System Specification, Verification and Synthesis (SSVS) – CS 4830/7485, Fall 2019

13: Formal Verification: Symbolic Methods

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SYMBOLIC METHODS

Symbolic Methods: Why?

Motivation: attack the state explosion problem.

A seminal paper: *Symbolic model checking: 10^{20} states and beyond.* [Burch et al., 1990].

10^{20} is less than 2^{67}, so far from adequate for real-world systems.

Nevertheless: a great leap forward at that time.

Ken McMillan

Symbolic Representation of State Spaces

Key idea:

Instead of reasoning about individual states, reason about **sets** *of states.*

How do we represent a set of states?

Symbolic representation:

Set = *predicate.*

Set of states = *predicate on state variables.*

Symbolic Representation of Sets of States

Examples:

1. Assume 3 state variables, p, q, r, of type boolean.

$$S_1 : \quad p \vee q$$

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2. Assume 3 state variables, x, i, b, of types real, integer, boolean.

$$S_2 : \qquad x > 0 \wedge (b \rightarrow i \geq 0)$$

How many states are in S_2?

Symbolic Representation of Transition Relations

Symbolic Representation of Transition Relations

Key idea:

> *Use a predicate on* **two copies** *of the state variables: unprimed (current state) + primed (next state).*

If $\vec{x}$ is the vector of state variables, then the transition relation R is a predicate on $\vec{x}$ and $\vec{x}'$:

$$R(\vec{x}, \vec{x}')$$

e.g., for three state variables, x, i, b:

$$R(x, i, b, x', i', b')$$

Symbolic Representation of Transition Relations

Examples:

1. Assume one state variable, p, of type boolean.

$$R_1 : \qquad (p \rightarrow \neg p') \land (\neg p \rightarrow p')$$

Which transition relation does this represent? Is it a relation or a function (deterministic)?

Symbolic Representation of Transition Relations

Examples:

1. Assume one state variable, p, of type boolean.

$$R_1 : \qquad (p \rightarrow \neg p') \wedge (\neg p \rightarrow p')$$

   Which transition relation does this represent? Is it a relation or a function (deterministic)?

2. Assume one state variable, n, of type integer.

$$R_2 : \qquad n' = n + 1 \vee n' = n$$

   Which transition relation does this represent? Is it a relation or a function (deterministic)?

Symbolic Representation of Kripke Structures

Kripke structure:

$$(P, S, S_0, L, R)$$

Symbolic representation:

$$(P, \mathit{Init}, \mathit{Trans})$$

where

- $P = \{x_1, x_2, ..., x_n\}$: set of (boolean) state variables, also taken to be the atomic propositions.[1]
- Predicate $\mathit{Init}(\vec{x})$ on vector $\vec{x} = (x_1, ..., x_n)$ represents the set S_0 of initial states.
- Predicate $\mathit{Trans}(\vec{x}, \vec{x}')$ represents the transition relation R.

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Basis of the language of SMV/NuSMV/NuXMV.

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Example: NuSMV model

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MODULE inverter(input)
VAR
  output : boolean;
INIT
  output = FALSE
TRANS
  next(output) = !input | next(output) = output
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What is the Kripke structure defined by this NuSMV program?

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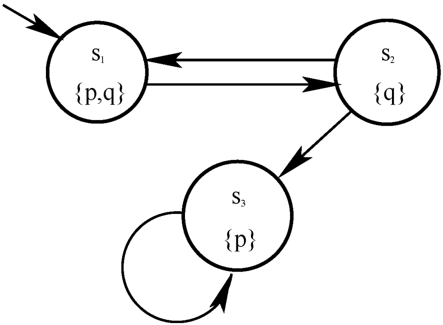
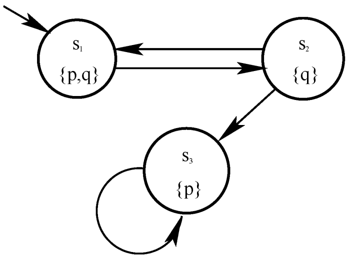
What about P and L?

Example: Kripke Structure



Represent this symbolically.

A subtlety



Transition relation – symbolic representation 1:

$$(s = s_1 \rightarrow s' = s_2) \wedge (s = s_2 \rightarrow (s' = s_1 \vee s' = s_3)) \wedge (s = s_3 \rightarrow s' = s_3)$$

Transition relation – symbolic representation 2:

$$(s = s_1 \wedge s' = s_2) \vee (s = s_2 \wedge (s' = s_1 \vee s' = s_3)) \vee (s = s_3 \wedge s' = s_3)$$

Which one is the right one?

A subtlety: a bit of propositional logic

Consider the two formulas:

$$\phi_1 = (a \to b) \wedge (c \to d)$$

$$\phi_2 = (a \wedge b) \vee (c \wedge d)$$

- Generally, they are not equivalent:
  - $\phi_1 \not\Rightarrow \phi_2$, e.g., when $a = c = 0$.
  - $\phi_2 \not\Rightarrow \phi_1$, e.g., when $a = b = c = 1$, $d = 0$.
- BUT:
  - $\phi_1 \Rightarrow \phi_2$ when $a \vee c$ is valid.
  - ϕ_1 and ϕ_2 are equivalent when both $a \vee c$ and $a \oplus c$ (a XOR c) are valid.

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So, if you cover all the cases for the current state s, and the cases are all mutually exclusive, both forms are equivalent.

SYMBOLIC REACHABILITY ANALYSIS

Recall: Symbolic Representation of Kripke Structures

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- Predicate $Trans(\vec{x}, \vec{x}')$ represents the transition relation R.

Recall: Symbolic Representation

- Set of states = predicate $\phi(\vec{x})$ on vector of state variables $\vec{x}$. E.g.:
  - $Init(x, y, z) : x \wedge \neg y$
  - $Bad(x_1, x_2) : x_1 = crit \wedge x_2 = crit$
- Transition relation = predicate $Trans(\vec{x}, \vec{x}')$ on state variables and next-state variables. E.g.:
  - $Trans(x, y, x', y') : x' = x + 1 \wedge (y' = 0 \vee y' = 1)$

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  - Union of two sets represented by ϕ_1 and ϕ_2:

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  - Complement of a set represented by ϕ:

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  - Complement of a set represented by ϕ: $\neg\phi$.

Symbolic Reachability Analysis

Main idea:

- Start with set of initial states S_0.
- Compute $S_1 := S_0 \cup \{$all 1-step successors of $S_0\} = S_0 \cup \mathbf{post}(S_0)$.
- Compute $S_2 := S_1 \cup \{$all 1-step successors of $S_1\} = S_1 \cup \mathbf{post}(S_1)$.
- ...
- Until $S_{k+1} = S_k$.
- S_k contains all reachable states.

Computing Successors Symbolically

Given a set of states represented as a predicate $\phi(\vec{x})$.

We want to compute a new predicate ϕ', representing the set of **all 1-step successors** of states in $\phi(\vec{x})$.

Predicate Transformer

- Successors can be computed by a **predicate transformer** :

$$\mathbf{succ}\big(\phi(\vec{x})\big) := \big(\exists \vec{x} : \phi(\vec{x}) \wedge \mathit{Trans}(\vec{x}, \vec{x}')\big)[\vec{x}' \rightsquigarrow \vec{x}]$$

  - $\exists \vec{x} : \phi(\vec{x}) \wedge \mathit{Trans}(\vec{x}, \vec{x}')$: successors of states in ϕ
  - $[\vec{x}' \rightsquigarrow \vec{x}]$: renames variables so that resulting predicate is over current state variables

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Example:

$$\begin{array}{rcl} \phi & = & 0 \leq x \leq 5 \\ Trans & = & x \leq x' \leq x + 1 \\ \mathbf{succ}(\phi) & = & (\exists x : 0 \leq x \leq 5 \wedge x \leq x' \leq x + 1)[x' \rightsquigarrow x] \end{array}$$

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Predicate Transformer

$$\mathbf{succ}\big(\phi(\vec{x})\big) := \big(\exists \vec{x} : \phi(\vec{x}) \wedge \mathit{Trans}(\vec{x}, \vec{x}')\big)[\vec{x}' \leadsto \vec{x}]$$

How to do quantifier elimination automatically?

In the case of propositional logic, quantifier elimination is simple. Suppose x is a boolean variable:

$$\exists x : \phi \quad \Leftrightarrow$$

Predicate Transformer

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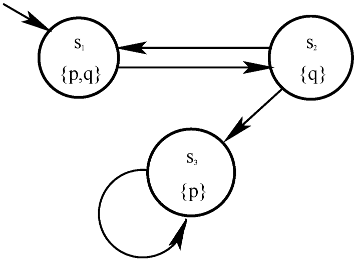
$$\exists x : \phi \quad \Leftrightarrow \quad \phi[x \rightsquigarrow 0] \vee \phi[x \rightsquigarrow 1]$$

where $\phi[x \rightsquigarrow 0]$ is the formula obtained by ϕ after replacing all free occurrences of x by 0 (false), and similarly for $\phi[x \rightsquigarrow 1]$.

Predicate Transformer: Another Example



$$\mathbf{succ}(p \wedge q) \quad = \quad (\exists p, q : p \wedge q \wedge \mathit{Trans})[p' \leadsto p, q' \leadsto q]$$

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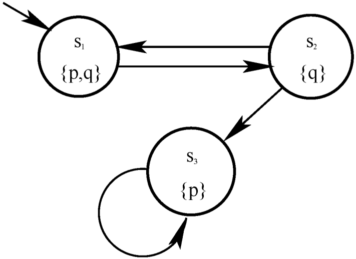

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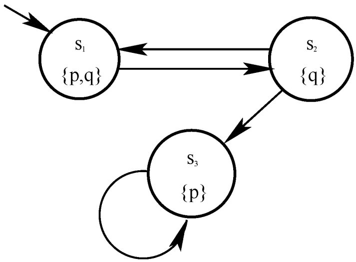

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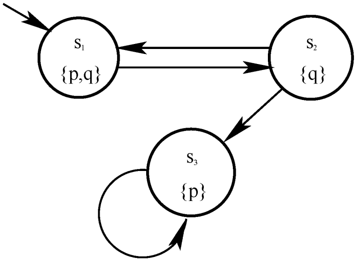

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\end{aligned}
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succ vs post

- **post** takes a set of states and returns a set of states:

$$\mathbf{post} : 2^S \to 2^S$$

where S is the set of states of the transition system.

- **succ** takes a formula and returns a formula:

$$\mathbf{succ} : \text{ Formula } \to \text{ Formula}$$

Symbolic Reachability Analysis Algorithm

$Reachable := Init$;
terminate := false;
repeat
    $tmp := Reachable \vee \mathbf{succ}(Reachable)$;
    if $tmp \Leftrightarrow Reachable$ **then**
        terminate := true;
    else
        $Reachable := tmp$;
    end if
until terminate
return $Reachable$;

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Does the algorithm terminate? Why?

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    if tmp ⇔ Reachable then
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    else
        Reachable := tmp;
    end if
until terminate
return Reachable;
```

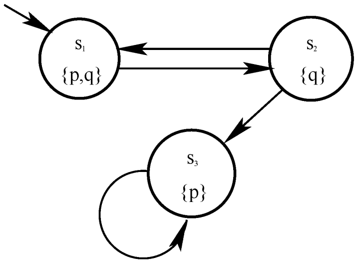
Does the algorithm terminate? Why?

Quiz: modify the algorithm to make it check reachability of a set of bad states characterized by predicate Bad.

Symbolic Reachability Algorithm: checking for Bad states

$Reachable := Init$;
terminate := false;
error := false;
repeat
    $tmp := Reachable \vee \mathbf{succ}(Reachable)$;
    if $tmp \Leftrightarrow Reachable$ **then**
        terminate := true;
    else
        $Reachable := tmp$;
    end if
    if SAT$(Reachable \wedge Bad)$ **then**
        error := true;
    end if
until terminate or error
return ($Reachable$,error);

Symbolic Reachability: Example



Let's check this system symbolically!

We want to check that all reachable states satisfy $p \lor q$.

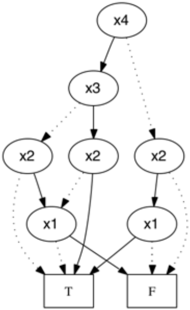
In temporal logic parlance:

$$\text{CTL:} \quad \mathbf{AG}(p \lor q)$$
$$\text{LTL:} \quad \mathbf{G}(p \lor q)$$

Symbolic Model-Checking: Implementation

- For finite-state systems, boolean variables can be used to encode state.
- All predicates then become boolean expressions.
- Efficient data structures for boolean expressions:
  - **BDDs (Binary Decision Diagrams)**
- Efficient algorithms for implementing logical operations (conjunction, disjunction, satisfiability check, ...) on BDDs.
- Note: logical operations correspond to set-theoretic operations:
  - Conjunction: intersection
  - Disjunction: union
  - Satisfiability check: emptiness check
  - ...

Example: BDD



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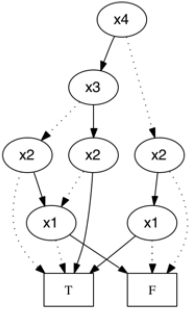
Example: BDD



Can you guess which boolean expression this BDD represents?

$$x_4\big(\overline{x_3}(\overline{x_2} + x_2\overline{x_1}) + x_3(\overline{x_2}\,\overline{x_1} + x_2)\big) + \overline{x_4}x_2x_1$$

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