CS 4800: Algorithms & Data

Lecture 6

January 26, 2018

Randomized algorithms

Events and probabilities

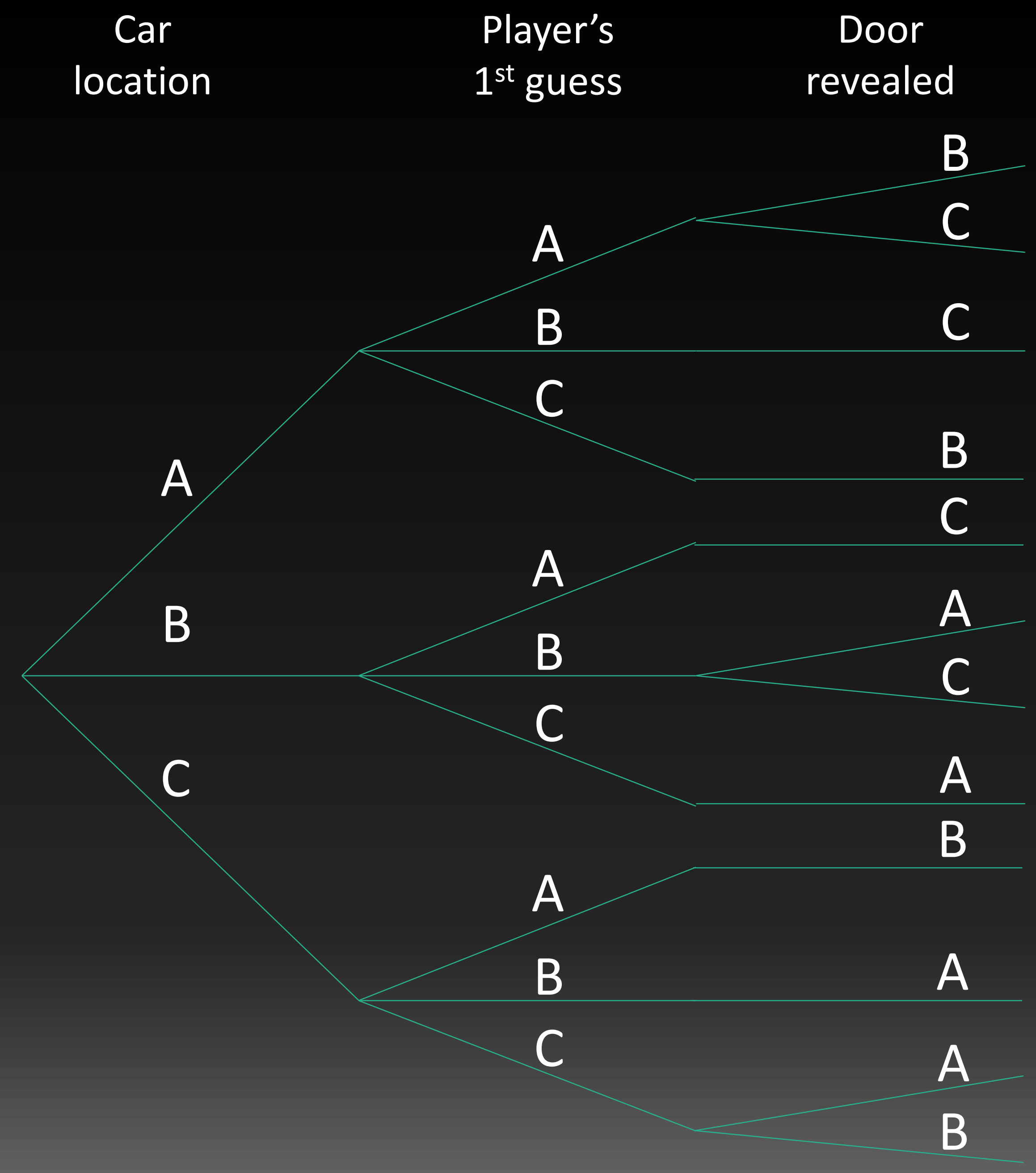
Suppose you're on a game show, and you're given the choice of three doors (A, B, C). Behind one door is a car, behind the others, goats. You pick a door, say A, and the host, who knows what's behind the doors, opens another door, say C, which has a goat. He says to you, "Do you want to pick door B?" Is it to your advantage to switch your choice of doors?

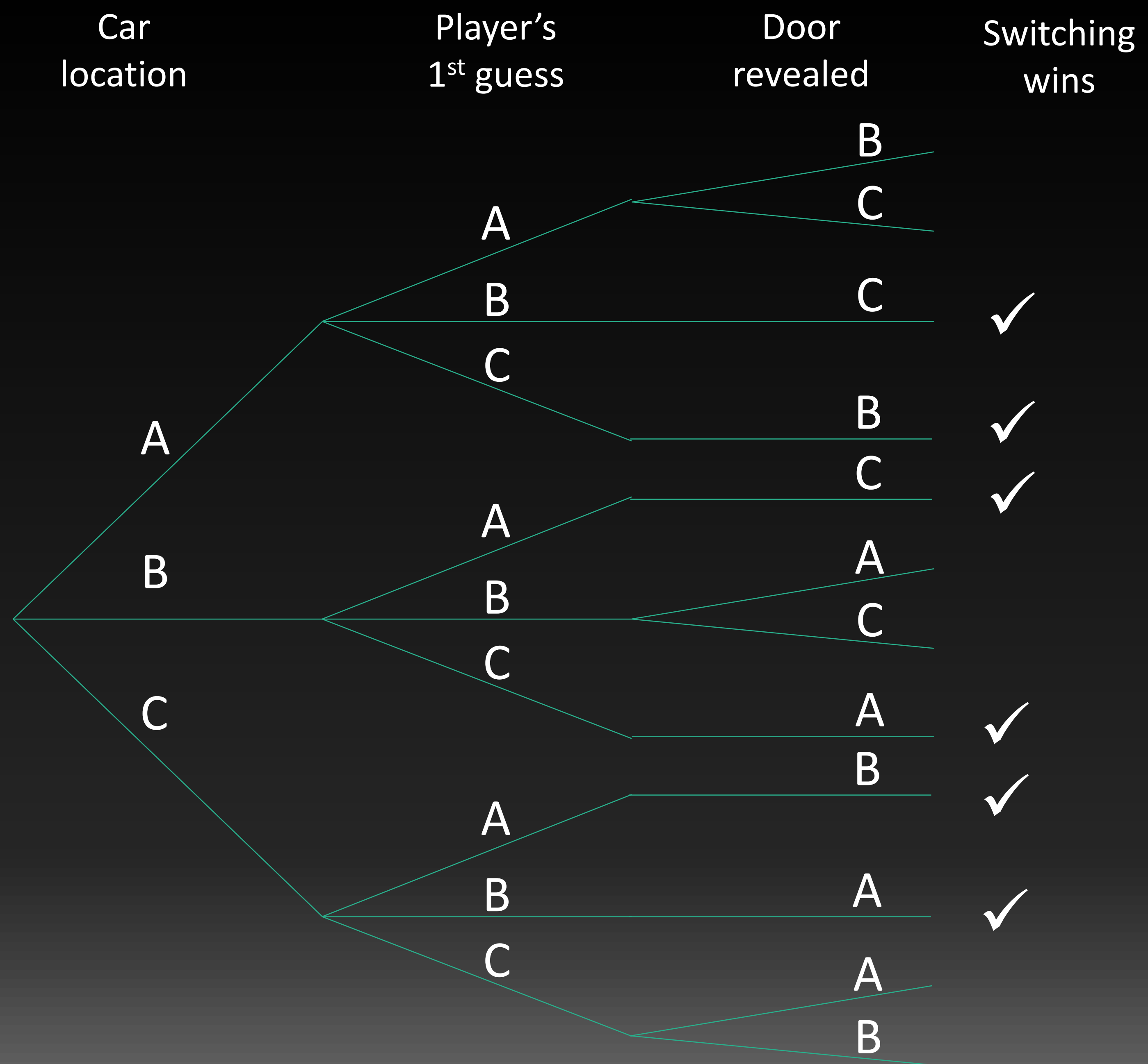
Assumptions

- Car is equally likely to be behind each door
- Player is equally likely to pick each door
- After player picks, host opens a different door with a goat behind
- If the host has choices, he is equally likely to pick each of them

Sample space

- Randomly determined quantities:
  - Car location
  - Door chosen by player
  - Door opened by host
- Every possible combination is an **outcome**
- Set of all outcomes is **sample space**


Car location
Player's 1st guess
Door revealed
A
B
C
A
B
C
A
B
C
A
B
C
B
C
C
B
C
A
C
A
B
A
A
B



Car location
Player's 1st guess
Door revealed
Switching wins
A
B
C
A
B
C
A
B
C
A
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B
C
C
B
C
A
C
A
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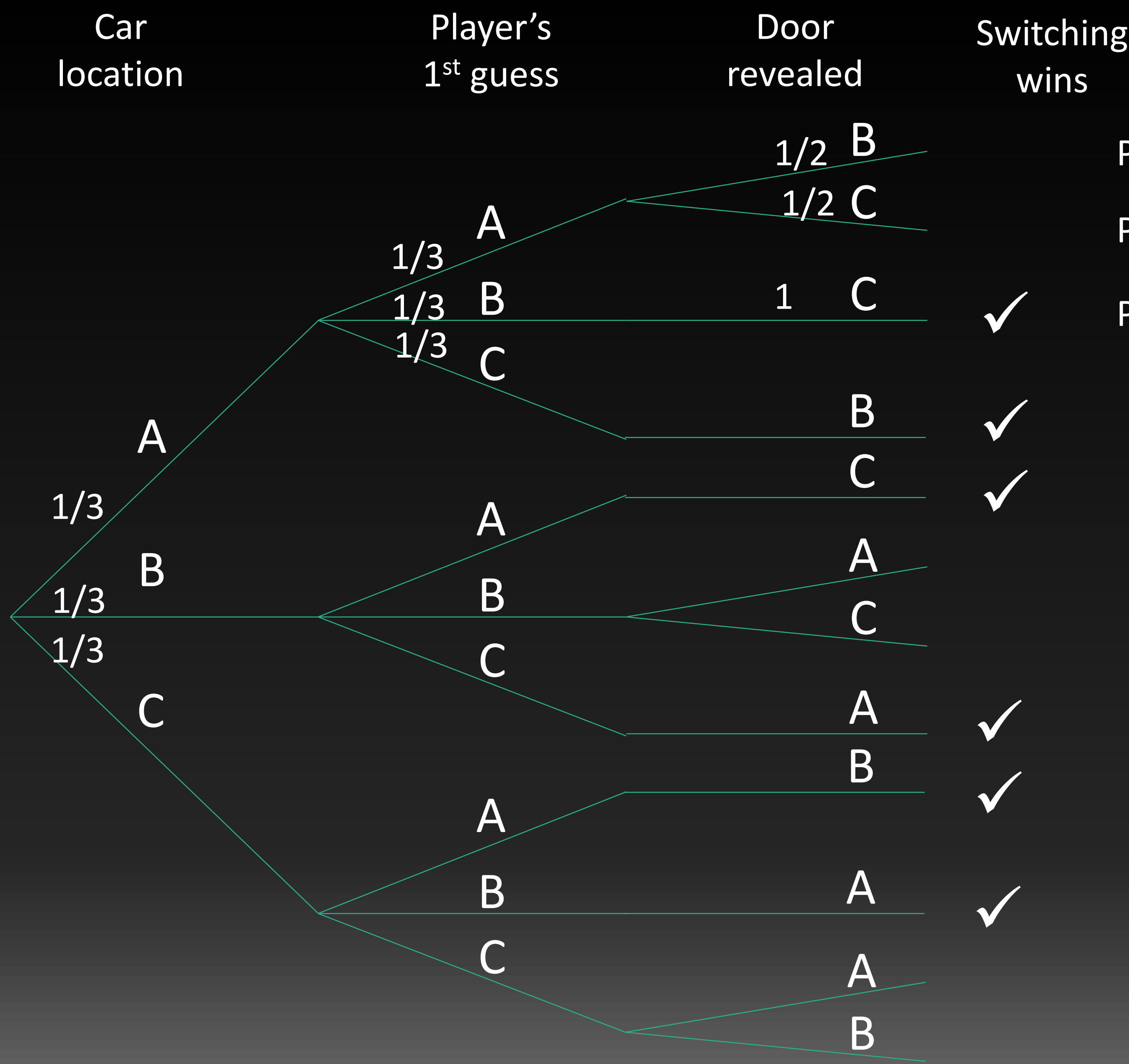
A
A
B



Prob. of (A,A,B) = 1/18

Prob. of (A,A,C) = 1/18

Prob. of (A,B,C) = 1/9

Random variables

- Random variable R is a function
  $R: \{sample\ space\} \to \mathbb{R}$
- Outcomes of 2 fair coin tosses
- R=#heads in the outcome
- R(HH) = 2
- R(HT) = 1
- Pr[R = 1] = Pr[HT]+Pr[TH] = ¼ + ¼ = ½

Expectation

- R is random variable on sample space S
- $E[R] = \sum_{outcome\ w} R(w)\Pr[w]$
- R: #heads in 2 fair coin tosses
- $E[R] = R(TT)\Pr[TT] + R(TH)\Pr[TH] + R(HT)\Pr[HT] + R(HH)\Pr[HH]$
- $E[R] = 0 + \frac{1}{4} + \frac{1}{4} + \frac{2}{4} = 1$

Linearity of expectation

Claim. For any variables R_1, R_2,

$$E[R_1 + R_2] = E[R_1] + E[R_2]$$

Proof. Let $R = R_1 + R_2$.

$$E[R] = \sum_{outcome\ w \in S} R(w) \Pr[w]$$

$$= \sum_{outcome\ w \in S} (R_1(w) + R_2(w)) \Pr[w]$$

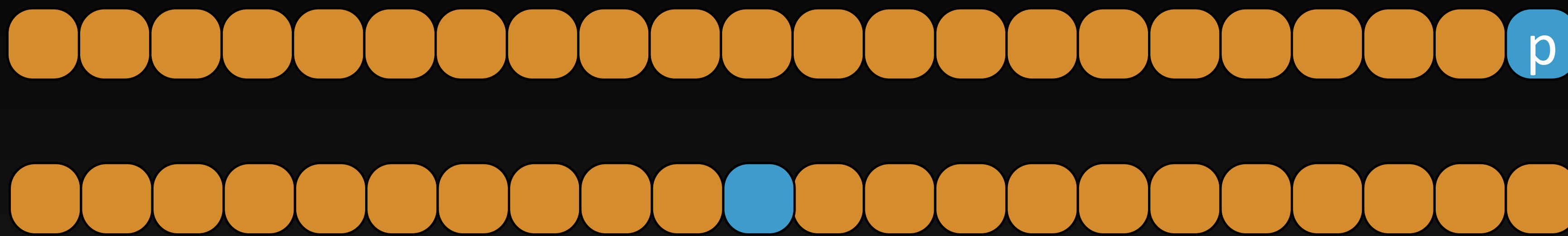
$$= \sum_{w \in S} R_1(w) \Pr[w] + \sum_{w \in S} R_2(w) \Pr[w]$$

$$= E[R_1] + E[R_2]$$

Application

- R: #heads in 2 fair coin tosses
- $R = R_1 + R_2$ where R_1=#heads in 1st coin toss
- $E[R_1]=1/2$ (head with probability ½, tail with probability ½)
- $E[R_2]=1/2$
- E[R] = ½ + ½ = 1
- E[#heads in 100 coin tosses] = ?
- 100 coin tosses. E[#times where two consecutive coins are different] = ?

Quicksort



- Pick an element p
- Partition the list using p as pivot
  - Left half are elements < p
  - Right half are elements > p
- Recursively sort both halves

How to pick good pivot p?

Picking good pivot

- Can run median algorithm to use median as pivot
- $O(n)$ time to find pivot
- $T(n) = 2T(n/2) + O(n)$
- Solution?
- Cons: Constant in $O(n)$ is large
- New idea: use random pivot

Running time with random pivot

- Suffices to count number of comparisons
- V_i: i[th] smallest value in array A[1…n]
- X_{ij}: random variable that is 1 if we compare V_i and V_j and 0 otherwise
- $\#comparisons = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij}$
- $E[\#comparisons] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$

When do we compare V_i and V_j?

- If V_k is picked as pivot and $V_i < V_k < V_j$
  - V_i goes left, V_j goes right
  - We do not compare V_i and V_j
- In general, we compare V_i and V_j if and only if the first pivot chosen from $\{V_i, V_{i+1}, \ldots, V_j\}$ is either V_i or V_j.
- By symmetry, the probability of this is $\frac{2}{j-i+1}$
- $E[X_{ij}] = \frac{2}{j-i+1}$

Running time of Quicksort

- $E[\#comparisons] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$

$$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{k=1}^{n-1} \sum_{i=1}^{n-k} \frac{2}{k+1} \quad (reorder\ sums, j = i + k)$$

$$= \sum_{k=1}^{n-1} \frac{2(n-k)}{k+1}$$

$$= \sum_{k=1}^{n-1} \left(\frac{2n+2}{k+1} - 2 \right)$$

$$= (2n+2) \sum_{k=1}^{n-1} \frac{1}{k+1} - 2(n-1)$$

Harmonic number $< \ln(n)$