Advanced Algorithms, Fall 2009	**Problem Set 4**
Instructor: Emanuele Viola	
Assigned: November 10	Due: November 24 before class

Guidelines If you write "I do not know how to solve this problem" then you get 1/4 of the score for the problem. If you write nonsense then you get 0.

As we are going to learn in this class, *time* and *space* are very valuable resources. Strive to give effective, compact solutions. Your solutions should touch on all the main points, but long calculations or discussions are not required nor sought.

Do not worry if you sometimes "do not get it." The problems are meant to stimulate you, not to overwork you. I need your solutions on paper (not file). To hand them in: give it to me or slide them under my door West Village H (246).

On this problem set you must work on your own

Problem 1. Eigenvalue bound In class we saw a randomized log-space algorithm for connectivity on undirected graphs. For the proof that the algorithm runs in polynomial time we claimed without proof a bound on the second largest eigenvalue of the adjacency matrix of the graph. In this exercise you will prove this bound (except for a few facts related to the existence of certain eigenvalues and eigenvectors).

Let G be a connected, undirected graph with degree $d \leq n$ at every node. Let A be its normalized adjacency matrix. We allow for G to have self-loops and multiple edges. One way to visualize this is to note that $a_{i,j} = B/d$ for the integer number B of edges between i and j. Note each row and each column of A sums to 1. (If self-loops or multiple edges bother you, you can assume that your graph has none, in which case $a_{i,j}$ is either $1/d$ or 0.)

You can assume that there are n real eigenvalues $\lambda_1, \ldots, \lambda_n$ with associated n real eigenvectors $v_1, \ldots, v_n$ such that: $1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \lambda_4 \ldots$ (note the first inequality is strict), the vectors v_i have each length 1 and they are orthogonal (and so pairwise independent), and $Av_i = \lambda_i v_i$.

(1) Prove that: $\lambda_2 = \max_x < Ax, x >$, where the maximum is taken over vectors x of length 1 that are orthogonal to the uniform distribution, i.e., $\sum_i x_i = 0$.

(2) Prove that:

$$1 - \lambda_2 = \frac{1}{2} \min_x \sum_{i,j} a_{i,j}(x_i - x_j)^2,$$

where the minimum is again taken over vectors x of length 1 that are orthogonal to the uniform distribution. Hint: Write $1 =< x, x >$ and use (1).

(3) Assume that G is connected. Prove that $\lambda_2 \leq 1 - 1/n^c$ for a universal constant c.

For the algorithm seen in class one in fact needs $|\lambda_2| \leq 1 - 1/n^c$, but this stronger statement than (3) can be obtained by considering A^2 (you are not required to do this).

Problem 2. CLRS Problem 26-5 Maximum flow by scaling Note: c is integer-valued.

Problem 3. Equivalent duality Use the duality theorem seen in class to obtain the following alternative form of duality. If both the following are satisfiable, then their optima coincide:

Primal: $\min cx$ subject to $Ax = b, x \geq 0$.

Dual: $\max by : A^T y \leq c$.

Problem 4. von Neumann's minmax theorem Let A be an $n \times n$ matrix. Prove:

$$\min_x \max_y yAx = \max_y \min_x yAx,$$

where the maximums and minimums are taken over probability distributions x, y, i.e., $x, y \in R^n$ such that $x_j, y_j \geq 0$, for every j, and $\sum_j x_j = \sum_j y_j = 1$). (We can interpret x and y as probability distributions for two players (i) and (ii) over their set of strategies $\{1, \ldots, n\}$ – x, y are called mixed strategies – and A as a payoff matrix. yAx is then the expected payoff when (i) plays x, and (ii) plays y. In this interpretation the theorem asserts that if (i) has a mixed strategy x that achieves an expected payoff of at most t no matter what (ii) plays, than (ii) has a mixed strategy y that achieves payoff at least t no matter what (i) plays.)

Hint: Rewrite the desired equation so that each side of the equation involves only one min or max (not an alternation as is currently).