INDU

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\dots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

weak ind

step $n \to n+1$

old customer

$$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 \Rightarrow \sum_{k=1}^{n+1} k^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

new customer

proof:

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 \overset{IH}{=} \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \quad ? \text{ VS } \quad \left(\frac{(n+1)(n+2)}{2}\right)^2$$

$\times 4$

$\div (n+1)^2$

$n^2 + (n+1)\cdot 4 \quad ? \quad (n+2)^2$

$n^2+4n+4 \quad ? \quad n^2+2\cdot 2\cdot n+2^2$ ✓

+ BASE CASE $n=1$

$\boxed{\text{IND 16}}$

Filonacci

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
0,	1,	1,	2,	3,	5,	8,	13,	21,	34,	55 ...

$F_0 = 0, \; F_1 = 1$ base

recursive definition

$$\boxed{F_{n+1} = F_n + F_{n-1}} \quad \forall n \geq 1$$

guess : $F_n \simeq a^n$ exponential

if true, $a^{n+1} = a^n + a^{n-1}$ $\Big| \div a^{n-1}$

$$\boxed{a^2 = a + 1}$$

quad roots $a = \boxed{\dfrac{1+\sqrt{5}}{2}}$ or $\boxed{\dfrac{1-\sqrt{5}}{2}}$

φ Golden Ratio. $\qquad \bar{\varphi}$ or ϕ conjugate

$$\varphi^2 = \varphi + 1 \Rightarrow \varphi^{n+1} = \varphi^n + \varphi^{n-1}$$

$$\bar{\varphi}^2 = \bar{\varphi} + 1 \Rightarrow \bar{\varphi}^{n+1} = \bar{\varphi}^n + \bar{\varphi}^{n-1}$$

Theorem

Fib close form

$$\boxed{F_n = \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}}} = \frac{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n}{\sqrt5} \simeq \frac{\varphi^n}{\varphi - \bar{\varphi}} \quad \text{approx}$$

proof by induction

step

$$\left.\begin{aligned} F_n &= \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}} \\ F_{n-1} &= \frac{\varphi^{n-1} - \bar{\varphi}^{n-1}}{\varphi - \bar{\varphi}} \end{aligned}\right\} \Rightarrow F_{n+1} = \frac{\varphi^{n+1} - \bar{\varphi}^{n+1}}{\varphi - \bar{\varphi}}$$

STRONG IND

proof

new webinar

$$F_{n+1} = F_n + F_{n-1} \overset{IH}{=} \frac{\varphi^n - \bar{\varphi}^n}{\varphi - \bar{\varphi}} + \frac{\varphi^{n-1} - \bar{\varphi}^{n-1}}{\varphi - \bar{\varphi}} =$$

$$= \frac{\overbrace{\varphi^n + \varphi^{n-1}}^{\varphi^{n+1}} - \overbrace{(\bar{\varphi}^n + \bar{\varphi}^{n-1})}^{\bar{\varphi}^{n+1}}}{\varphi - \bar{\varphi}} = \frac{\varphi^{n+1} - \bar{\varphi}^{n+1}}{\varphi - \bar{\varphi}} \checkmark$$

base case:

$$F_1 = \frac{\varphi^1 - \bar{\varphi}^1}{\varphi - \bar{\varphi}} = 1 \qquad F_0 = \frac{\varphi^0 - \bar{\varphi}^0}{\varphi - \bar{\varphi}} = \frac{0}{\varphi - \bar{\varphi}} = 0$$

IND 17 Fibonacci F_n

$$M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

2x2 matrix

Induction Step.

$$\underbrace{\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}}_{A_{n+1}} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \underbrace{\begin{array}{cc} 1 & 2 \\ \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \end{array}}_{A_n}$$

2x2 2x2 2x2

$$\begin{array}{c} 1 \\ \\ \\ 2 \end{array} \begin{bmatrix} 1 \cdot F_n + 1 \cdot F_{n-1} & 1 \cdot F_{n-1} + 1 \cdot F_{n-2} \\ = F_{n+1} & = F_n \\ 1 \cdot F_n + 0 \cdot F_{n-1} & 1 \cdot F_{n-1} + 0 \cdot F_{n-2} \\ = F_n & = F_{n-1} \end{bmatrix}$$

2x2

$$A_n = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}$$

already proved the induction step

$$\boxed{A_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot A_n}$$

$$A_{n+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{n}$$

· base A_1

$$\boxed{A_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \cdot \quad \text{Base}}$$

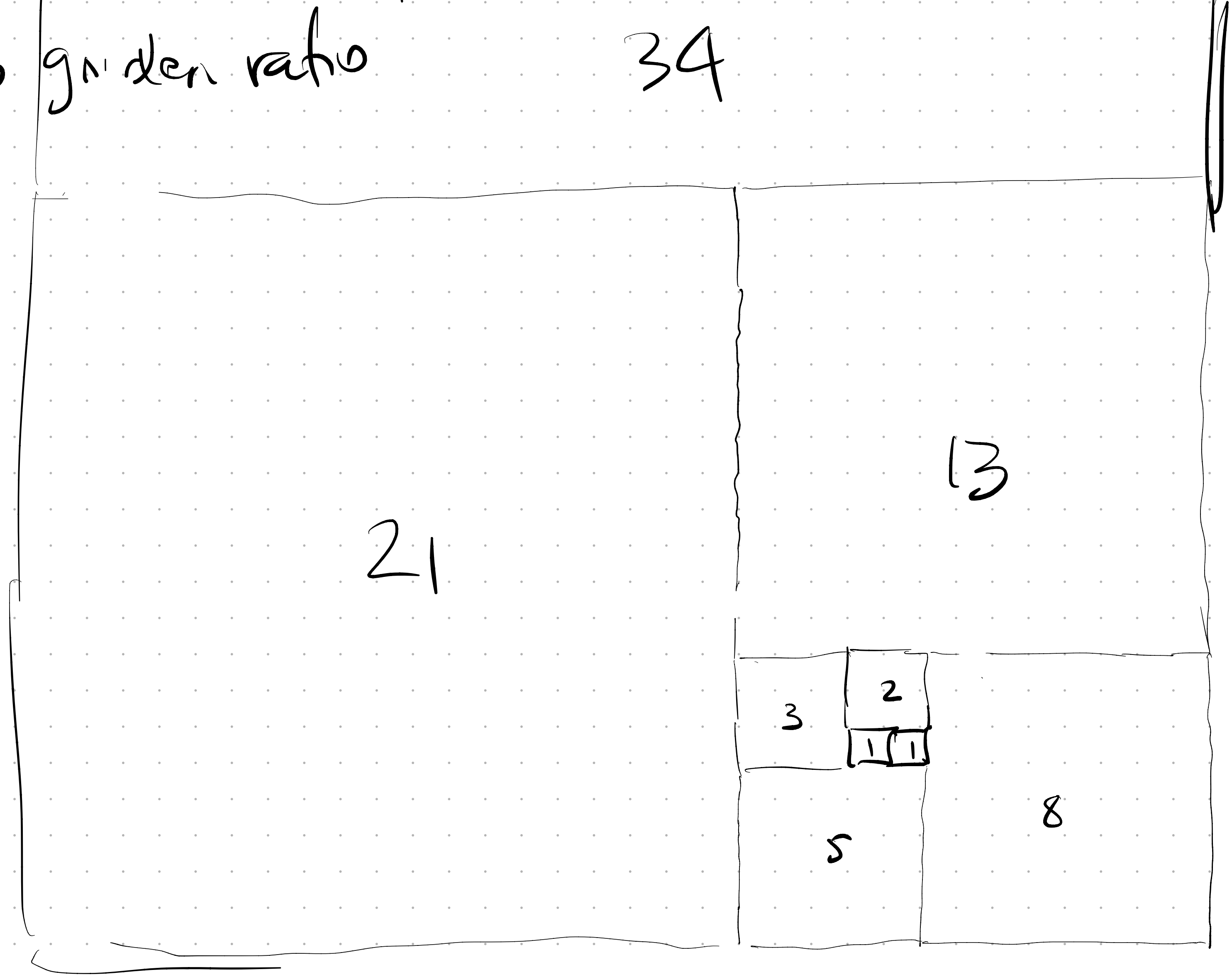
use repeated sq.

compute F_n in $\simeq \log(n)$ time.

Fibonacci properties

0, 1, 1, 2, 3, 5, 8, 13, 21, 34

- golden ratio



- Exercise $F_{n+1} = \varphi \cdot F_n + \bar{\varphi}^n$ (- induction
-easier w/out induction)

IND 18

old cust new cust

- $\varphi^n = F_n \cdot \varphi + F_{n-1} \xrightarrow{\text{ind step}} \varphi^{n+1} = F_{n+1} \cdot \varphi + F_n$

proof: $\varphi^{n+1} = \varphi^n \cdot \varphi \overset{IH}{=} (F_n \cdot \varphi + F_{n-1})\varphi =$

$\varphi^2 = 1+\varphi$

$= F_n \varphi^2 + F_{n-1}\varphi = F_n (1+\varphi) + F_{n-1} \cdot \varphi =$

$= F_n + F_n\varphi + F_{n-1}\varphi = \varphi(F_n + F_{n-1}) + F_n =$

$= \varphi \cdot F_{n+1} + F_n$ ✓

base case? exercise

IND 19 Count S_n = # strings of n bits without consecutive 1^s?

$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{1}\ \underline{0}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}$ cannot be 1

n bit

base cases

$S_1 = F_3$

$S_2 = F_4$

n small (say n=10) => cases

Solution: S_n follows fibonacci: $S_n = F_{n+2}$

ind step $\left.\begin{array}{l} S_n = F_{n+2} \\ S_{n-1} = F_{n+1} \end{array}\right\} \Rightarrow S_{n+1} = F_{n+3}$

STRONG INDUCTION

proof

S_{n+1}: # valid strings of n+1 bits : 2 cases (disjoint)

- $\underline{0}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{n \text{ bits}}$ #count = S_n

- $\underline{1}\ \underline{0}\ \underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{n-1 \text{ bits}}$ # count = S_{n-1}

$S_{n+1} = S_n + S_{n-1} \Rightarrow$ Fibonacci

• Exercise $F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$ "diagonals of Pascal Δ"

• Exercise S_n = # ordered decompositions of n into sums of 1 and 2. $\boxed{S_n = F_{n+1}}$

ex $n=5$:

1+1+1+1+1
1+1+1+2
1+1+2+1
1+2+1+1
2+1+1+1
2+2+1
2+1+2
1+2+2

IND20 $r^0 + r^1 + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$ $r \neq 1$

$\sum_{k=0}^{n} r^k$

base case? $n=0$
$n=1$

IND STEP $\sum_{k=0}^{n+1} r^k = \frac{r^{n+2}-1}{r-1}$

proof

$$\sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1} =$$

$$\overset{IH}{=} \frac{r^{n+1}-1}{r-1} + r^{n+1} \underset{\text{WANT}}{\overset{?}{=}} \frac{r^{n+2}-1}{r-1} \quad \Big| \times (r-1)$$

$$r^{n+1} - 1 + (r-1)r^{n+1} \underset{\text{want}}{\overset{?}{=}} r^{n+2} - 1$$

$$\cancel{r^{n+1}} - 1 + r^{n+2} - \cancel{r^{n+1}} \underset{\text{want}}{\overset{?}{=}} r^{n+2} - 1$$

✓

IND 21 $\quad n > m$

example $n=5 \quad m=2$

$$\sqrt{2\sqrt{3\sqrt{4\sqrt{5}}}} < 3$$

new want

$$\sqrt{m\sqrt{(m+1)\sqrt{(m+2)\sqrt{\cdots\sqrt{n}}}}} < m+1$$

R_m

IND STEP backwards $m+1 \to m$

old want

$$\sqrt{(m+1)\sqrt{(m+2)\sqrt{\cdots\sqrt{n}}}} < m+2$$

R_{m+1}

proof

$$R_m = \sqrt{m \cdot R_{m+1}} \overset{IH}{\le} \sqrt{m(m+2)} \overset{?}{<} m+1$$

want

$$m^2+2m \overset{?}{<} m^2+2m+1 \quad \checkmark$$

want

IND 22 $3^{n+1} \mid 2^{3^n}+1 \overset{\text{IND STEP}}{\Longrightarrow} \underbrace{3^{n+2} \mid 2^{3^{n+1}}+1}_{\text{new custom}}$

IH: $\boxed{2^{3^n}+1 = 3^{n+1}\cdot k} \Rightarrow$ - - - -

proof

$$2^{3^{n+1}}+1 = 2^{3^n\cdot 3}+1 = \left(2^{3^n}\right)^3+1$$

$$2^{3^n} = 3^{n+1}\cdot k - 1$$

$$\overset{IH}{=} (3^{n+1}\cdot k - 1)^3 + 1$$

$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

$$\boxed{a = 3^{n+1}\cdot k \qquad b = 1}$$

$$= (3^{n+1}\cdot k)^3 - 3(3^{n+1}\cdot k)^2 + 3\cdot 3^{n+1}\cdot k \cancel{-1} \cancel{+1}$$

$$= 3^{3(n+1)}\cdot k^3 - 3^{(n+1)\cdot 2+1}\cdot k + 3^{n+2}\cdot k$$

$$= 3^{n+2}\,(\text{something}) \Rightarrow 3^{n+2} \mid 2^{3^{n+1}}+1 \ \checkmark$$

base case: $n=1$ $9 \mid 2^3+1$ ✓

IND23

$a_i > 0$ reals

quad mean, arith mean, geom mean, harmonic mean

$$\sqrt{\frac{\sum a_i^2}{n}} \geq \frac{\sum a_i}{n} \geq \sqrt[n]{\prod a_i} \geq \frac{n}{\sum \frac{1}{a_i}}$$

$\sum_{i=1}^{n} a_i = a_1 + a_2 \dots + a_n$ sum

$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot a_3 \dots \cdot a_n$ product

- easy to prove with convexity arguments

$()^2, \times n^2$

First ineq by induction $n \to n+1$

$$\boxed{n \sum_{i=1}^{n} a_i^2 \geq \left(\sum_{i=1}^{n} a_i\right)^2} \Rightarrow (n+1) \sum_{i=1}^{n+1} a_i^2 \geq \left(\sum_{i=1}^{n+1} a_i\right)^2$$

proof:

$$(n+1) \sum_{i=1}^{n+1} a_i^2 = n \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i^2 + \boxed{(n+1) a_{n+1}^2}$$

$$= n \sum_{i=1}^{n} a_i^2 + a_{n+1}^2 + \sum_{i=1}^{n} (a_i^2 + a_{n+1}^2) \overset{?}{\geq} \left(\sum_{i=1}^{n} a_i + a_{n+1}\right)^2$$

$(a+b)^2 = a^2+b^2+2ab$

$$\cancel{n\sum_{i=1}^{n} a_i^2} + \cancel{a_{n+1}^2} + \sum_{i=1}^{n}\left(a_i^2 + a_{n+1}^2\right) \overset{?}{\geq} \cancel{\left(\sum_{i=1}^{n} a_i\right)^2} + \cancel{a_{n+1}^2} + 2\left(\sum_{i=1}^{n} a_i\right)a_{n+1}$$

IH

$$\sum_{i=1}^{n}\left(a_i^2 + a_{n+1}^2\right) \overset{?}{\geq} \sum_{i=1}^{n} 2\cdot a_i \cdot a_{n+1}$$

$$\sum_{i=1}^{n}\left(a_i^2 + a_{n+1}^2 - 2a_i a_{n+1}\right) \overset{?}{\geq} 0$$

$$\sum_{i=1}^{n}\left(a_i - a_{n+1}\right)^2 \overset{?}{\geq} 0$$ ✓ true

$\text{sum}(\;)^2 \geq 0$