Induction "Every day Sun comes up"

$\forall n \geq 1$ $P(n)$: "Sun comes up on day n"

$n=0$	$n=1$	$n=2$	...	$n = m$ present day	$m+1$ tomorrow	...
Sun forms	Sun up	Sun up	...	Sun up	Sun up	...

prove $P(n)$ by induction

- $P(0) \Rightarrow P(1) \Rightarrow P(2) \ldots \Rightarrow P(m) \Rightarrow P(m+1) \Rightarrow \ldots$
  induction steps (implications, chain of implications)

  in general
  $P(m) \Rightarrow P(m+1)$
  $\forall m$

- true at start ($n=0$)

- ind step : $\boxed{\begin{array}{c} P(n) \\ \text{Sun came up day } n \end{array}}$ $\Longrightarrow$ $\boxed{\begin{array}{c} P(n+1) \\ \text{Sun up day } n+1 \end{array}}$

ind hypot — ind conclusion

proof :)

Sun up day n
Earth rotates on itself in 24h
$\Big\}$ ⇒ 24h later sun will be up (day n+1)

INDUCTION TYPES

- Sum / Series / Arithmetic properties
- Strong induction
- Structural Induction (not algebra)

IND 1

$$\underbrace{\sum_{k=1}^{n} k}_{S_n} = \frac{n(n+1)}{2}$$

Predict $P[n]$: $S_n = \frac{n(n+1)}{2}$ $\forall n \geq 1$

ind step $P(n) \Rightarrow P(n+1)$ weak induction

$$\underbrace{S_n = \frac{n(n+1)}{2}}_{\text{old customer}} \Longrightarrow \underbrace{S_{n+1} = \frac{(n+1)(n+2)}{2}}_{\text{new customer}}$$

ind hyp

proof

start new customer

$$S_{n+1} = \sum_{k=1}^{n+1} k = \underbrace{\sum_{k=1}^{n} k}_{S_n} + (n+1)$$

use ind hyp

$$= \frac{n(n+1)}{2} + n+1$$

wishful thinking

$$\frac{n(n+1)}{2} + (n+1) \overset{?}{\underset{\text{want}}{=}} \frac{(n+1)(n+2)}{2} \quad |\times 2$$

$$n(n+1) + 2n+2 \overset{?\text{ want}}{=} (n+1)(n+2) \quad \Big| \quad (\;)$$

$$n^2+n+2n+2 \overset{\text{want}}{=} n^2+n+2n+2 \quad \checkmark$$

Alternative (not $\overset{\text{want}}{=}$)

$$\frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2} \quad \checkmark$$

base case (plug in, verify) $n=1$ $\quad S_1 = \frac{1 \cdot 2}{2}$

$$1 = \frac{1 \cdot 2}{2} \quad \checkmark$$

IND 2

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

S_n

Ind Ste

$$\underbrace{\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}}_{P(n)} \Longrightarrow \underbrace{\sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \frac{n+1}{n+2}}_{P(n+1)}$$

proof

new custome

$$S_{n+1} = \sum_{i=1}^{n+1} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \frac{1}{i(i+1)} + \frac{1}{(n+1)(n+2)} =$$

ind hyp

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \underset{=}{\text{want}} \frac{n+1}{n+2} \quad \Big| \begin{matrix} \times(n+2) \\ \times(n+1) \end{matrix}$$

$$n(n+2) + 1 \underset{=}{\text{want}} (n+1)^2$$

$$n^2 + 2n + 1 \underset{=}{\text{want}} n^2 + 2n + 1 \quad \checkmark$$

base case $n=1$ $\quad \frac{1}{1\cdot 2} = \frac{1}{2}$ ✓

no induction proof

$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) =$$

$$= \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1} =$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

IND3

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad n \geq 1$$

$1^2+2^2+3^2+\dots+n^2$

S_n

$P(n)$: $S_n = \frac{n(n+1)(2n+1)}{6}$

ind step $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \Longrightarrow \sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

$2(n+1)+1$

base case

$n=1$

$$\sum_{k=1}^{1} k^2 = 1^2 = \frac{1 \cdot 2\,(2+1)}{6} \checkmark$$

proof: $\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 =$

ind hyp

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 \stackrel{?}{=} \frac{(n+1)(n+2)(2n+3)}{6}$$

$\div(n+1)$

$\times 6$

$$n(2n+1) + 6(n+1) \stackrel{?}{=} (n+2)(2n+3)$$

$$2n^2+n+6n+6 \stackrel{?}{=} 2n^2+4n+3n+6$$

$$2n^2+7n+6 \stackrel{?}{=} 2n^2+7n+6 \checkmark$$

IND4 Strong induction

$\forall n \geq 12$ $\underbrace{\text{n cents can be obtaine with coin } 4c, 5c}_{P(n)}$

ind step $\quad P(n) \Rightarrow P(n+4)$

old customer $\quad$ new customer

proof $\quad$ n+4 cents = n cents + 4 cents

ind hyp P(n)

coins needed (n+4 cents) = Coins(n cents) + 1 coin (4c)

$P(12) \Rightarrow P(16) \Rightarrow P(20) \Rightarrow P(24) \Rightarrow \ldots$

$P(13) \Rightarrow P(17) \Rightarrow P(21) \Rightarrow P(25) \Rightarrow \ldots$

$P(14) \Rightarrow P(18) \Rightarrow P(22) \Rightarrow P(26) \Rightarrow \ldots$

$P(15) \Rightarrow P(19) \Rightarrow P(23) \Rightarrow P(27) \ldots$

Base cases:

$12 = 4+4+4$

$14 = 5+5+4$

$13 = 4+4+5$

$15 = 5+5+5$

IND5 $\displaystyle\sum_{i=1}^{n} (-1)^i \cdot i^2 = (-1)^n \frac{n(n+1)}{2}$

$-1^2+2^2-3^2+4^2-5^2 \ldots -$

ind step

$$\sum_{i=1}^{n} (-1)^i i^2 = (-1)^n \frac{n(n+1)}{2} \xRightarrow{\text{weak}} \sum_{i=1}^{n+1} (-1)^i i^2 = (-1)^{n+1} \frac{(n+1)(n+2)}{2}$$

proof

$$\sum_{i=1}^{n+1} (-1)^i i^2 = \sum_{i=1}^{n} (-1)^i i^2 + (-1)^{n+1}(n+1)^2$$

new customer

exercise base case

ind hyp

$$(-1)^n \frac{n(n+1)}{2} + (-1)^{n+1}(n+1)^2 \stackrel{?}{=} (-1)^{n+1} \frac{(n+1)(n+2)}{2}$$

$\div (-1)^n$

$\times 2$

$\div (n+1)$

$$n + (-1)(n+1) \cdot 2 \stackrel{?}{=} (-1)(n+2)$$

$$n - 2n - 2 \stackrel{?}{=} -n - 2 \quad ✓$$

IND 6

$$n! > 3^n > 2^n > n^2 > n\log_2(n) > n > \log_2(n)$$

$\forall n \geq 7$

proof (partial)

$$n! > 3^n > 2^n > n^2 > n\log_2 n > n > \log_2(n)$$

ind step

$$(n+1)! > 3^{n+1} > 2^{n+1} > (n+1)^2 > (n+1)\log_2(n+1) > n+1 > \log_2(n+1)$$

proof

$$(n+1)! = n!(n+1) \overset{IH}{>} 3^n(n+1) > 3^n \cdot 3 = 3^{n+1}$$

$$3^n \cdot 3 > 2^n \cdot 3 > 2^n \cdot 2 = 2^{n+1}$$

$$2^{n+1} = 2^n \cdot 2 \overset{IH}{>} n^2 \cdot 2 = n^2 + n^2 \geq n^2 + 7n =$$

$$= n^2 + 2n + 5n > n^2 + 2n + 1 = (n+1)^2$$

$$2^{n+1} > n^2 \Rightarrow (n+1) > \log_2(n^2) \geq \log_2(7n) = \log_2(n+6n) \geq \log_2(n+1)$$

$$(n+1) > \log_2(n+1) \Rightarrow (n+1)^2 > (n+1)\log(n+1)$$

base case $n = 7, 8$

(IND 7) $5 \mid 8^n - 3^n \quad \forall n \geq 1$

ind step $5 \mid 8^n - 3^n \Longrightarrow 5 \mid 8^{n+1} - 3^{n+1}$

proof $8^{n+1} - 3^{n+1} = \underbrace{8}_{5+3} \cdot 8^n - 3 \cdot 3^n =$

old customer

new customer

$= 5 \cdot 8^n + \boxed{3 \cdot 8^n - 3 \cdot 3^n}$

$= 5 \cdot 8^n + 3(8^n - 3^n) =$

IH: multiple of 5

$= 5 \cdot 8^n + 5k = 5(8^n + k) =$ multiple of 5

base case $n=1$ $5 \mid 8-3$ ✓

IND8 $10^n = a^2+b^2$ $n \geq 1$

weak

$P(n) \Rightarrow P(n+1)$

fails

strong induction

ind step $P(n) \Rightarrow P(n+2)$

proof : $10^{n+2} = 10^n \cdot 10^2 \overset{IH}{=} (a^2+b^2) \cdot 10^2 =$

$= (10a)^2 + (10b)^2$ sum of two squares.

$\boxed{\begin{matrix} P(1) \\ P(2) \end{matrix}}$

$P(1) \Rightarrow P(3) \Rightarrow P(5) \Rightarrow \dots$

$P(2) \Rightarrow P(4) \Rightarrow P(6) \Rightarrow P(8) \Rightarrow \dots$

base cases

$n=1$ $10 = 3^2+1^2$

$n=2$ $100 = 6^2+8^2$ ✓