Substitution

We want to define a notion of substitution so that if we substitute term t for variable x in formula φ, obtaining φ', then φ' says about t what φ says about x.

Substitution is know to be error-prone.

Consider $\varphi = \exists z z + z \equiv x$.

$\langle \mathcal{N}, \beta \rangle \models \varphi$ iff ???.

Replacing x by y gives, $\varphi' = \exists z z + z \equiv y$, where $\langle \mathcal{N}, \beta \rangle \models \varphi$ iff $\beta.y$ is even.

What about replacing x by z? This gives $\varphi' = \exists z z + z \equiv z$, but $\mathcal{N} \models \varphi$, so here we have a problem.

In order to get a φ' which expresses about z what φ expresses about x, we can first replace bound occurences of z by a new variable u in φ, and then proceed as before.

Substitution for Terms

1. $x\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = x \text{if } x \neq x_0, \ldots, x \neq x_r$

2. $x\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = t_i \text{if } x = x_i$

3. $c\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = c$

4. $[ft'_1 \ldots t'_n]\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = ft'_1\frac{t_0 \ldots t_r}{x_0 \ldots x_r} \ldots t'_n\frac{t_0 \ldots t_r}{x_0 \ldots x_r}$

Substitution for Formulas

1. $[t'_1 \equiv t'_2]\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = t'_1\frac{t_0 \ldots t_r}{x_0 \ldots x_r} \equiv t'_2\frac{t_0 \ldots t_r}{x_0 \ldots x_r}$
2. $[Rt'_1 \ldots t'_n]\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = Rt'_1\frac{t_0 \ldots t_r}{x_0 \ldots x_r} \ldots t'_n\frac{t_0 \ldots t_r}{x_0 \ldots x_r}$
3. $[\neg\varphi]\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = \neg[\varphi\frac{t_0 \ldots t_r}{x_0 \ldots x_r}]$
4. $(\varphi \vee \psi)\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = (\varphi\frac{t_0 \ldots t_r}{x_0 \ldots x_r} \vee \psi\frac{t_0 \ldots t_r}{x_0 \ldots x_r})$
5. Suppose $x_{i_1}, \ldots, x_{i_s} (i_1 < \cdots < i_s)$ are exactly the variables x_i among the $x_0, \ldots, x_r$ such that $x_i \in \mathit{free}(\exists x \varphi)$ and $x_i \neq t_i$ Then, set

$$[\exists x \varphi]\frac{t_0 \ldots t_r}{x_0 \ldots x_r} = \exists u[\varphi\frac{t_{i_1} \ldots t_{i_s} u}{x_{i_1} \ldots x_{i_s} x}],$$

where u is x if x does not occur in $t_{i_1} \ldots t_{i_s}$; otherwise u is the first variable in the list $v_0, v_1, v_2, \ldots$ which does not occur in $\varphi, t_{i_1} \ldots t_{i_s}$.

Substitution Examples

1. $[Pv_0fv_1v_2]\frac{v_2v_0v_1}{v_1v_2v_3} = ?$

2. $[\exists v_0Pv_0fv_1v_2]\frac{v_4fv_1v_1}{v_0v_2} = ?$

3. $[\exists v_0Pv_0fv_1v_2]\frac{v_0v_2v_4}{v_1v_2v_0} = ?$

Substitution Lemma

Let $\mathcal{J} = \langle \mathbf{U}, \beta \rangle$ with $a_0, \ldots, a_r \in A$. Then:

$\beta\frac{a_0 \ldots a_r}{x_0 \ldots x_r}(y) = \beta.y$ if $y \neq x_0, \ldots, y \neq x_r$

$\beta\frac{a_0 \ldots a_r}{x_0 \ldots x_r}(y) = a_i$ if $y = x_i$

$\mathcal{J}\frac{a_0 \ldots a_r}{x_0 \ldots x_r} = \langle \mathbf{U}, \beta\frac{a_0 \ldots a_r}{x_0 \ldots x_r} \rangle$

Here then is the main result about substitution.

Lemma 1 *1. For every term* t, $\mathcal{J}(t\frac{t_0 \ldots t_r}{x_0 \ldots x_r}) = \mathcal{J}\frac{\mathcal{J}(t_0) \ldots \mathcal{J}(t_r)}{x_0 \ldots x_r}(t)$

2. For every formula φ, $\mathcal{J} \models \varphi\frac{t_0 \ldots t_r}{x_0 \ldots x_r}$ *iff* $\mathcal{J}\frac{\mathcal{J}(t_0) \ldots \mathcal{J}(t_r)}{x_0 \ldots x_r} \models \varphi$

Proof By induction on terms and formulas. □

Formalizations

Your book contains examples of simple formalizations; let's look at one. How would you formalize an equivalence relation?

1. $\forall x Rxx$
2. $\forall x \forall y (Rxy) \rightarrow (Ryx)$
3. $\forall x \forall y \forall z ((Rxy \quad \wedge \quad Ryz) \rightarrow Rxz)$

The way I would write them is:

1. $\langle \forall x :: xRx \rangle$
2. $\langle \forall x, y :: xRy \Rightarrow yRx \rangle$
3. $\langle \forall x, y, z :: xRy \wedge yRz \quad \Rightarrow \quad xRz \rangle$

Define a new quantifier "there exists exactly one", written $\exists^{=1} x \varphi$.

Uses of the Coincidence Lemma

Definition 1 *Let S and S' be symbol sets with $S \subseteq S'$; let $\mathbf{U} = \langle A, \mathbf{a} \rangle$ be an S-structure and $\mathbf{U}' = \langle A', \mathbf{a}' \rangle$ be an S'-structure. We call $\mathbf{U}$ the S-*reduct *of $\mathbf{U}'$ (and conversely $\mathbf{U}'$ an expansion of $\mathbf{U}$) iff $A = A'$ and $\mathbf{a}$ and $\mathbf{a}'$ agree on S. We write $\mathbf{U} = \mathbf{U}'|_S$.*

For example, the ordered group $\mathbb{Z}^<$ of integers as an $S_{gr}^<$-structure, written $\langle \mathbb{Z}, +, 0, < \rangle$, is an expansion of the group of integers as an S_{gr}-structure, written $\langle \mathbb{Z}, +, 0 \rangle$.

If $\mathbf{U} = \mathbf{U}'|_S$, then for every S-formula, φ, whose free variables are among $v_0, \ldots, v_{n-1}$, and for all $a_0, \ldots, a_{n-1} \in A$,

$$\mathbf{U} \models \varphi[a_0, \ldots, a_{n-1}] \text{ iff } \mathbf{U}' \models \varphi[a_0, \ldots, a_{n-1}]$$

Proof Apply the coincidence lemma.

Isomorphism

Definition 2 *Let* $\mathbf{U}$ *and* $\mathcal{B}$ *be* S*-structures.*

1. $\pi : A \to B$ *is called an* isomorphism *of* $\mathbf{U}$ *onto* $\mathcal{B}$ $(\pi : \mathbf{U} \cong \mathcal{B})$ *iff*
   (a) π *is a bijection of* A *onto* B.
   (b) *For* n*-ary* $R \in S$ *and* $a_1, \ldots, a_n \in A$: $R^{\mathbf{U}} a_1 \ldots a_n$ *iff* $R^{\mathcal{B}} \pi.a_1 \ldots \pi.a_n$.
   (c) *For* n*-ary* $f \in S$ *and* $a_1, \ldots, a_n \in A$:
   $\pi(f^{\mathbf{U}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\pi.a_1, \ldots, \pi.a_n)$.
   (d) *For* $c \in S$, $\pi(c^{\mathbf{U}}) = c^{\mathcal{B}}$.
2. $\mathbf{U}$ *and* $\mathcal{B}$ *are* isomorphic *iff there is an isomorphism* $\pi : \mathbf{U} \cong \mathcal{B}$.

For example, the S_{gr}-structure $\langle \mathbb{N}, +, 0 \rangle$ (not a group) is isomorphic to the S_{gr}-structure $\langle E, +^E, 0 \rangle$, consisting of the even numbers and addition on them.

Give a suitable π.

Isomorphism

Lemma 2 *If* $\mathbf{U}$ *and* $\mathcal{B}$ *are isomorphic* S*-structures then for all* S*-sentences* φ, $\mathbf{U} \models \varphi$ *iff* $\mathcal{B} \models \varphi$.

Corollary 1 *If* $\pi : \mathbf{U} \cong \mathcal{B}$ *then for* $\varphi \in L_n^S$ *and* $a_0, \ldots, a_{n-1} \in A$, $\mathbf{U} \models \varphi[a_0, \ldots, a_{n-1}]$ *iff* $\mathcal{B} \models \varphi[\pi.a_0, \ldots, \pi.a_{n-1}]$

Note that isomorphic structures cannot be distinguished in L_0^S.

Is it the case that S-structures which satisfy the same S-sentences are isomorphic?

This is a major question and we will see that the answer is ???

Substructures

Definition 3 *Let* $\mathbf{U}$ *and* $\mathcal{B}$ *be* S*-structures.* $\mathbf{U}$ *is called a* substructure *of* $\mathcal{B}$ *(written* $\mathbf{U} \subset \mathcal{B}$*) if:*

1. $A \subseteq B$

2. (a) For n-ary $R \in S$, $R^{\mathbf{U}} = R^{\mathcal{B}} \cap A^n$.
   (b) For n-ary $f \in S$, $f^{\mathbf{U}} = f^{\mathcal{B}}|_{A^n}$ (the restriction of $f^{\mathcal{B}}$ to A^n).
   (c) For $c \in S$, $c^{\mathbf{U}} = c^{\mathcal{B}}$.

For example $\langle \mathbb{Z}, +, 0 \rangle$ is a substructure of $\langle \mathbb{Q}, +, 0 \rangle$.

If $\mathbf{U} \subset \mathcal{B}$, then A is S-*closed* in $\mathcal{B}$. Conversely, every $X \subseteq B$ which is S-closed in $\mathcal{B}$ is the domain of exactly one substructure of $\mathcal{B}$ ($[X]^{\mathcal{B}}$).

For example, $\{2n | n \in \mathbb{N}\}$ is S_{gr}-closed, but $\{2n+1 | n \in \mathbb{N}\}$ is not.

Substructures

For arbitrary X, we obtain $[X]^{\mathcal{B}}$ by starting with X and applying the functions in the symbol set of $\mathcal{B}$ over and over until a fixed point is reached.

Lemma 3 *Let* $\mathbf{U}$ *and* $\mathcal{B}$ *be* S*-structures with* $\mathbf{U} \subset \mathcal{B}$ *and let* $\beta : \{v_n | n \in \mathbb{N}\} \to A$ *be an assignment in* $\mathbf{U}$*. Then the following holds for every* S*-term* t*:*

$$\langle \mathbf{U}, \beta \rangle (t) = \langle \mathcal{B}, \beta \rangle (t)$$

and for every quantifier-free S*-formula* φ*:*

$$\langle \mathbf{U}, \beta \rangle \models \varphi \text{ iff } \langle \mathcal{B}, \beta \rangle \models \varphi$$

What about quantified formulas?

Universal Formulas

Definition 4 *The set of* universal formulas *is the least set closed under the following rules.*

1. Every quantifier-free formula is a universal formula.
2. If φ, ψ are universal, so are $\varphi \wedge \psi$ and $\varphi \vee \psi$.
3. If φ is universal, so is $\forall x \varphi$.

Lemma 4 *Let* $\mathbf{U}$ *and* $\mathcal{B}$ *be* S*-structures with* $\mathbf{U} \subset \mathcal{B}$ *and* $\varphi \in L_n^S$ *be universal. Then the following holds for all* $a_0, \ldots, a_{n-1} \in A$:
If $\mathcal{B} \models \varphi[a_0, \ldots, a_{n-1}]$ *then* $\mathbf{U} \models \varphi[a_0, \ldots, a_{n-1}]$.

Corollary 2 *If* $\mathbf{U} \subset \mathcal{B}$ *then the following holds for every universal sentence* φ*: If* $\mathcal{B} \models \varphi$ *then* $\mathbf{U} \models \varphi$.