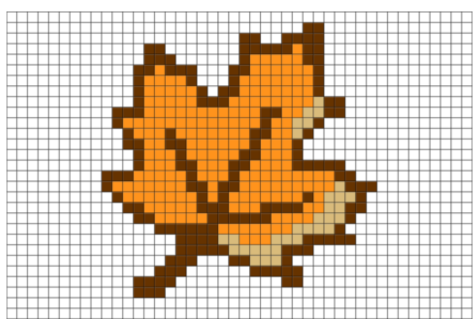
H10.

- The assignment is due at Gradescope on April 13 at 1:35pm. Late assignments will not be accepted. Submit early and often.
- You are permitted to study with friends and discuss the problems; however, *you must write up you own solutions, in your own words*. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please do list all your collaborators in your submission for each problem.
- Finding solutions to homework problems on the web, or by asking students not enrolled in the class is strictly prohibited.
- We require that all homework submissions are prepared in Latex. If you need to draw any diagrams, however, you may draw them with your hand.

PROBLEM 1 *Gold Bullion face*

In the ruins of Pompeii, I remember seeing the House of the Tragic Poet with a famous mosaic floor proclaiming visitors to "Beware of the Dog." In Boston, a less tragic and wealthier poet has commissioned a mosaic using 1kg bars of solid gold, specifically the type CreditSuisse mints in the dimension 80mmx40mm.

Design an algorithm that takes as input a grid of 40mmx40mm squares that are either colored or white. The algorithm determines if the colored squares in the design can be *entirely covered* with gold bullion bars. Note that gold bars can never be split in half! Each gold bar covers exactly two of the squares. As an example, consider the gold bars on the left, and the pixel art on the right. Can the leaf be covered in gold?



Hint: reduce the question to the bipartite matching problem by observing that if we impose a checkerboard pattern on the grid, each gold bar covers one white square and one black square.

Solution:

PROBLEM 2 *Matrix rounding*

Suppose we are given a large matrix $A[1 \ldots m][1 \ldots n]$ of population data (each entry is a non-negative real number). We want to publish matrix A, but need to simplify it for the public by making the entries integers by replacing each entry x in A with either $\lceil x \rceil$ or $\lfloor x \rfloor$. However, the matrix represents important population data, and we do not want to change the sums of entries in any row or column. For example:

$$\begin{bmatrix} 1.2 & 3.4 & 2.4 \\ 3.9 & 4.0 & 2.1 \\ 7.9 & 1.6 & 0.5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 \\ 4 & 4 & 2 \\ 8 & 1 & 1 \end{bmatrix}$$

We will use max flow to design an efficient algorithm that either rounds A in this fashion, or reports correctly that no such rounding is possible.

(a) First the algorithm checks if the sums of every row and column is integral. What can we conclude if some row or column does not sum to an integer?

Solution:

(b) From now on, we assume that all rows and columns sum to integers. Create a maximum flow network similar to the one for bipartite matching to model our problem: one side has vertices corresponding to rows of A and one side has vertices corresponding to columns of A. Describe your graph and show how to use maximum flow to check if rounding is possible. Prove that your algorithm is correct by showing that (1) if there is a rounding then the maximum flow is at least a certain value, and (2) if the maximum flow attains that value then there is a rounding.

Solution: