Sequences

$(a_n)_{n=1 \ldots \infty}$

$a_1, a_2, a_3 \ldots \; a_{100}, \ldots$

$n=0$ ok to start

$a_0, a_1, a_2, \ldots$

Series

= sum ~~of~~ the first m terms

$$\sum_{k=1}^{m} a_k = a_1 + a_2 \ldots + a_m$$

- $m = \infty$ sum all sequence

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 \ldots$$

- arithmetic (constand diff or delta) $a_{k+1} - a_k = \text{const}$ step

$-7, -1, 5, 11, 17, 23, ?=29, ?=35, ?=41$

$+6 \quad +6 \quad +6 \quad +6 \quad +6$

general form $= a_n = a \cdot n + b$

$a = \text{step}$

$a, b = \text{coef}$

$a_n = a \cdot n + b$ starting at $n=0$

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
b,	$a+b$,	$2a+b$,	$3a+b$,	$4a+b$
7,	-1,	5,	11,	17

$\Rightarrow \; b = -7, \; a = +6$

$a_n = a \cdot n + b$ starting at $n=1$

$n=1$	$n=2$	$n=3$	$n=4$
$a+b$	$2a+b$	$3a+b$	$4a+b$
-7	-1	5	11

$-19, -13,$

$\Rightarrow \begin{cases} a = +6 \text{ step} \\ b = -13 \end{cases}$

Arithmetic series $\sum_{k=1}^{n} k = 1+2+3 \cdots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (a \cdot k + b) = a \sum_{k=1}^{n} k + \sum_{k=1}^{n} (b) =$$

$$= a \frac{n(n+1)}{2} + n \cdot b$$

Geometric ratio = constant $a_{k+1} = a_k \cdot \underline{\text{ratio}}$

$3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}$ ratio $= r = \frac{1}{2}$

$a_n = c \cdot r^n$

c = initial value

r = ratio (multiplicative)

- Start from $n=1$

$c \cdot r$	$c \cdot r^2$	$c r^3$	$c \cdot r^4$	$c \cdot r^5$	$r = \frac{1}{2}$
3	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{16}$	$c = 6$

- Star from $n=0$

c	$c \cdot r$	$c \cdot r^2$	$c \cdot r^3$	$c \cdot r^4$	$r = \frac{1}{2}$
$12, 6, 3$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{16}$	$c = 3$

Geometric Series

$$\boxed{1+r+r^2+r^3+\cdots+r^n = \frac{r^{n+1}-1}{r-1} \quad \boxed{\text{if } r \neq 1}}$$

$$\sum_{k=0}^{n} a_k = \qquad \Longleftrightarrow \quad (r-1)(1+r+r^2\cdots+r^n) = r^{n+1}-1$$

proof

$$\begin{array}{l} \cancel{r}+\cancel{r^2}+\cancel{r^3}+r^4\cdots+r^{n-1}+\cancel{r^n}+r^{n+1} \\ -1-\cancel{r}-\cancel{r^2}-r^3\cdots-r^{n-1}-\cancel{r^n} \\ \hline -1 \quad +r^{n+1} \quad \checkmark \end{array}$$

$$=\sum_{k=0}^{n}(c\cdot r^k)$$

$$=c\sum_{r=0}^{n} r^k = c(r^0+r^1+r^2\cdots+r^n) = c\,\frac{r^{n+1}-1}{r-1}$$

if $r \neq 1$

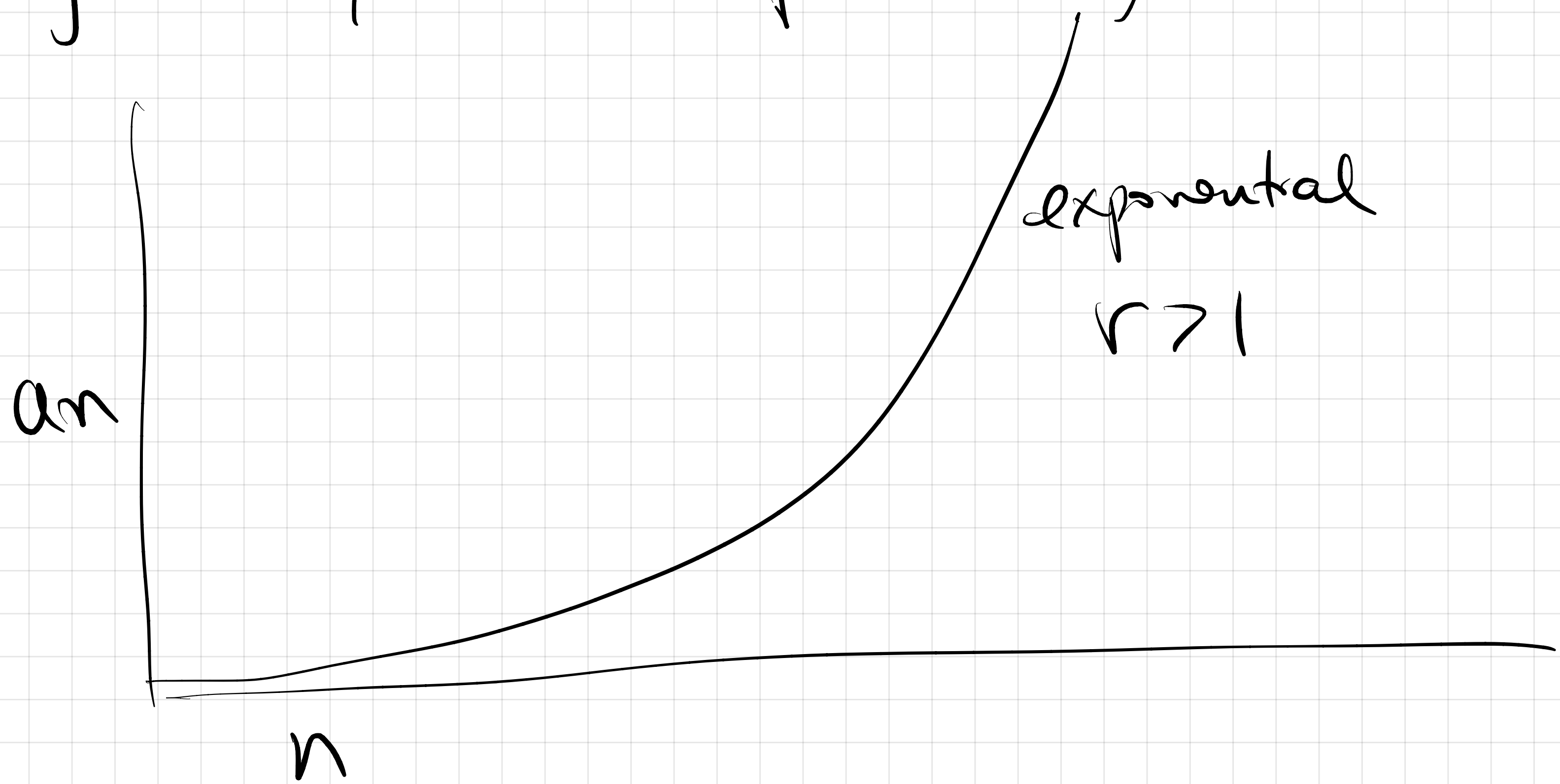
if $r=1$ manually: $\underbrace{c+c+c\cdots c}_{n+1} = c(n+1)$

example

$2, 2\cdot 3, 2\cdot 3^2, 2\cdot 3^3, 2\cdot 3^4, 2\cdot 3^5$ $\quad c=2$ $\quad r=3$

$2, 6, 18, 54, 162, 486$?

geom sequence = exponential "growth" for $r > 1$



- $r < 1 \Rightarrow a_n$ goes to 0 very fast (exponential)

a_n
$r<1$
n

Quadratic Sequence $a_n = a \cdot n^2 + b \cdot n + c$

a, b, c coef n = index

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$		
1	3	6	10	15	21	...

delta: 2 3 4 5 6

arithmetic: 1 1 1 1

Find a, b, c: solve linear system (3 eq, 3 var a, b, c)

$n=0$: $a_n = a \cdot n^2 + bn + c \iff \boxed{1 = a \cdot 0^2 + b0 + c}$

$n=1$: $a_n = a n^2 + bn + c \iff \boxed{3 = a \cdot 1^2 + b \cdot 1 + c}$

$n=2$: $\iff \boxed{6 = a \cdot 2^2 + b \cdot 2 + c}$

$c=1$

$a+b+c=3 \Rightarrow a+b=2 \xrightarrow{\cdot 2} 2a+\cancel{2b}=4$

$4a+2b+c=6 \Rightarrow 4a+\cancel{2b}=5$

$2a=1 \Rightarrow a=\frac{1}{2}$

$b=2-\frac{1}{2}=\frac{3}{2}$

$n=3: \; a_n=an^2+bn+c=\frac{1}{2}\cdot 9+\frac{3}{2}\cdot 3+1=$

$=\frac{9+9+2}{2}=\frac{20}{2}=10 \checkmark$

$a_n = a \cdot n^2 + bn + c$ $\qquad (n+1)^2 = n^2 + 1 + 2n$

$$\Delta_n = a_{n+1} - a_n = a(n+1)^2 + b \cdot (n+1) + \cancel{c} - a \cdot n^2 - bn - \cancel{c}$$

$$= a\left((n+1)^2 - n^2\right) + b(n+1-n)$$

$$= a(2n+1) + b$$

$$= (2a) \cdot n + (a+b)$$

$$\text{const step} \cdot n + \text{base} \quad \text{arithmetic}$$

$$\frac{\partial(an^2+bn+c)}{\partial n} = 2an + b$$

Quad Series

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (a \cdot k^2 + b \cdot k + c) =$$

$$= \sum_{k=1}^{n} a k^2 + \sum_{k=1}^{n} b \cdot k + \sum_{k=1}^{n} c =$$

$$= a \sum_{k=1}^{n} k^2 + b \sum_{k=1}^{n} k + nc =$$

$1^2 + 2^2 + 3^2 + \dots n^2$

$$= a \cdot \frac{n(n+1)(2n+1)}{6} + b \cdot \frac{n(n+1)}{2} + nc$$

Harmonic Series $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{n-1} + \frac{1}{n}$

delta $H_{n+1} - H_n = \frac{1}{n+1}$

$\simeq \ln(n) + \text{const}$

$a_n = \left(1+\frac{1}{n}\right)^n$ $\quad b_n = \left(1+\frac{1}{n}\right)^{n+1}$

same limit "e"

$$\underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n} < \underbrace{\left(1+\frac{1}{n+1}\right)^{n+1}}_{a_{n+1}} < e < \underbrace{\left(1+\frac{1}{n+1}\right)^{n+2}}_{b_{n+1}} < \underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{b_n}$$

a_n increasing

b_n decreasing

$$\frac{b_{n+1}}{a_{n+1}} = 1+\frac{1}{n+1} = \frac{n+2}{n+1} \xrightarrow{\lim} 1$$

upper $\lim_{n\to\infty} a_n = e = \text{lower limit } b_n$

$e \simeq 2.78$

$2.78 \simeq \left(1+\frac{1}{k}\right)^k$ (for large k) $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$$\underbrace{\left(1+\frac{1}{k}\right)^{k}}_{a_k} < e < \underbrace{\left(1+\frac{1}{k}\right)^{k+1}}_{b_k}$$

take log (defined nat log)

$\log_e(e)=1$

$$k\log\frac{k+1}{k} < 1 < (k+1)\log\frac{k+1}{k}$$

$$\cancel{k}\left(\log(k+1)-\log(k)\right) < 1 < (k+1)\left(\log(k+1)-\log(k)\right)$$

$$\frac{1}{k+1} < \log(k+1)-\log(k) < \frac{1}{k}$$

write all of them $k = 1, 2, 3, \ldots n$

$$k=1 \qquad \frac{1}{2} < \log(2) - \log(1) < \frac{1}{1}$$

$$k=2 \qquad \frac{1}{3} < \log(3) - \log(2) < \frac{1}{2}$$

$$k=3 \qquad \frac{1}{4} < \log(4) - \log(3) < \frac{1}{3}$$

$$\vdots$$

$$\log(n) - \log(n-1)$$

$$k=n \qquad \frac{1}{n+1} < \log(n+1) - \log(n) < \frac{1}{n}$$

Sum 3 cols

$$\frac{1}{2}+\frac{1}{3} \cdots + \frac{1}{n+1} < \log(n+1) - \underbrace{\log(1)}_{0} < \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$$

$$H_{n+1} < \log(n+1) < H_n$$

$$-1 + H_n + \frac{1}{n+1} < \log(n+1) < H_n$$

$$-1+\frac{1}{n+1} < \log(n+1) - H_n < 0$$

$$H_n \simeq \log(n)$$