Probabilities Intro.

- spaces, events, uniform prob. → requires uniform prob ad space

$\Rightarrow$ prob (informal) $= \boxed{\dfrac{\text{count numerator}}{\text{count denominator}}} = \dfrac{\text{count favorable}}{\text{count all}}$

- random variables, joint, conditional

$\Rightarrow$ not simple counts + fractions

functions (R.V) like expectation, variance, entropy

- $\boxed{\text{non-uniform distributions}}$: prob $\neq \dfrac{\text{numerator}}{\text{denominator}}$

doesn't work

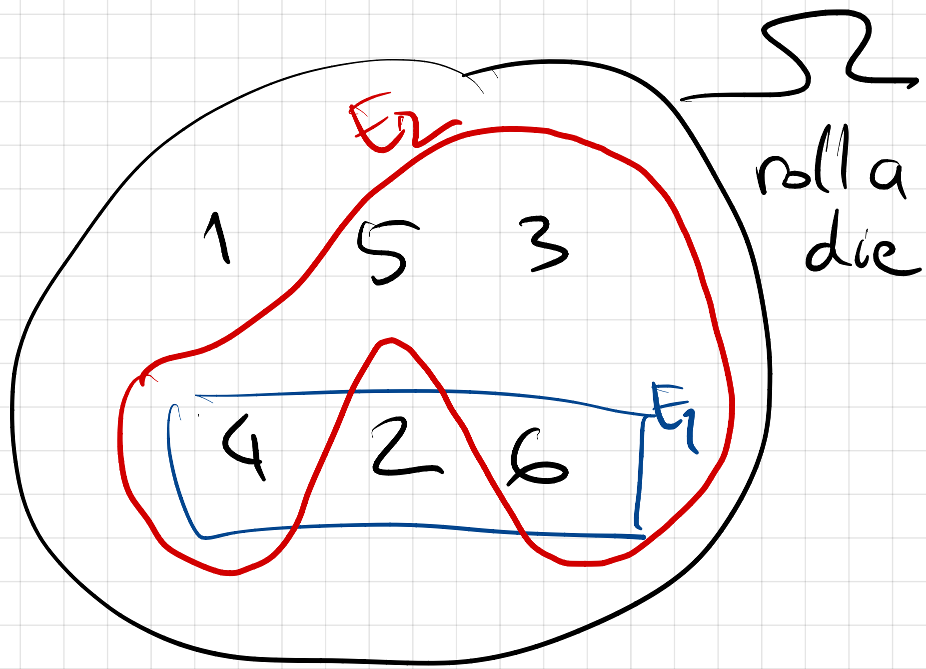
probability • random experiment $\Rightarrow$ outcome

set of outcomes Ω

outcome $w \in \Omega$

Event: subset $E \subset \Omega$

E outcomes "favorable"



probab = measure

$p(\text{outcome}) \in \mathbb{R}^+$

$0 \leq p(w) \leq 1$

total

$$\sum_{w \in \Omega} p(w) = 1$$

uniform $p(w) = \frac{1}{|\Omega|}$

E_1 = even outcome $= \{2,4,6\}$

$p(E_2) = 4/6 = 2/3$

E_2: outcome $\geq 3 = \{3,4,5,6\}$

$$P(E_1) = \sum_{w \in E_1} p(w)$$

uniform $p \Rightarrow P(E_1) = \frac{3}{6} = \frac{1}{2}$

$\boxed{\text{2 fair die roll}}$

$$\Omega = \{ (1,1), (1,2), \ldots (1,6)$$
$$(2,1)\ (2,2) \ldots (2,6)$$
$$\vdots$$
$$(6,1)\ (6,2) \ldots (6,6) \}$$

$$|\Omega| = 36$$

E = sum of 7

$$= \{ (1,6), (2,5), \ldots (6,1) \} \qquad |E| = 6$$

$$P(E) = 6/36 = 1/6$$

E_2 = sum > 8

= 9 or 10, 11, 12

$$P(E_2) = \frac{10}{36}$$

36	46	56	66
45	55	65	
~~54~~ 54	64		
63			

10

$\boxed{\text{deck of cards}}$ 4 suits ♠ ♣ ♢ ♡

values $\underbrace{2, 3, 4 \dots 10}_{\text{vals}}, \underbrace{J, Q, K, A}_{\text{faces}}$

random exp : pick a card

E_1 = "face card" $= \dfrac{16}{52}$

2 suits

E_2 = " value red between" $= \dfrac{\overset{♡}{9} + \overset{♢}{9}}{52}$

2 and 10

15 red balls 10 blue balls in urn.

- draw 1 at random $\quad P(\text{red}) = \dfrac{15}{25}$

- draw 3 at once (without repetition)

$$p(3\text{ reds}) = \frac{\binom{15}{3} \text{ favorable possib}}{\binom{25}{3} \text{ all possib of 3 out of 25}}$$

- draw 3 with replacement

$$p(3\text{ reds}) = \frac{15^3 \rightarrow \text{all red possib with rep}}{25^3 \rightarrow \text{all possib w/ repet}}$$

Lecture 12 ?

Random Variables

Conditional probability ; joint probability

Bayes Theorem

Expectation

Before

$$\text{prob} = \frac{\text{count favorable}}{\text{count total}}$$

= arith average

Random Variables

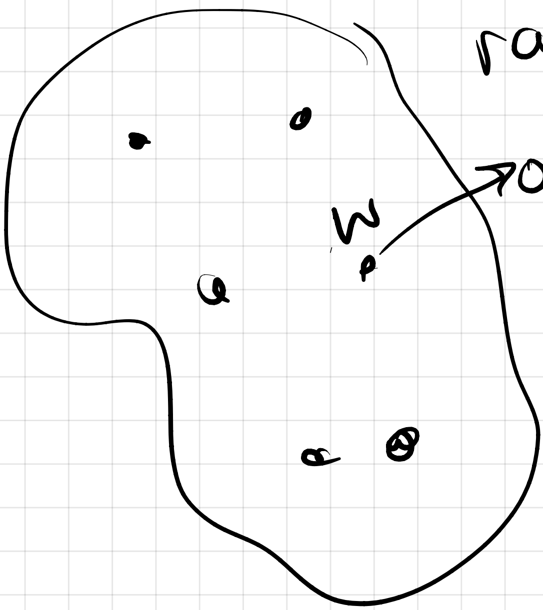
= variables with a set of possible values (we not sure of value)

$X, A, B, Y, Z \ldots$

- each value has a probability ($\simeq$ chance of happening)

$\Rightarrow$ distribution of possible values.

Ω = space



rand exp

$\Downarrow$

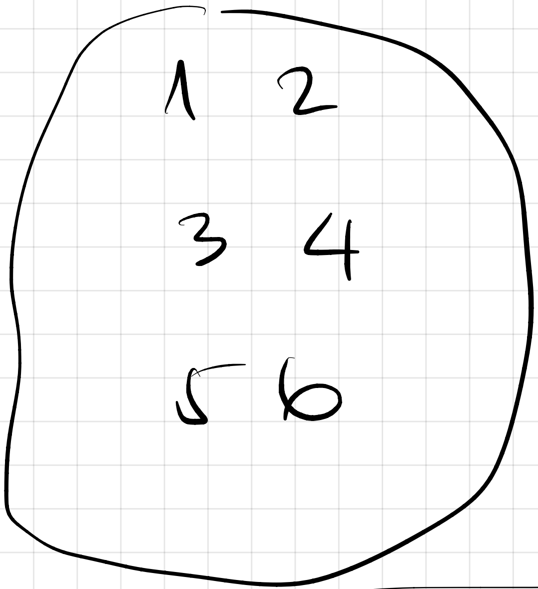
outcome

$X = $ R.V.

$X = $ function of outcome

Ex: rand exp	outcome	X
pick a student	the student	age(student)
pick 3 cards	3 cards	sum(values cards)
flip coin n=100 times	flip values ex. HTHHTTTH...	# of Heads

Roll a die.



E_1 = outcome is even

$X(\text{outcome}) = \begin{cases} \text{even} & \text{prob} = 1/2 \\ \text{odd} & \text{prob} = 1/2 \end{cases}$

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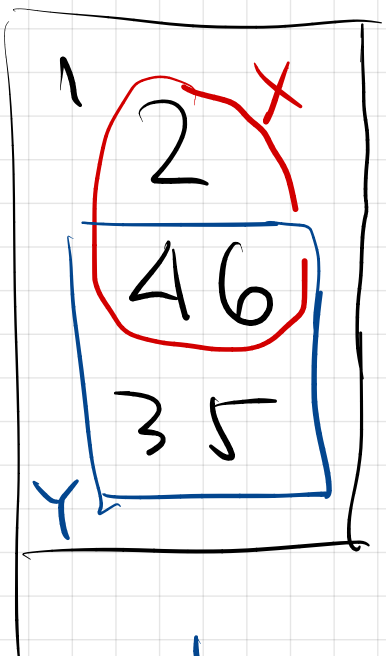
2 R.V. on same space $\equiv$ 2 properties or functions (outcomes)

maybe $\equiv$ 2 events

generating process (experiment or sampling) $\Rightarrow$ outcomes

Look at outcome through R.V.

roll a die (R.V ≃ events)



X: outcome is even? → T 1/2, → F 1/2

Y: outcome ≥ 3 → T 4/6, → F 2/6

X, Y R.V. binary

outcomes as X, Y properties/values ⟹ $\boxed{\text{joint table}}$

X (2 possib) Y (2 possib) ⟹ table 2x2 = 4

	X=T	X=F
Y=T	2/6 2 ("4" "6")	2/6 2 ("3" "5")
Y=F	1/6 1 ("2")	1/6 1 ("1")

$\text{prob}(X=T \wedge Y=T) =$

$= \text{prob}(X=T, Y=T) = \frac{2}{6}$

$P(X=F, Y=T) = 2/6$

$p(X=T, Y=F) = 1/6$

Marginalization

For Y

only care about Y?

Sum up each row (summed up by diff x)

	X=T	X=F	
Y=T	2/6 2 ("4" "6")	2/6 2 ("3" "4 5")	4/6
Y=F	1/6 1 ("2")	1/6 1 ("1")	2/6
	3/6	3/6	

$p(Y=T) = \frac{4}{6}$

$p(Y=F) = 2/6$

For X

only care about x?

- sum up each column (over diff values of Y)

$p(X=T) = 3 \qquad P(X=F) = 3$

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joint table $\equiv$ prob (x = val_x , Y = val_y) joint probability

and

chance of { x = val_x AND y = val_y }

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probability of X = val_x given I know val_y

ex: prob X=T given Y=F

not random

Restriction of space to only Y=F

	$X=T$	$X=F$
$Y=F$	1	1

condition

$P(X=T \mid Y=F) = 1/2$ (given)

$P(X=F \mid Y=F) = 1/2$

conditional probability

$$P(X=T \mid Y=F) = \frac{\text{numerator}}{\text{denominator}}$$ in the cases (row) with $Y=F$

$$= \frac{P(X=T, Y=F)}{P(Y=F)}$$

context / restriction / condition

$$P(X \mid \bar{Y}) = \frac{P(X, \bar{Y})}{P(\bar{Y})}$$

When there is no confusion, instead of $P(X=T)$ write $P(X)$ or $P(T)$

ex : X binary $P(X) = P(X=T)$; $P(\bar{X}) = P(\neg X) = P(X=F)$

ex X color $\in \{$Red, Blue, Green$\}$ $P(\text{red}) = P(X=\text{red})$

$Y=T$ given

$X=T$	$X=F$
2	2

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{P(X=T \text{ and } Y=T)}{P(Y=T)}$$

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general X, Y R.V

$$\boxed{P(X|Y) = \frac{P(X,Y)}{P(Y)}}$$

true always

$\forall\ val_x \in \Omega_X,\ val_y \in \Omega_Y$

$$P(X=val_x \mid Y=val_y) = \frac{P(X=val_x \text{ AND } Y=val_y)}{P(Y=val_y)}$$

	$X=val_x$	$X \neq val_x$
$Y=val_y$	count	count

example 2 100 objects

$X = \text{color} \in \{R, B, G\}$

$Y = \text{shape} \in \{\square, \bigcirc\}$

joint table 2x3

	R	B	G	P(Y) distribut
□	$\frac{25}{100}$	$\frac{10}{100}$	$\frac{21}{100}$	0.56
○	$\frac{17}{100}$	$\frac{4}{100}$	$\frac{23}{100}$	0.44
P(X) distribution	0.42	0.14	0.44	

there are 25 objects Red □

23 objects Green ○

$.25 + .10 + .21 + .17 + .04 + .23 = 1$

conditional $P(Y=\square \mid X=\text{Red}) = P(\square \mid \text{Red})$

$$= \frac{P(Y=\square, X=R)}{P(X=R)} = \frac{P(\square, \text{Red})}{P(\text{Red})} = \frac{25}{42}$$

$$= \frac{p(\square, R)}{p(\square, R) + p(O, R)} = \frac{25}{25+17}$$

conditional $p(X = \text{Red} \mid Y = \square) = P(\text{Red} \mid \square) =$

look at first row → corresponds to all $\square$

	X=R	X=B	X=G	total
Y= $\square$	25	10	21	56

$$p(\text{red} \mid \square) = \frac{p(\text{red}, \square)}{p(\square)} = \frac{25}{56}$$

$$= \frac{p(\text{red}, \square)}{p(\square, \text{red}) + p(\square, \text{Blue}) + p(\square, \text{Green})} = \frac{25}{25+10+21}$$

$$\boxed{P(X|Y) = \frac{P(X,Y)}{P(Y)}}$$

⇒ Bayes theorem
allows cond. prob calculation

by inverting conditionals

$$P(X,Y) = P(X|Y) \cdot P(Y)$$
$$\| $$
$$P(Y,X) = P(Y|X) \cdot P(X)$$

$$P(X|Y) \cdot P(Y) = P(Y|X) \cdot P(X)$$

$$\boxed{P(X|Y) = \frac{P(Y|X) \cdot P(X)}{P(Y)}}$$

$$\boxed{P(Y|X) = \frac{P(Y,X)}{P(X)}}$$

$$P(Y|X) = P(Y)$$
$$\Updownarrow$$

<u>definition</u> X, Y independent if $P(X|Y) = P(X)$

"Knowing Y does not influence change distribution of X"

property

if X, Y independent $\Longleftrightarrow$ $\forall$ cell joint table = product of marginals

$$\underbrace{P(X,Y)}_{\text{joint}}^{\text{cell}} = \overbrace{P(X) \cdot P(Y)}^{\text{marginals}}$$

$$P(X=val_x \text{ and } Y=val_y) = P(X=val_x) \cdot P(Y=val_y)$$

2 dice 6-faces 1:6 uniform D6

20-faces 1:20 uniform D20

pick one of them at random

roll $\Rightarrow$ 5

befor outcome

$P(D6) = 1/2$

Prob (picked D6) = ?

after we see outcome

D = which die $\to$ 6, 20

E = outcome $\to$ 5, not 5

Bayes Th

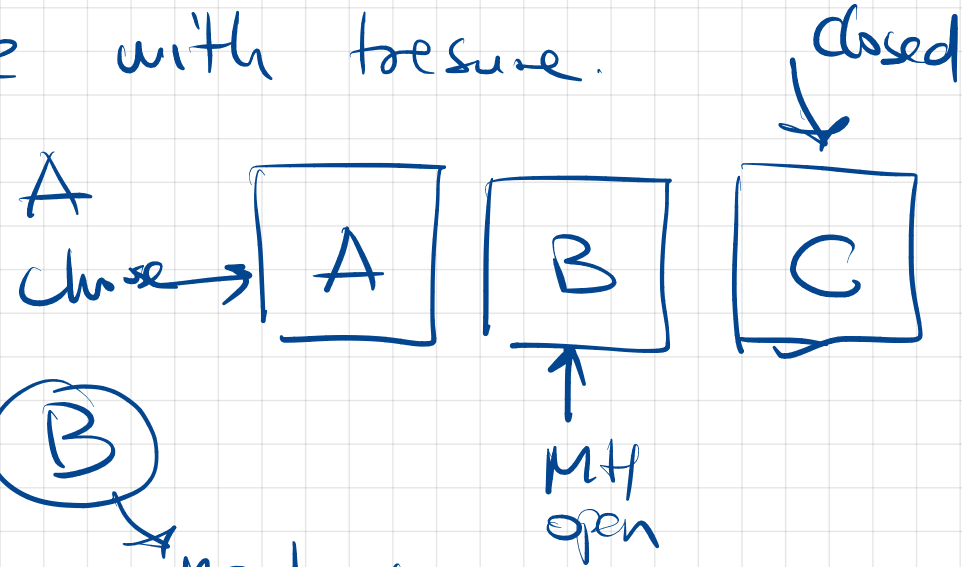
$$\text{Prob}(D = D6 \mid E = 5) = \frac{p(5 \mid D6) \cdot P(D6)}{P(5)}$$

$$= \frac{1/6 \cdot 1/2}{P(5, D6) + p(5, D20)} = \frac{1/6 \cdot 1/2}{p(5|D6) \cdot p(D6) + p(5|D_{20}) \cdot p(D_{20})}$$

$$\frac{1}{6} \cdot \frac{1}{2} + \frac{1}{20} \cdot \frac{1}{2}$$

Monty Hall 3 doors one with tresure.

You pick uniformly randomly door A
(don't open)



MH opens at random ~~a~~ door B → no treasure
- not the one picked ~~A~~
- not the one with treasure

Choice: T = treasure O = open MH

stick with A
$$P(T=A \mid O=B) = \frac{\overbrace{\boxed{P(O=B \mid T=A)}}^{1/2} \cdot \overbrace{\boxed{P(T=A)}}^{1/3}}{P(O=B)} = \frac{1/2 \cdot 1/3}{P(O=B)}$$

switch to C
$$P(T=C \mid O=B) = \frac{\overbrace{\boxed{P(O=B \mid T=C)}}^{1} \cdot \overbrace{\boxed{P(T=C)}}^{1/3}}{P(O=B)} = \frac{1/3}{P(O=B)}$$

$$P(T=C \mid O=B) + P(T=A \mid O=B) = 1 \Rightarrow P(O=B) = 1/2$$
$$P(T=C \mid O=B) = 2/3 \qquad P(T=A \mid O=B) = 1/3$$

virus

test for virus not perfect "false positives"

$$P(V) = \overset{V,\ T}{P}(\text{virus}) = P(\text{infected with virus}) = \frac{1}{10^4}$$

$$P(\text{pos test} \mid \text{virus infection}) = P(T \mid V) = \frac{99}{100}$$

$$P(\text{pos test} \mid \text{no virus}) = P(T \mid \bar{V}) = 1/1000$$

false positive

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if test = pos, chance of being infected?

$$P(V \mid T) = ? = \frac{P(T \mid V) \cdot P(V)}{P(T)} = \frac{99/100 \cdot 1/10{,}000}{P(T)}$$

$$\frac{99/10^6}{P(T \mid V) \cdot P(V) + P(T \mid \bar{V}) \cdot P(\bar{V})} = \frac{99/10^6}{99/10^6 + 1/1000 \cdot (1 - 1/10^4)}$$

cabs 60% white 40% yellow

accident; witness says "cab was yellow"

witness tells truth 80%; lies 20% indep of cab color

Q: What prob(cab = Yellow) = ?

R.V cab color: Y vs $\bar{Y}$

Second R.V: A GOOD witness says cab = yellow

$\bar{A}$: says cab = white"

match

match

Second RV — NO GOOD

witness tells Truth W

Lies $\bar{W}$

tells yellow

no best

cab is yellow

$$\boxed{P(Y \mid A)} = \frac{P(A \mid Y) \cdot P(Y)}{P(A)} =$$

Lies

$$= \frac{80\% \cdot 40\%}{P(A, Y) + P(A, \bar{Y})} = \frac{0.8 \times 0.4}{P(A \mid Y) \cdot P(Y) + P(A \mid \bar{Y}) \cdot P(\bar{Y})}$$

marginal

$$= \frac{0.8 \times 0.4}{0.8 \times 0.4 + 0.2 \times 0.6}$$ → cat is white

$$\boxed{P(A|Y) = \frac{P(A,Y)}{P(Y)}}$$

out of "Y possib" (restricted)

P(A,Y) = the ones with A

$$\text{cond} = \frac{\text{joint}}{\text{marginal}}$$

Alien Planet 3 Parties Red Blue Purple
4 states E W N S

Exercise

each region elects 2 senators at random (from that region)

	Red	Blue	Purple
E	12	20	8
W	16	14	18
N	9	18	14
S	22	10	12

Task: if we meet a purple elected senator, what is chance its from S region?

$P(X=S \mid Y=P, G=T)$

$Pr(S \mid P, G) =$

$= \dfrac{Prob(P, G \mid S) \cdot P(S)}{P(P, G)}$

Hints $X =$ Region $\{N, S, W, E\}$

$Y =$ Party $\{R, B, P\}$

$G =$ elected (senator) $\{T, F\}$

Birthday paradox $n = 244$ people, bdays are at random

probab different bdays?

"no collisions"

preview

$$\boxed{1+x \simeq e^{x} \quad \Big| \quad x \text{ close to } 0}$$

probab no collisions, product rule

$$\underset{1^{st}}{1} \times \underset{2^{nd}}{\frac{364}{365}} \times \underset{3^{rd}}{\frac{363}{365}} \times \cdots \times \underset{n}{\frac{366-n}{366}}$$

$$= \left(1-\frac{0}{365}\right)\left(1-\frac{1}{365}\right) \cdots \left(1-\frac{n-1}{365}\right)$$

$$\simeq e^{-0} \cdot e^{-1/365} \cdot e \cdots e^{-\frac{n-1}{365}}$$

$$= e^{-(0+1+2 \cdots + (n-1))/365}$$

$$= e^{-\frac{n(n-1)}{2}/365} \simeq 0.5 \quad (n = 244)$$