

- The assignment is due on January 31 in class. Late assignments will not be accepted. Submit early and often.
- You are permitted to study with friends and discuss the problems; however, *you must write up your own solutions, in your own words*. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please do list all your collaborators in your submission for each problem.
- Finding solutions to homework problems on the web, or by asking students not enrolled in the class is strictly prohibited.
- We require that all homework submissions are prepared in Latex. If you need to draw any diagrams, however, you may draw them with your hand.

PROBLEM 1 *Approximate counter*

In class we analyzed an approximate counter that counts up to n using $O(\log \log n)$ bits: initialize X to 0 and for every update, we increase X with probability 2^{-X} . By averaging many such estimators, we obtained a $1 \pm \epsilon$ approximation with constant probability. Here we consider a variation of the algorithm. We still initialize the counter X to 0 but for each update, we increase X with probability $1/(1+a)^X$. Note that our estimator for n has to change from $2^X - 1$ to something else and this is part of the problem.

Question: How small do we need to set a so that our estimate \hat{n} of n satisfies $|n - \hat{n}| \leq \epsilon n$ with probability at least 9/10 when we return the output of only a single estimator instead of averaging many estimators as in class.

PROBLEM 2 *Fast ℓ_2 estimator*

Recall the AMS sketch from class for estimating the ℓ_2 norm of the input vector $x \in \mathbb{R}^n$: a random $m \times n$ matrix Π with entries $\pm 1/\sqrt{m}$ is drawn for $m = O(1/\epsilon^2)$ and $\|x\|^2$ is estimated as $\|\Pi x\|^2$. Then with probability at least 2/3, we have

$$(1 - \epsilon)\|x\|_2^2 \leq \|\Pi x\|_2^2 \leq (1 + \epsilon)\|x\|_2^2 \tag{1}$$

The update time of this algorithm is $O(1/\epsilon^2)$ because when we add δ to coordinate x_i we need to update every coordinate of Πx . An approach to improve the update time is to use sparser random matrix Π . Imagine picking Π as follows: for each $i \in \{1, 2, \dots, n\}$ we pick a uniformly random number $h_i \in \{1, \dots, m\}$. We then set $\Pi_{h_i, i} = \pm 1$ for each $i \in \{1, \dots, n\}$ and all other entries of Π are set to 0. This Π has the advantage that when we update coordinate x_i , we only need to change coordinate h_i of Πx and thus, the update time is constant. Show that this Π still satisfies equation (1) with probability 2/3 for $m = O(1/\epsilon^2)$.