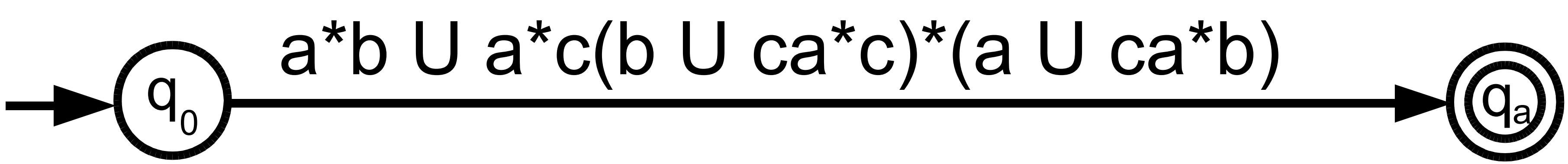


Eliminate q_3:

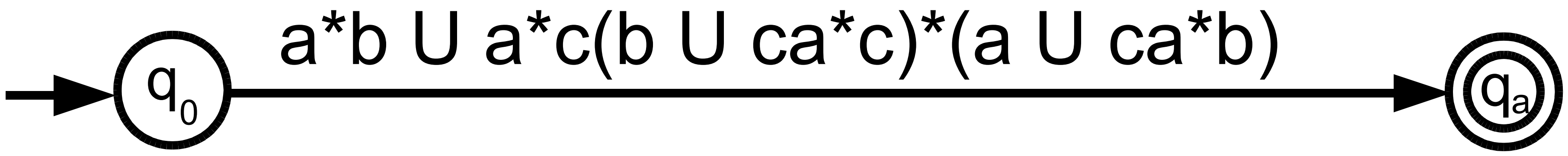
when no more paths through q_3, start over

(and simplify REs)

don't forget: $\emptyset^* = \varepsilon$



ANOTHER Example: DFA → GNFA → RE



Only two states remain:

RE = a*b U a*c(b U ca*c)*(a U ca*b)

Recap:

Here "$\Rightarrow$" means "can be converted to"

$$\text{RE} \Leftrightarrow \text{DFA} \Leftrightarrow \text{NFA}$$

Any of the three recognize exactly

the regular languages (initially defined using DFA)

These conversions are used every time you enter an RE, for example for pattern matching using *grep*

- The RE is converted to an NFA
- Then the NFA is converted to a DFA
- The DFA representation is used to pattern-match

Optimizations have been devised,
but this is still the general approach.

What language is NOT regular?

Is $\{ 0^n 1^n : n \geq 0 \} = \{\varepsilon, 01, 0011, 000111, \dots \}$ regular?

Pumping lemma:

L regular language $\Rightarrow$ $\boxed{\begin{array}{l} \exists\ p \geq 0 \\ \forall\ w \in L,\ |w| \geq p \\ \exists\ x,y,z : w = xyz,\ |y| > 0,\ |xy| \leq p \\ \forall\ i \geq 0 : xy^iz \in L \end{array}}$

Recall $y^0 = \varepsilon$, $y^1 = y$, $y^2 = yy$, $y^3 = yyy$, ...

Pumping lemma:

L regular language $\Rightarrow$ $\boxed{\begin{array}{l} \exists\ p \geq 0 \\ \forall\ w \in L,\ |w| \geq p \\ \exists\ x,y,z : w = xyz,\ |y| > 0,\ |xy| \leq p \\ \forall\ i \geq 0 : xy^iz \in L \end{array}}$

We will not see the proof. But here's the idea:

$p := |Q|$ for DFA recognizing L

If $w \in L$, $|w| \geq p$, then during computation

2 states must be the same $q \in Q$

y = portion of w that brings back to q

can repeat y and still accept string

Pumping lemma:

L regular language $\Rightarrow$ $\boxed{\begin{array}{l} \exists\ p \geq 0 \qquad\qquad\qquad\qquad A \\ \forall\ w \in L, |w| \geq p \\ \exists\ x,y,z : w = xyz, |y| > 0, |xy| \leq p \\ \forall\ i \geq 0 : xy^iz \in L \end{array}}$

Useful to prove L NOT regular. Use contrapositive:

L regular language $\Rightarrow$ A

same as

(not A) $\Rightarrow$ L not regular

Pumping lemma (contrapositive)

$\forall$ p $\geq$0 not A

$\exists$ w $\in$ L, |w| $\geq$ p

$\forall$ x,y,z : w = xyz, |y| > 0, |xy| $\leq$ p

$\exists$ i $\geq$ 0 : $xy^iz \notin$ L

$\Rightarrow$ L not regular

To prove L not regular it is enough to prove not A

Not A is the stuff in the box.

Proving something like

∀ bla ∃ bla ∀ bla ∃ bla bla

means winning a game

Theory is all about winning games!

Example NAME THE BIGGEST NUMBER GAME

- Two players:

  You, Adversary.

- Rules:

  First Adversary says a number.

  Then You say a number.

  You win if your number is bigger.

Can you win this game?

Example NAME THE BIGGEST NUMBER GAME

- Two players:

  You, Adversary.

- Rules:

  First Adversary says a number.

  Then You say a number.

  You win if your number is bigger.

You have winning strategy:

if adversary says x, you say x+1

Example NAME THE BIGGEST NUMBER GAME

- Two players:

  You, Adversary. $\exists$, $\forall$

- Rules:

  First Adversary says a number. $\forall x \exists y : y > x$

  Then You say a number.

  You win if your number is bigger.

You have winning strategy: Claim is true

if adversary says x, you say x+1

Another example:

Theorem: $\forall$ NFA N $\exists$ DFA M : L(M) = L(N)

We already saw a winning strategy for this game

What is it?

Another example:

Theorem: $\forall$ NFA N $\exists$ DFA M : L(M) = L(N)

We already saw a winning strategy for this game

The power set construction.

Games with more moves:

Chess, Checkers, Tic-Tac-Toe

You can win if

$\forall$ move of the Adversary

$\exists$ move You can make

$\forall$ move of the Adversary

$\exists$ move You can make

…

: You checkmate

Pumping lemma (contrapositive)

$\forall$ p $\geq$0
$\exists$ w $\in$ L, |w| $\geq$ p
$\forall$ x,y,z : w = xyz, |y| > 0, |xy| $\leq$ p
$\exists$ i $\geq$ 0 : $xy^iz \notin L$

$\Rightarrow$ L not regular

Rules of the game:

Adversary picks p,

You pick w $\in$ L of length $\geq$ p,

Adversary decomposes w in xyz, where |y| > **0**, |xy|$\leq$p

You pick i $\geq$ 0

Finally, you win if $xy^iz \notin L$

Theorem: $L := \{0^n 1^n : n \geq 0\}$ is not regular

Proof:

$\forall\ p \geq 0$

$\exists\ w \in L, |w| \geq p$

$\forall\ x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\ i \geq 0 : xy^i z \notin L$

Use pumping lemma

Adversary moves p

You move $w := 0^p 1^p$

Adversary moves x,y,z

You move $i := 2$

You must show $xyyz \notin L$:

Since $|xy| \leq p$ and $w = xyz = 0^p 1^p$, y only has 0

So $xyyz = 0^{p + |y|} 1^p$

Since $|y| > 0$, this is not of the form $0^n 1^n$ DONE

Theorem: L := {w : w has as many 0 as 1} not regular

Same Proof:

Use pumping lemma

Adversary moves p

You move w := ?

$\forall$ p $\geq$0

$\exists$ w $\in$ L, $|w| \geq p$

$\forall$ x,y,z : w = xyz, $|y| > 0$, $|xy| \leq p$

$\exists$ i $\geq$ 0 : $xy^iz \notin L$

Theorem: L := {w : w has as many 0 as 1} not regular

Same Proof:

Use pumping lemma

Adversary moves p

You move w := $0^p\ 1^p$

Adversary moves x,y,z

You move i := ?

$\forall$ p $\geq$0

$\exists$ w $\in$ L, |w| $\geq$ p

$\forall$ x,y,z : w = xyz, |y| > 0, |xy| $\leq$ p

$\exists$ i $\geq$ 0 : $xy^iz \notin$ L

Theorem: L := {w : w has as many 0 as 1} not regular

$\forall$ p $\geq$0

$\exists$ w $\in$ L, |w| $\geq$ p

$\forall$ x,y,z : w = xyz, |y| > 0, |xy| $\leq$ p

$\exists$ i $\geq$ 0 : $xy^iz \notin$ L

Same Proof:

Use pumping lemma

Adversary moves p

You move w := 0^p 1^p

Adversary moves x,y,z

You move i := 2

You must show xyyz $\notin$ L:

Since |xy|$\leq$p and w = xyz = 0^p 1^p , y only has 0

So xyyz = ?

Theorem: L := {w : w has as many 0 as 1} not regular

$\forall$ p $\geq$ 0

$\exists$ w $\in$ L, |w| $\geq$ p

$\forall$ x,y,z : w = xyz, |y| > 0, |xy| $\leq$ p

$\exists$ i $\geq$ 0 : $xy^iz \notin$ L

Same Proof:

Use pumping lemma

Adversary moves p

You move w := $0^p\ 1^p$

Adversary moves x,y,z

You move i := 2

You must show xyyz $\notin$ L:

Since |xy|$\leq$p and w = xyz = $0^p\ 1^p$, y only has 0

So xyyz = $0^{p+|y|}\ 1^p$

Since |y| > 0, not as many 0 as 1 DONE

Theorem: $L := \{0^j 1^k : j > k\}$ is not regular

Proof:

Use pumping lemma

Adversary moves p

You move w := ?

$\forall\ p \geq 0$

$\exists\ w \in L, |w| \geq p$

$\forall\ x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Theorem: L := $\{0^j 1^k : j > k\}$ is not regular

Proof:

Use pumping lemma

Adversary moves p

You move w := $0^{p+1} 1^p$

Adversary moves x,y,z

You move i := ?

$\forall\ p \geq 0$

$\exists\ w \in L, |w| \geq p$

$\forall\ x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Theorem: $L := \{0^j 1^k : j > k\}$ is not regular

Proof:

$\forall\ p \geq 0$

$\exists\ w \in L, |w| \geq p$

$\forall\ x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Use pumping lemma

Adversary moves p

You move $w := 0^{p+1} 1^p$

Adversary moves x,y,z

You move $i := 0$

You must show $xz \notin L$:

Since $|xy| \leq p$ and $w = xyz = 0^{p+1} 1^p$, y only has 0

So $xz = 0^{p + 1 - |y|} 1^p$

Since $|y| > 0$, this is not of the form $0^j 1^k$ with $j > k$

Theorem: L := $\{uu : u \in \{0,1\}^* \}$ is not regular

Proof:

Use pumping lemma

Adversary moves p

You move w := ?

$\forall$ p ≥ 0

$\exists$ $w \in L$, $|w| \geq p$

$\forall$ x,y,z : $w = xyz$, $|y| > 0$, $|xy| \leq p$

$\exists$ $i \geq 0$: $xy^iz \notin L$

Theorem: $L := \{uu : u \in \{0,1\}^*\}$ is not regular

Proof:

Use pumping lemma

Adversary moves p

You move $w := 0^p 1\ 0^p\ 1$

Adversary moves x,y,z

You move i := ?

$\forall\ p \geq 0$

$\exists\ w \in L,\ |w| \geq p$

$\forall\ x,y,z : w = xyz,\ |y| > 0,\ |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Theorem: L := {uu : u $\in$ {0,1}* } is not regular

Proof:

$\forall$ p $\geq$0

$\exists$ w $\in$ L, |w| $\geq$ p

$\forall$ x,y,z : w = xyz, |y| > 0, |xy| $\leq$ p

$\exists$ i $\geq$ 0 : $xy^iz \notin$ L

Use pumping lemma

Adversary moves p

You move w := 0^p 1 0^p 1

Adversary moves x,y,z

You move i := 2

You must show xyyz $\notin$ L:

Since |xy|$\leq$p and w = xyz = 0^p 1 0^p 1 , y only has 0

So xyyz = $0^{p + |y|}$ 1 0^p 1

Since |y| > 0, first half of xyyz only 0, so xyyz $\notin$ L

Theorem: $L := \{ 1^{n^2} : n \geq 0 \}$ is not regular

Proof:

Use pumping lemma

Adversary moves p

You move $w :=$?

$\forall\ p \geq 0$

$\exists\ w \in L, |w| \geq p$

$\forall\ x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Theorem: L := { 1^{n^2} : $n \geq 0$ } is not regular

Proof:

Use pumping lemma

Adversary moves p

You move w := 1^{p^2}

Adversary moves x,y,z

You move i := ?

$\forall$ $p \geq 0$

$\exists$ $w \in L$, $|w| \geq p$

$\forall$ $x,y,z : w = xyz$, $|y| > 0$, $|xy| \leq p$

$\exists$ $i \geq 0 : xy^iz \notin L$

Theorem: L := { 1^{n^2} : $n \geq 0$ } is not regular

Proof:

Use pumping lemma

Adversary moves p

You move w := 1^{p^2}

Adversary moves x,y,z

You move i := 2

You must show xyyz $\notin$ L:

Since $|xy| \leq p$, $|xyyz| \leq$?

$\forall$ $p \geq 0$

$\exists$ $w \in L$, $|w| \geq p$

$\forall$ x,y,z : $w = xyz$, $|y| > 0$, $|xy| \leq p$

$\exists$ $i \geq 0$: $xy^iz \notin L$

Theorem: $L := \{ 1^{n^2} : n \geq 0 \}$ is not regular

Proof:

$\forall\ p \geq 0$

$\exists\ w \in L, |w| \geq p$

$\forall\ x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Use pumping lemma

Adversary moves p

You move $w := 1^{p^2}$

Adversary moves x,y,z

You move $i := 2$

You must show $xyyz \notin L$:

Since $|xy| \leq p$, $|xyyz| \leq p^2 + p < (p+1)^2$

Since $|y| > 0$, $|xyyz| >$?

Theorem: $L := \{ 1^{n^2} : n \geq 0 \}$ is not regular

Proof:

$\forall\ p \geq 0$

$\exists\ w \in L,\ |w| \geq p$

$\forall\ x,y,z : w = xyz,\ |y| > 0,\ |xy| \leq p$

$\exists\ i \geq 0 : xy^iz \notin L$

Use pumping lemma

Adversary moves p

You move $w := 1^{p^2}$

Adversary moves x,y,z

You move i := 2

You must show $xyyz \notin L$:

Since $|xy| \leq p$, $|xyyz| \leq p^2 + p < (p+1)^2$

Since $|y| > 0$, $|xyyz| > p^2$

So $|xyyz|$ cannot be … what ?

Theorem: $L := \{ 1^{n^2} : n \geq 0 \}$ is not regular

Proof:

$\forall\, p \geq 0$

$\exists\, w \in L, |w| \geq p$

$\forall\, x,y,z : w = xyz, |y| > 0, |xy| \leq p$

$\exists\, i \geq 0 : xy^iz \notin L$

Use pumping lemma

Adversary moves p

You move $w := 1^{p^2}$

Adversary moves x,y,z

You move $i := 2$

You must show $xyyz \notin L$:

Since $|xy| \leq p$, $|xyyz| \leq p^2 + p < (p+1)^2$

Since $|y| > 0$, $|xyyz| > p^2$

So $|xyyz|$ cannot be a square. $xyyz \notin L$

Big picture

- All languages
- Decidable

  Turing machines
- NP
- P
- Context-free

  Context-free grammars, push-down automata
- Regular

  Automata, non-deterministic automata, regular expressions