

- The assignment is due at Gradescope on January 19 at 1:35pm. Late assignments will not be accepted. Submit early and often.
- You are permitted to study with friends and discuss the problems; however, *you must write up your own solutions, in your own words*. Do not submit anything you cannot explain. If you do collaborate with any of the other students on any problem, please do list all your collaborators in your submission for each problem.
- Finding solutions to homework problems on the web, or by asking students not enrolled in the class is strictly prohibited.
- We require that all homework submissions are prepared in Latex. If you need to draw any diagrams, however, you may draw them with your hand.

PROBLEM 1 *Asymptotic Notation Review*

Rank the following functions by order of growth; that is, find an ordering f_1, f_2, \dots, f_{12} of the functions satisfying $f_i = \Omega(f_{i+1}) \forall i \in \{1, \dots, 11\}$. You do not need to provide justification. Hint: use logarithms to simplify the functions.

$$\begin{array}{cccccc} n^3 & 7^{\log_2 n} & n! & 2^{(\log_2 n)^2} & \log_2(n!) & n^{1/(\log_2 n)} \\ \log_2(\log_2 n) & 2^n & (\log_2 n)^{\log_2 n} & \sqrt{n} & 2^{\log_3 n} & 3^{\sqrt{\log_2 n}} \end{array}$$

Solution:

PROBLEM 2 *Induction*

Consider the following theorem and proof.

Theorem 1 *In every set of $n \geq 1$ aardvarks, all aardvarks have the same weight.*

Proof. In the base case $n = 1$, because the set has only one aardvark, it has the same weight as itself. In the inductive step, assume that the theorem is true for $n = k$. We will prove the theorem for $n = k + 1$. Consider a set of $k + 1$ aardvarks a_1, \dots, a_k, a_{k+1} . By our assumption, the first k aardvarks have the same weight.

$$\underbrace{a_1, a_2, \dots, a_k}_{\text{same weight}}, a_{k+1}$$

Also by our assumption, the last k aardvarks have the same weight.

$$a_1, \underbrace{a_2, \dots, a_k, a_{k+1}}_{\text{same weight}}$$

Therefore, by transitivity, all aardvarks in the set have the same weight.

Thus, by induction, the theorem is true for all $n \geq 1$. \square

Is there any mistake in this proof?

Solution:

PROBLEM 3 *Karatsuba Example*

Carry out the Karatsuba algorithm for $20 \cdot 18$.

Solution:

PROBLEM 4 *Mysterious code*

You encounter the following mysterious piece of code in a cryptographic program.

```
1: function COMPUTE( $a, n$ )
2:   if  $n = 0$  then
3:     return the pair  $(1, a)$ 
4:   else if  $n = 1$  then
5:     return the pair  $(a, a \cdot a)$ 
6:   else if  $n$  even then
7:      $(u, v) \leftarrow \text{COMPUTE}(a, \lfloor n/2 \rfloor)$ 
8:     return  $(u \cdot u, u \cdot v)$ 
9:   else if  $n$  odd then
10:     $(u, v) \leftarrow \text{COMPUTE}(a, \lfloor n/2 \rfloor)$ 
11:    return  $(v \cdot u, v \cdot v)$ 
12:   end if
13: end function
```

(a) What is the result of $\text{COMPUTE}(a, 3)$?

Solution:

(b) What is the result of $\text{COMPUTE}(a, 4)$?

Solution:

(c) What does the code do in general? Prove your assertion by induction on n .

Solution:

(d) Write the recurrence for the running time as a function of n and show $T(n) = O(\log n)$ by induction. Assume that each multiplication takes one unit of time and you only need to count the number of multiplications.

Solution: