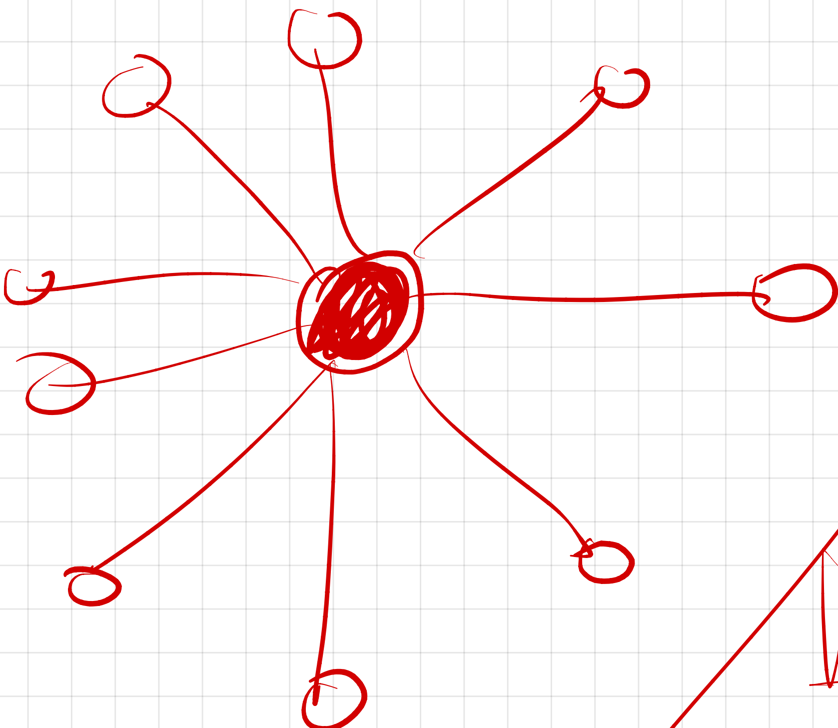
Vertex Cover: Find smallest set of vertices incident in all edges.

extreme: $|vc| = 1$



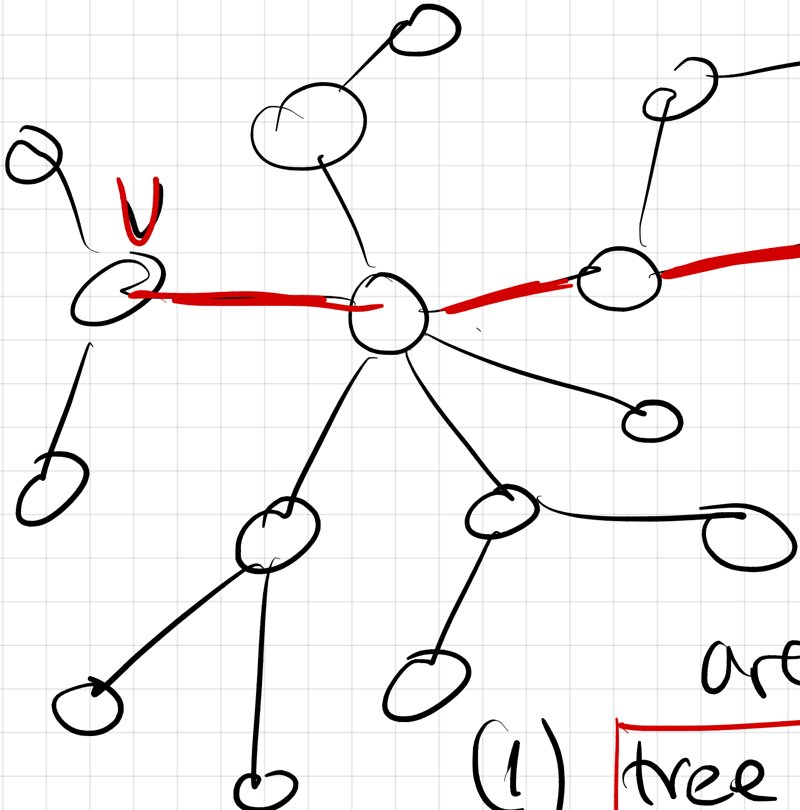
LECTURE 21

- Trees
- BST
- Skiplists → hw PB4

After THXGV

- week 11/29 – 12/3 full
- $\boxed{\text{week 12/6 – 12/10}}$
  - rec optional only Mon + Wed
  - playing poker
- project due 12/8
- hw PB4 due 12/11

Tree $T = G(V, E) = \text{tree}$ $\begin{cases} \text{connected} \\ \text{no cycles} \end{cases}$ &



exercise 1: $|E|$ in a tree is precisely $|V|-1$

Th exercise 2 following statements are equivalent:

(1) tree def: connected + no_cycles

(2) any two vertices (u, v) connected with unique path

(3) T minimal connected (remove any edge $\Rightarrow$ disconnect)

(4) T max acyclic (add any missing edge $\Rightarrow$ cycle)

(5) connected & $|E| = |V|-1$ || (6) acyclic & $|E| = |V|-1$

proof $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$

$1 \Rightarrow 2$) tree def $\Rightarrow$ $\forall u,v \in V$ $\exists$ path(u,v) unique
$u \rightsquigarrow v$

tree def $\Rightarrow$ connected $\Rightarrow$ $\exists$ path(u,v)

assume (hyp) path $u \rightsquigarrow v$ is not unique $\Rightarrow$ $\exists$ 2 dif paths $u \rightsquigarrow v$

2 dif paths (not directional)

found cycle

contradiction!

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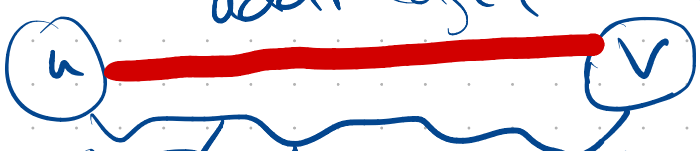
edge addl.

2) $\forall u,v$ $\exists$path(u,v) unique $\Rightarrow$ 3) Min connected $T +$ uv cycle.

addl edge (missing in T)



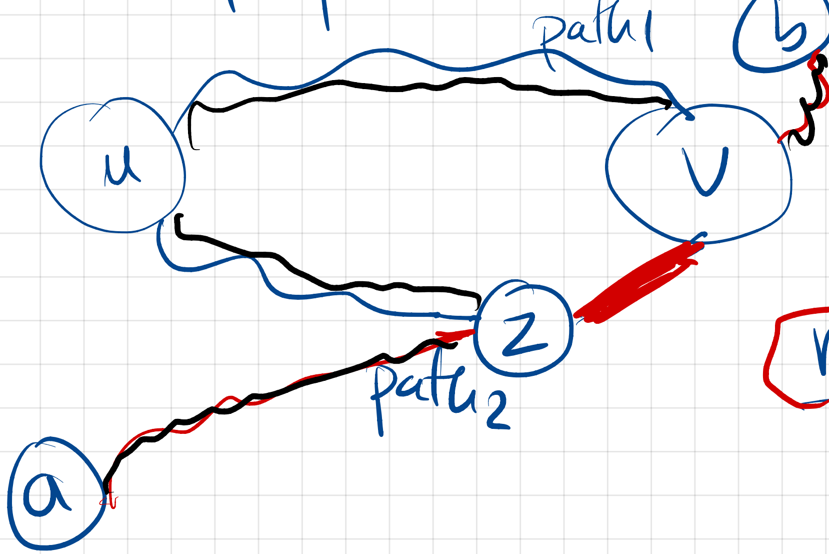
2) $\exists$path(u,v) $\nexists$ edge uv path

$\Rightarrow$ cycle

③ Min connected T $\Longrightarrow$ ② $\forall u,v$ $\underline{\exists \text{ path}(u,v)}$ unique

u,v T connected $\Rightarrow$ $\exists$ path(u,v). We want $\underline{\text{uniqueness}}$

assume 2 diff paths $u \to v$



Z = last node on path2 before V

path2(a,v) = path(a,z) + edge ZV

$\boxed{\text{remove}}$ edge ZV from T

$\Rightarrow$ T still connected

a,b $\in$ Vertices look at path(a,b)

case1 • path(a,b) $\not\ni$ $\boxed{\text{edge ZV}}$ removed $\Longrightarrow$ like before path(a,b)

case2 • path(a,b) $\ni$ $\boxed{\text{edge ZV}}$ removed $\Longrightarrow$ new path(a,b)

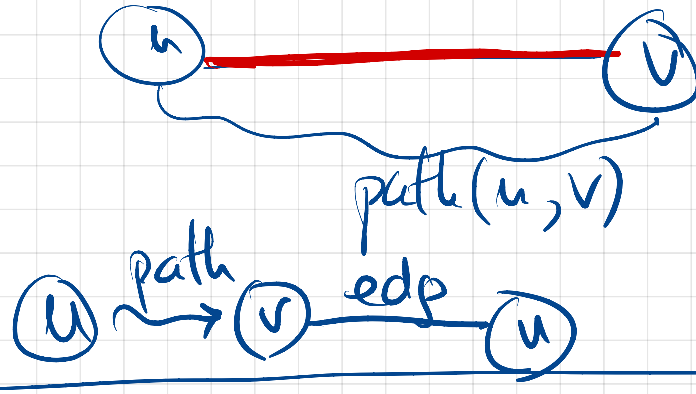
ⓐ $\to$ ⓩ $\to$ ⓤ $\to$ ⓥ $\to$ ⓑ

contradicts ③ hypothesis $\Longrightarrow$ path(u,v) unique

① tree def connected + no cycles ⟹ ④ max acyclic

T + edge cycle

add missing edge



⇓

cycle

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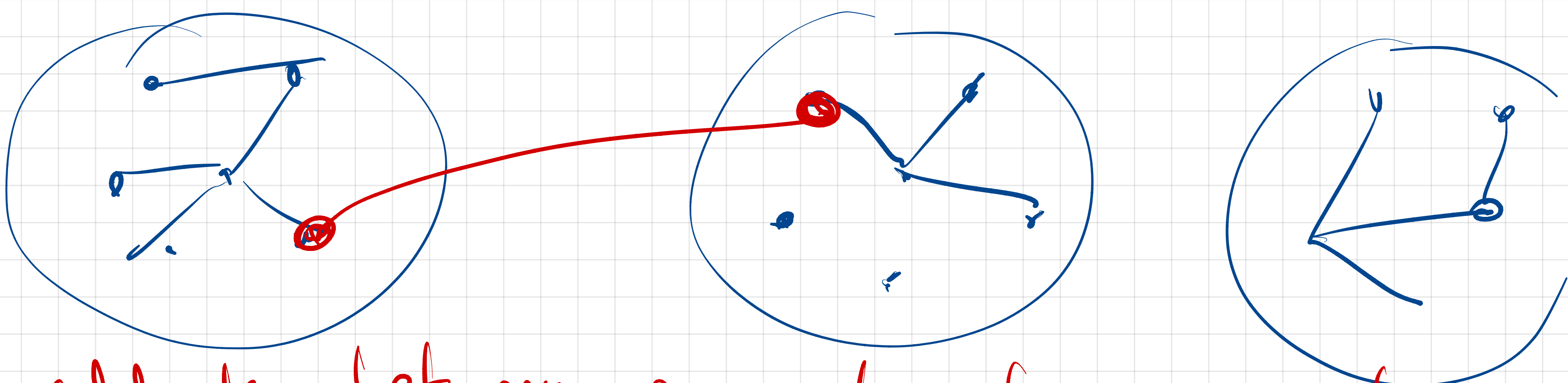
④ max acyclic ⟹ ① tree def connected + no cycles

T + new edge ⟹ cycles

- connected: Assume not ⟹ at least 2 disconnected compon.

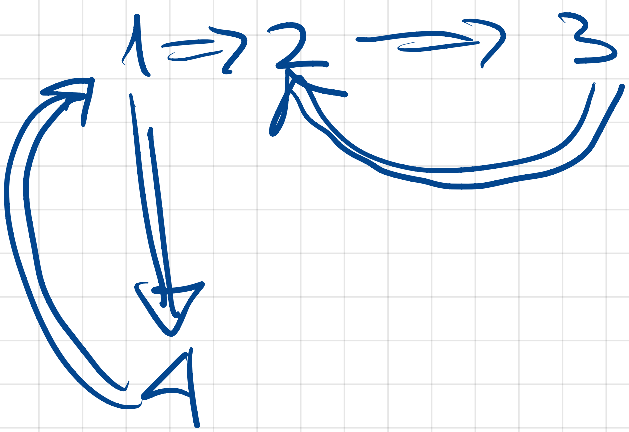


add edge between components → forms no cycle

contradicts hyp ⟹ connected

- no cycles

~~max~~ $\boxed{\text{acyclic}} \Rightarrow$ no cycles



Need one of these: $2 \to 1$

$2 \to 4$

$3 \to 1$

$3 \to 4$

(2) $\forall u,v\ \exists$ path(u,v) unique $\Longrightarrow$ (1) tree def: connected + no cycles

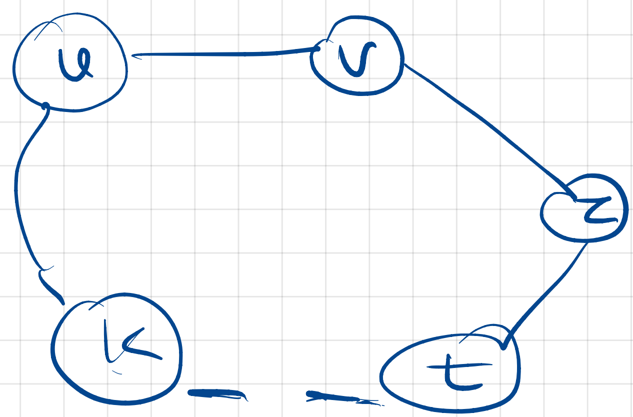
- connected: done path(u,v) exists

- Assume (hyp) $\exists$ cycle

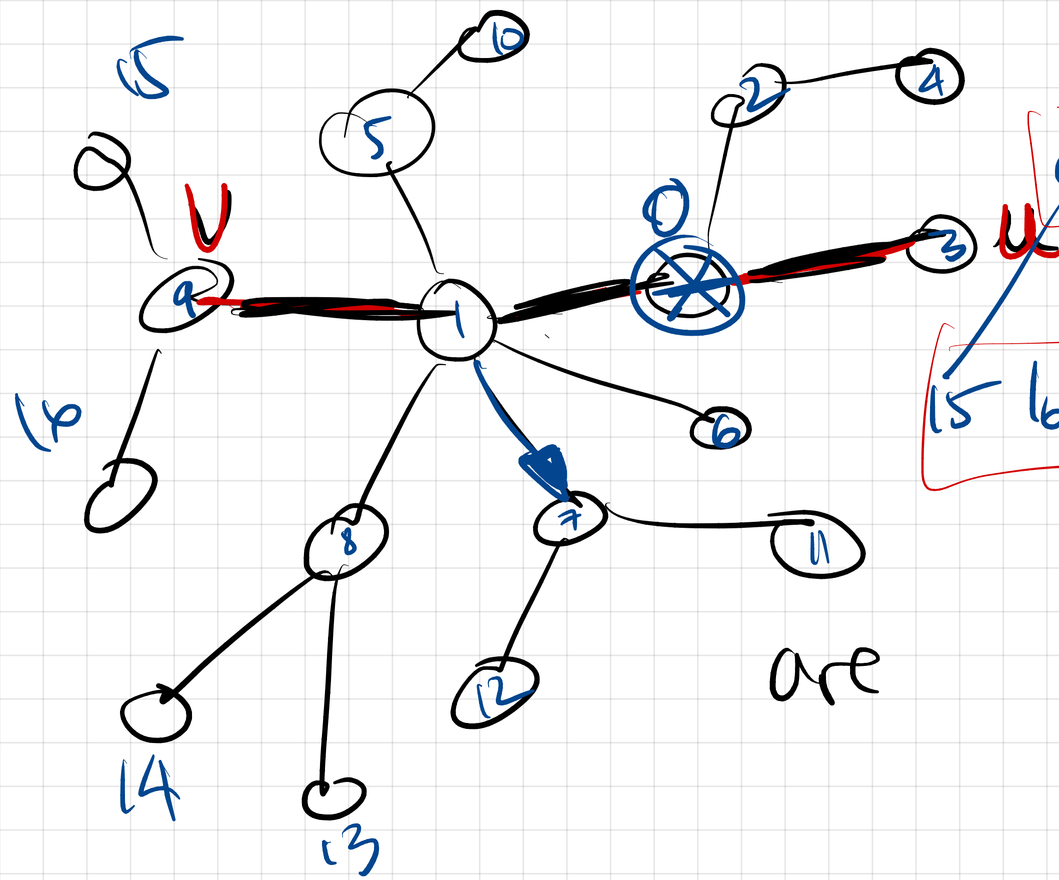
  2 dif paths u,v:

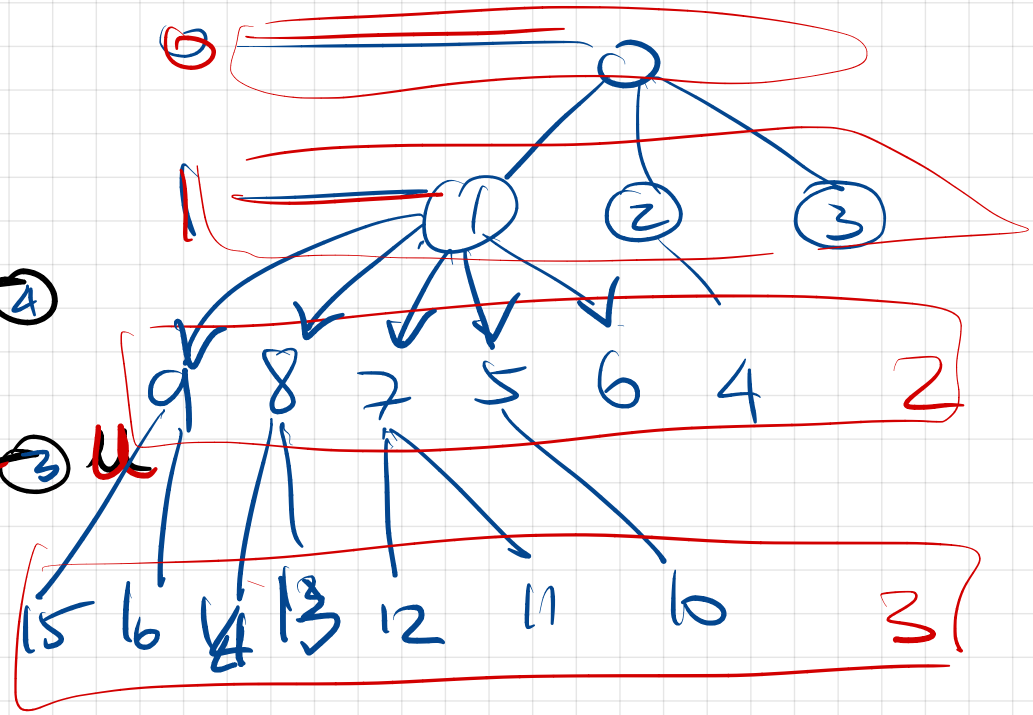
  $u - v$ edge (path 1)

  $u - k - \dots - t - z - v$ (path 2)



Contradiction $\Rightarrow$ tree. (connected, no cycles) pick root ⊛ → level 0





≃ BFS alg

are

Th tree ∃ 2 vertices of degree = 1

incident edges

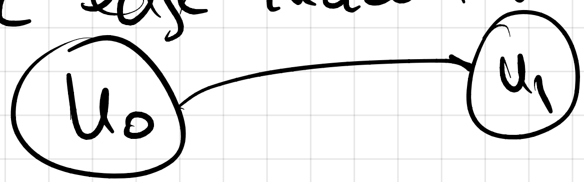
tree ⇒ $\deg(x) \geq 1$

Proof by contradiction: Assume $|\{u \in V \mid deg(u)=1\}| \le 1$

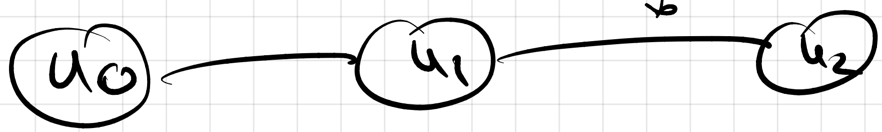
• either none such vertex, or there is one

Start build a path in that one if exists, or anywhere is no node $deg=1$ exists. . $Start = u_0$, $\forall$ other node $u \neq u_0$ $deg(u) \geq 2$

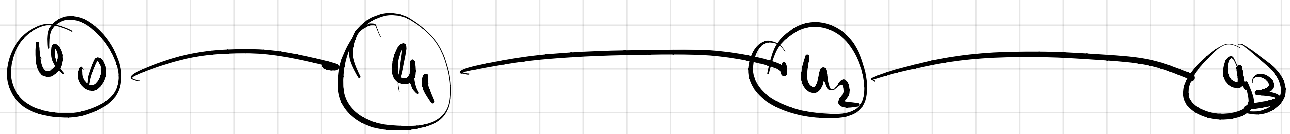
pick edge incident in u_0 — u_1



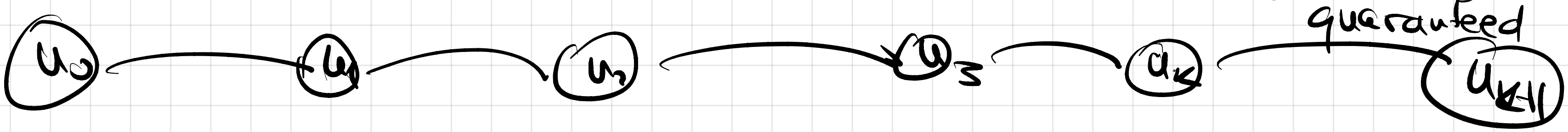
$deg(u_1) > 1 \Rightarrow u_1$ must have another edge to u_2



$deg(u_2) > 1 \Rightarrow u_2$ must have another edge (say u_3)



deg ≥ 2
new edge
guaranteed



This cannot go on forever!

- $\deg(u_k) = 1$ no further edge $\Rightarrow$ done found 2^{nd} vertex $\deg = 1$

OR

- u_k repeats $u_k =$ previous u_t $(t < k)$

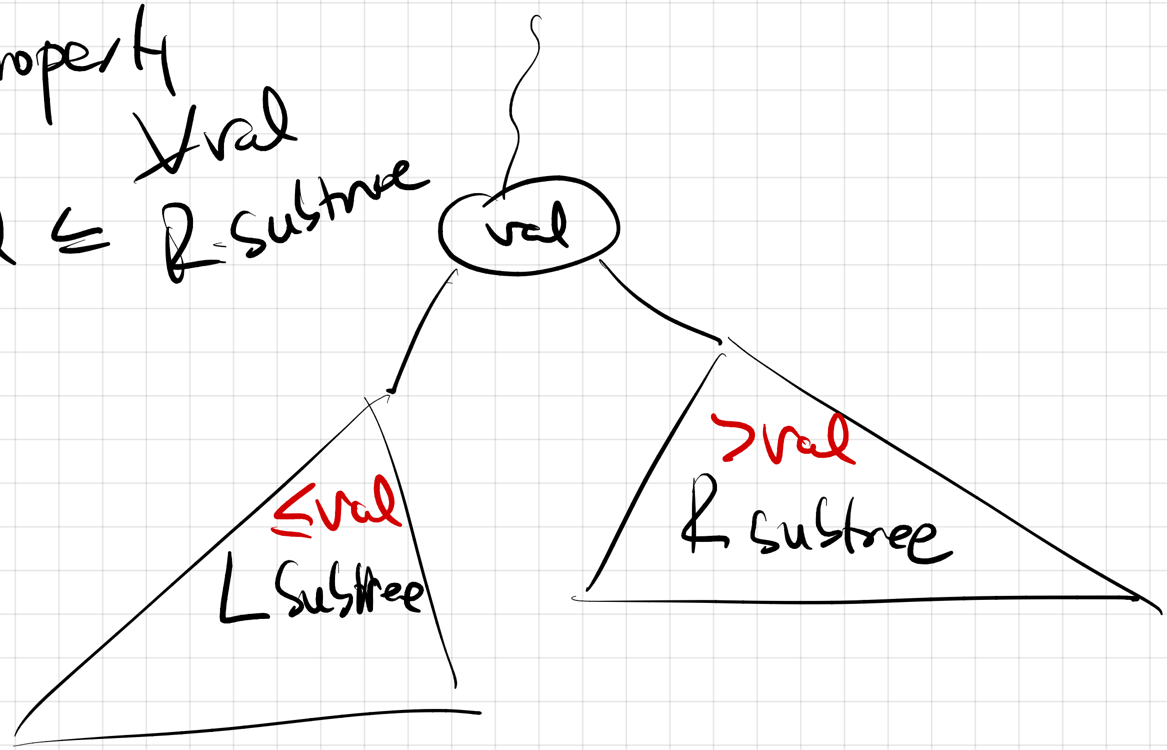
  $\Rightarrow$ cycle contradiction

Binary Search Tree

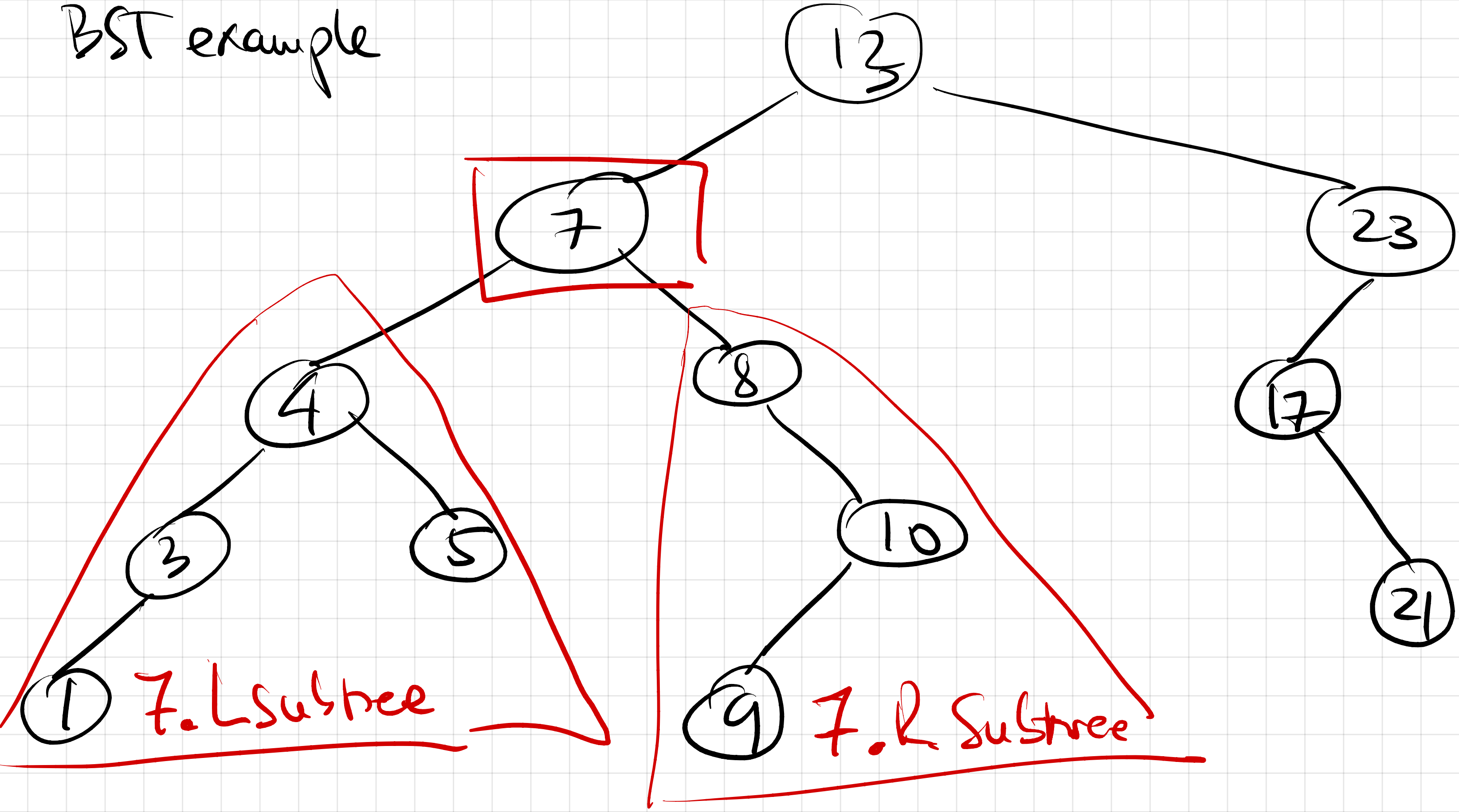
- store values in nodes (vertices)
- root
- binary: at most 2 children.

BST property

$\forall$val in L.subtree $\leq$ val $\leq$ $\forall$val R-subtree



BST example



BST sort :- insert all elements in BST (maintain BST prop)

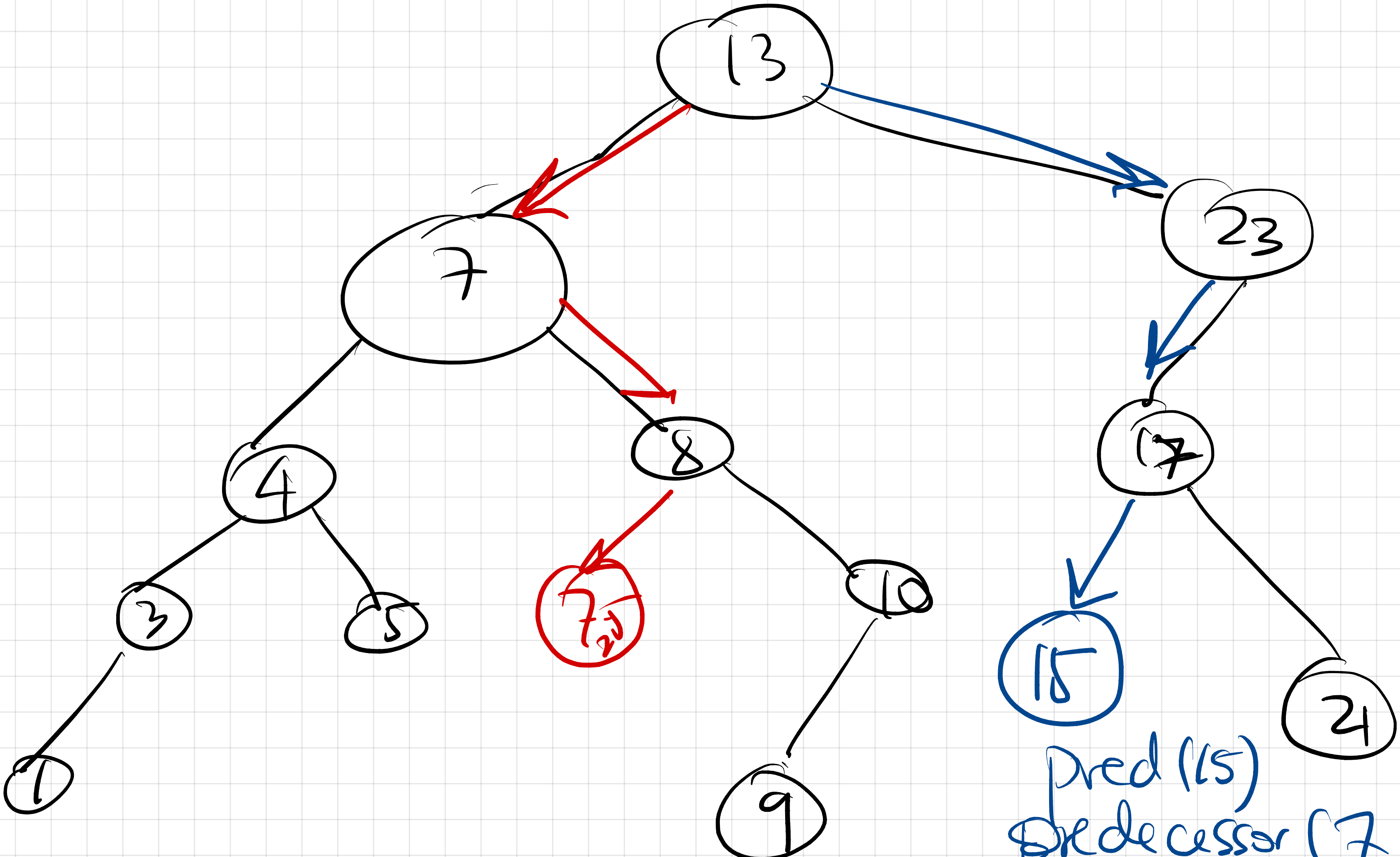
- read all values "in order" Left-val-Right

INORDER (node)

- inorder (node. L Subtree) If ∃ L. Subtree
- print "node. val"
- inorder (node R. Subtree) if ∃ R. Subtree.

Inorder(root) ⟹ produce all values sorted.

INSERT : navigate to the spot, add node



pred(15)
predecessor(7.5) = ?

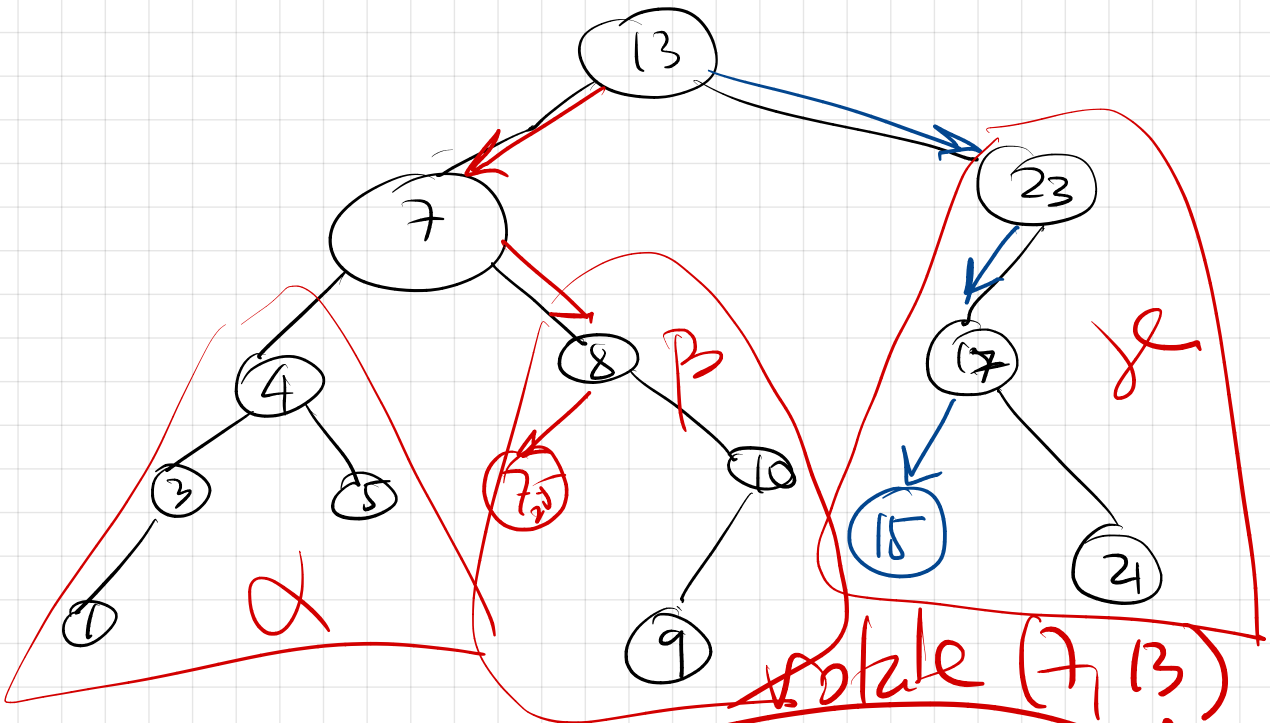
- min
- max

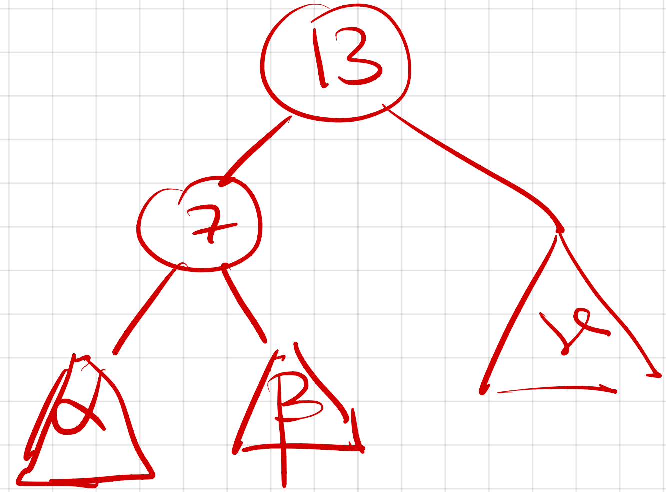
insert 7.5

insert 15

- delete
- rotate
- successor, predecessor

next val in sorted order





$\alpha \leq 7 \leq \beta \leq 13 \leq \gamma$

rotate (7, 13)

7 as root

STILL BST