- with repetition: $S = \{A, B, C, D\}$

sequence of 5 letters (rep. allowed): A B A A C

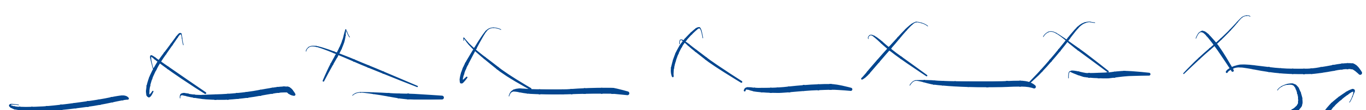
~~A~~ ~~A~~ C D C

4 possi 4 possb 4 possib

- any combination (sequence) valid

$$\#\text{sequences} = 4 \times 4 \times 4 \times 4 \times 4 = 4^5$$

- License plates 8 spots



26 ~~38~~ 36 ~~36~~ 8

$S = \{$ Letters (cap) + all digits $\}$ $\Rightarrow 36$

• $\boxed{\text{no repetitions}}$ $S = \{a, b, c, d, e\}$

$\Rightarrow$ max #spots $= |S|$
(longest sequence)

5 spots

$\boxed{\text{permutation (of all elements in } S)}$ $= 5!$

$\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$ $\underline{\quad}$

5 4 3 2 1 120

(one is used) (two are used) the only one left

no repetitions

a	b	c	d	e
b	c	d	a	e
e	d	b	c	a
⋮				

5-plets
sequence of length 5

no repetitions $S = \{a, b, c, d, e\}$ $n=5$

sequences of length $k=3$ $k \leq n$

$5 \cdot 4 \cdot 3$

$$\underset{n}{\underline{\quad}\;5} \quad \underset{n-1}{\underline{\quad}\;4} \quad \underset{n-2}{\underline{\quad}\;3}$$

$\boxed{\begin{matrix}(a\,b\,c)\\(b\,a\,c)\end{matrix}}$ diff
$(c\,a\,b)$
$(d\,a\,c)$
$(e\,d\,b)$

$$\frac{5!}{2!} \qquad \frac{5!}{(5-3)!}$$

general $S = \{1, 2, \ldots n\}$
want sequence of length $k \leq n$

$n \times (n-1) \times \ldots \times (n-k+1)$

$\underline{\quad} \times \underline{\quad} \times \underline{\quad} \times \underline{\quad} \times \underline{\quad}$

spot 1: n

spot 2: $n-1$ (one used)

$n-2$

spot k: $n-k+1$

choose seq of k IN ORDER out of n $= P(n,k) = \dfrac{n!}{(n-k)!} = \dfrac{n \cdot (n-1) \cdots (n-k+1)\,(n-k) \cdots 1}{(n-k)(n-k-1) \cdots 1}$

• no repetition. no -order

n items: 1, 2, ... n
K spots

$$\underset{\text{spot1}}{\underline{n}} \times \underset{\text{spot2}}{\underline{(n-1)}} \times \underset{\text{spot3}}{\underline{(n-2)}} \times \underline{\quad} \quad \underline{\quad} \times \underset{\text{spot}K}{\underline{(n-k+1)}}$$

remaining
n-K

write it down (output) AS A SET

n=5 K=3

sequen

a	b	c	d, e
b	a	c	d, e
c	a	b	
a	c	b	
b	c	a	
c	b	a	

$$6 = \frac{5!}{2!}$$

$$\frac{n!}{(n-k)!}$$

SETS

one set {a, b, c}
{b, a, c}
{c, a, b}
{a, c, b}
{b, c, a}
{c, b, a}

K!
occurrences of set {a, b, c}

$\#$ different sets $\times K! = \#$ diff seq

$$P(n,k) = \frac{n!}{(n-k)!}$$

$$\#\text{diff sets} = \frac{n!}{(n-k)!\,k!}$$

choose k items (no order, no repetition) out of n

$$\text{"n choose k"} = \boxed{\binom{n}{k}} = C(n,k) = nCk$$

choose a subset of size k out of a set of n elements

set = {1, 2, 3, ... n}

$$\binom{n}{k} = \#\text{different subsets of size } k \text{ (out of } n)$$

$$2^n = \underbrace{\binom{n}{0}}_{\substack{\#\text{subsets}\\ \text{size}=0}} + \underbrace{\binom{n}{1}}_{\substack{\#\text{subsets}\\ \text{size } 1}} + \underbrace{\binom{n}{2}}_{\substack{\#\text{subsets}\\ \text{size } 2}} + \underbrace{\binom{n}{3}}_{\substack{\#\text{subsets}\\ \text{size } 3}} + \cdots + \underbrace{\binom{n}{n-1}}_{\substack{\#\text{subsets}\\ \text{size } n-1}} + \underbrace{\binom{n}{n}}_{\substack{\#\text{subsets}\\ \text{size}=n}}$$

$\emptyset$

{1}
{2}
{3}
{n}

{1, 2}
{1, 3}
⋮
{n-1, n}

{1, 2, 3}
{1, 2, 4}
⋮
{n-2, n-1, n}

{1, 2, ... n-1}
⋮
{2, 3, ... n}

1

{1:n}

COUNT 1

Two decks of cards are mixed together, total 104 cards where each card appears exactly twice. How many distinct permutations are there of all 104 cards?

global: $\frac{104!}{\#\text{double counted}} = \frac{104! \text{ all permute}}{\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{52}}$

2^{52}

$2! = 2$

Constructive

_ _ _ $\overset{3}{X}$ _ $\overset{5}{X}$ _ _ _ 104 spots

choose 2 spots for (A♦ two) $\binom{104}{2} = \frac{104 \cdot 103}{2}$

choose 2 spots for (Q♣ two) $\rightarrow \binom{102}{2} = \frac{102 \cdot 101}{2}$

choose 2 spots for (3♡ two) $\rightarrow \binom{100}{2} = \frac{100 \cdot 99}{2}$

⋮

choose (last rem 2 spots) for las card ×2 $\rightarrow \binom{2}{2} = \frac{2 \cdot 1}{2}$

COUNT 2

A) How many ways to place a rock, knight and bishop on a chess board (8x8) such that no two of them are on the same row or column?

$\boxed{\binom{8}{3}\binom{8}{3}\cdot 3! \times}$ 3! please

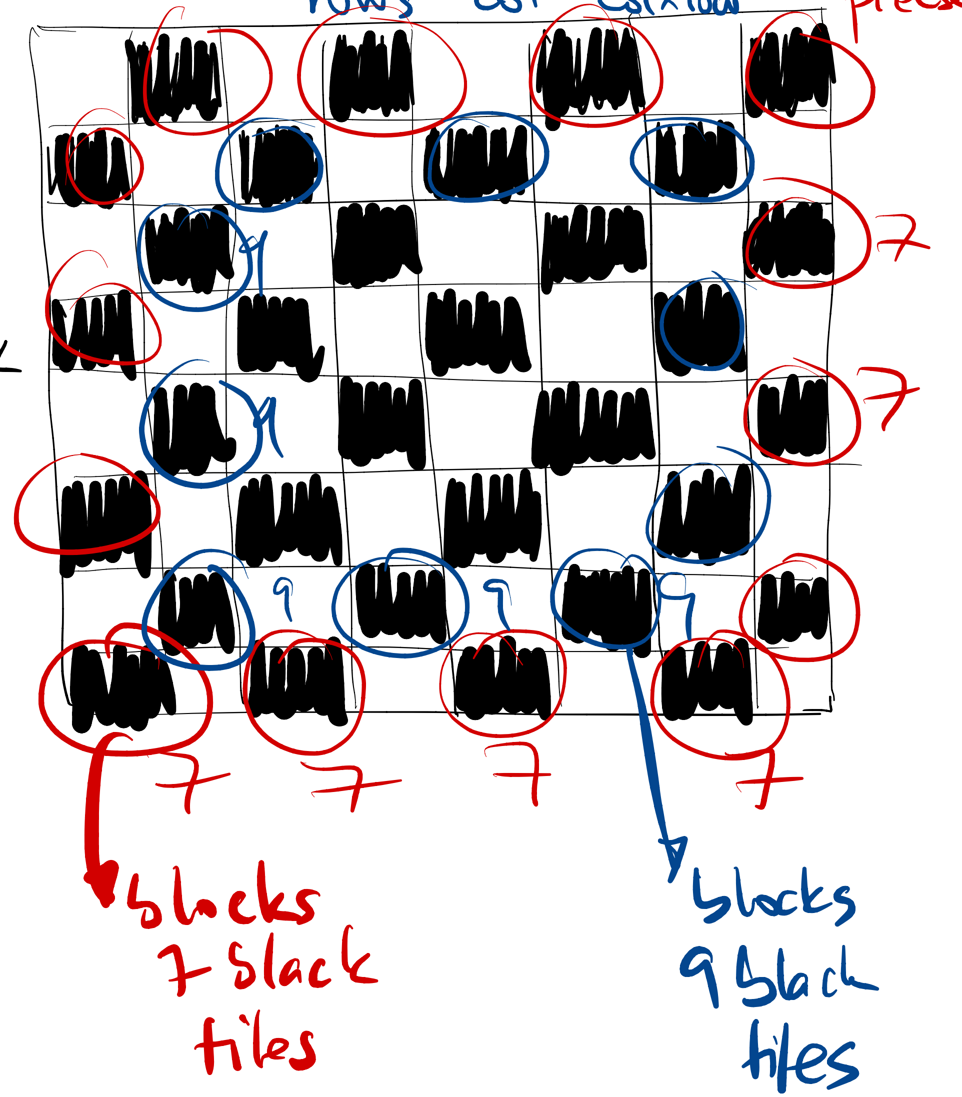
rows col col x row pos

$8^2 \times 7^2 \times 6^2$?

B★) How many ways to place 2 bishops that do not attack each other? [bishop attack on diagonals]

exercise

C) ~~★★★~~ 3 Bishops



COUNT 3

How many different permutations of "MISSISSIPPI"?

naive: $M_1, S_1, S_2, S_3, S_4, P_1, P_2, I_1, I_2, I_3, I_4$

11 objects: Permute them $\Rightarrow$ 11!

same $\left(\begin{matrix} M_1 S_1 I_1 S_2 I_2 I_3 \dots \\ M_1 S_2 I_2 S_1 I_3 I_4 \dots \end{matrix}\right.$ WRONG!

Sol 1: $\dfrac{\text{count all permutations } 11!}{\text{\# permutations make same word}}$

SIMSISIPISP = $\dfrac{\boxed{\text{\# permutations } S_i I_i M S_i \dots}}{\text{all those give same word}}$

4×4×2?

4!4!.2!?

2 5 7 9

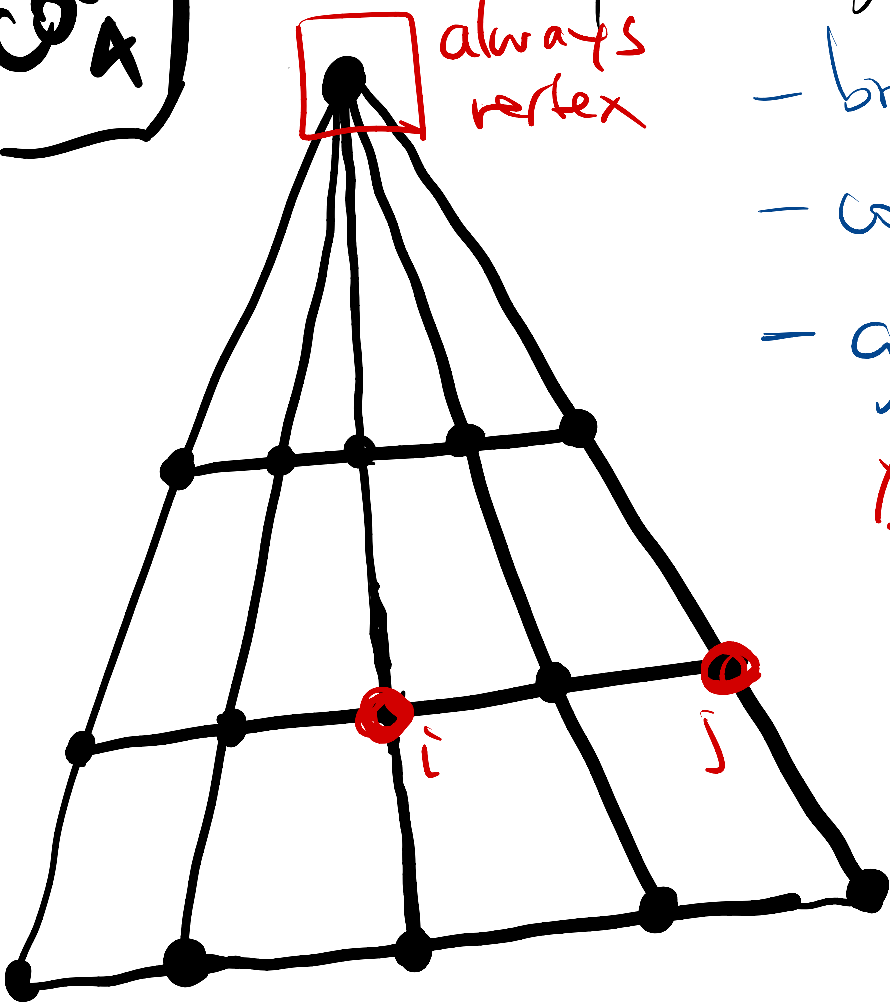
I-pos = {2,5,7,9} 4! to place 4-Is on those spots.

Constructive

- place the S : S S S S 1 choice
  seq 4
- place I-s in between S S S S
  Box 1 Box 5
  4 I-s into 5 boxes
  ⇓
  seq of of 8 S+I
- place P-s in between
  2 P-s in 9 boxes
  P+S+I
- place M ⟹ 11 choices
- product of choices ⟶ ? exercise next week

COUNT 4

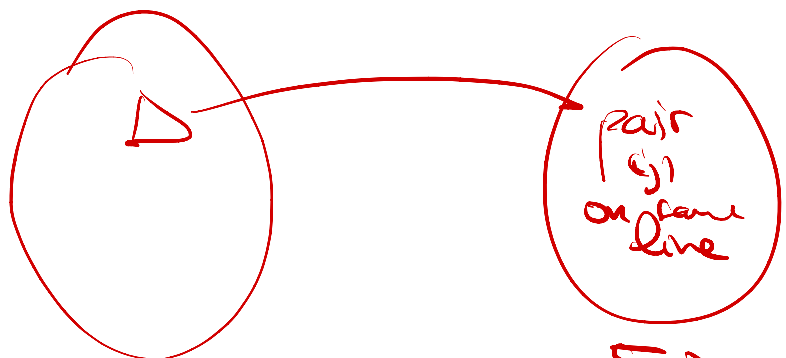
How many triangles?



Divide & conquer:

- break pb into choices/subpb
- count/solve those subpb/choices
- glue them together

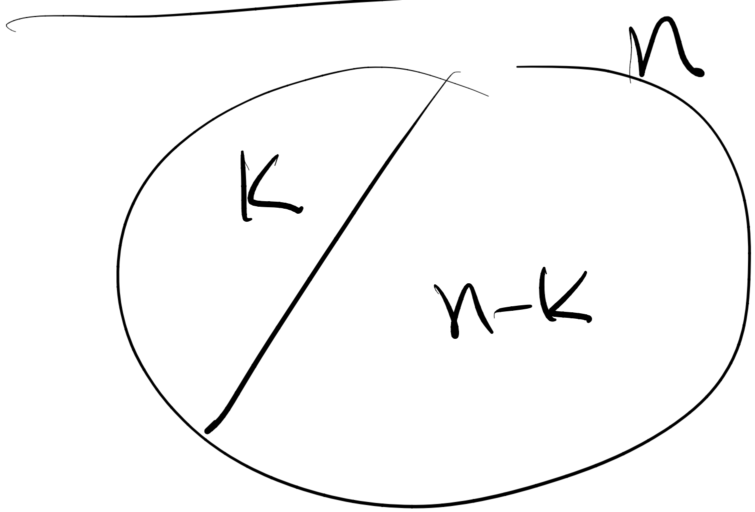
Δ = choose 2 dots on same horizontal line



$$\#\text{pairs } (i,j) \text{ on same line} = \underbrace{3}_{\text{choose a line}} \times \underbrace{\binom{5}{2}}_{\text{choose 2 dots out of 5 (no order, no repet)}} = 3 \cdot \frac{5!}{2!\,3!}$$

Th $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$ $\qquad \displaystyle\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

$$\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} = 2^n$$



choosing k-set
$\equiv$ choosing remaining n-k set

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\frac{n}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,(n-(n-k))!}$$

PB 4 Counting with cases (Sum Rule disjoint preferable OR PIE not disjoint)

$36 = \overset{T}{13 \text{ witches}} + \overset{W}{4 \text{ warlocks}} + \overset{V}{12 \text{ vampires}} + \overset{G}{7 \text{ goblins}}$

distinguishable

3 discussion sessions A, B, C ⟹ partition of all, $\boxed{12 \text{ each}}$

(A) How many ways to make sessions?

$$\underbrace{\binom{36}{12}}_{\text{choose session A}}\underbrace{\binom{24}{12}}_{\text{choose session B}}\underbrace{\binom{12}{12}}_{\text{session C}}$$

(B) Sessions with restriction: all warlocks same session.

$$\underbrace{\binom{32}{8}}_{\text{A session W}}\binom{24}{12}\binom{12}{12} + \binom{32}{12}\binom{28}{8}\binom{12}{12} + \text{warlocks in "C"}$$

nonW → 32; nonW → 32; nonW → 28

warlocks in B

© session ©= 2G + 2V + 4T + 4W

want © to break into 2 groups ("red" "blue") of size 4 each

Constraints:

goblins ∉ Blue ; vampires ∉ Red.

3 cases: #G in Red group. $\rightarrow$ 0, 1, 2

red 0G : $\binom{8^{\text{nonGV}}}{4}_{\text{red}} \times \binom{4^{\text{nonGV}} + 2^{V}}{4}_{\text{blue}}$

red 1G : $\binom{2}{1}_{\text{goblin}} \times \binom{8^{\text{nonGV}}}{3}_{\text{red}} \times \binom{5^{\text{nonGV}} + 2V}{4}_{\text{blue}}$

2G : $\binom{2}{2} \times \binom{8\,\text{nonGV}}{2} \times \binom{6\,\text{nonGV} + 2V}{4}_{\text{blue}}$

$\boxed{\text{Binomial theorem}}$ (coef) $\Rightarrow$ Pascal $\triangle$ $x, y \in \mathbb{R}$

$n=2$ $(x+y)^2 = 1x^2 + 2xy + y^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$

$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

$\binom{4}{2} = \frac{4!}{2!\cdot 2!} = \frac{3\cdot 4}{2}$

$\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$

$\binom{4}{1}x^{4-1}y^1$ $\binom{4}{2}x^{4-2}y^2$ $\binom{4}{3}x^{4-3}y^3$

$(x+y)(x+y)(x+y)(x+y) \rightarrow$ 16 terms (incl repetion)

4 factors ( )

$\rightarrow x^4$ — 1 way: x x x x

$\binom{4}{3}$ $\leftarrow$ $\boxed{x^3y^1}$ (3 x, 1 y) xxxy xxyx xyxx yxxx 4

3 x choice

distinct terms $\boxed{x^2y^2}$ xxyy xyxy xyyx yxxy yyxx yxyx 6

x^1y^3

$\binom{4}{2}$ choose 2 x

y^4

how many ways to choose j out of n

y choice in j paran ( )

x choice in n-j paran ( )

$$(x+y)^n = \sum_{j=0}^{n} \underbrace{\binom{n}{j} x^{n-j} y^j}_{\text{choose } j \text{ "y"},\ n-j \text{ "x"}} = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$$

2^n terms (with repetitions)

$x=1 \quad y=1$

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} \cdot 1^j = \sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

subsets $|\,|=0$

subsets $|\,|=1$

subsets all

$x=+1,\ y=-1$

$$0 = (1-1)^n = \sum_{j=0}^{n} \binom{n}{j} \boxed{1^{n-j}} \underline{\underline{(-1)^j}} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} \dots (-1)^n \binom{n}{n}$$

$n=3 \qquad 1-3+3-1=0$

$n=4 \qquad 1-4+6-4+1=0$

$n=5 \qquad 1-5+10-10+5-1=0$

exercise: $\binom{n}{k} = \binom{n}{n-k}$ choose subset of k "in"

$\iff$ choose n-k stay out

n-k $\}$ k

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

proof $\{1\ 2\ 3 \ldots n\}$

$\binom{n}{k}$ = # ways to choose a subset of k out $\{1,2,3,\ldots n\}$

= SUM RULE

case1 include last elem "n"

$\{\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{from } \{1,2,\ldots n-1\}}^{k-1}\ n\}$ $\binom{n-1}{k-1}$

case2 dont inclue "n"

$\{\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{\text{from } \{1,2,\ldots n-1\}}^{k \text{ elem}}\}$ $\binom{n-1}{k}$

verify (exercise) with factorials.

exercise

$$\binom{n}{k} = \binom{n}{n-k}$$

Pascal △

Binomial coef

n=1	1	1					$(x+y)^1$
2	1	2	1				$(x+y)^2$
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	-6	+15	-20	+15	-6	+1

exercise

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$

$$\binom{5}{1} = \binom{4}{0} + \binom{4}{1}$$

(*)

$x = p \quad y = 1-p$ $\quad p =$ prob of success (coin flip)

$1-p =$ prob failure

$$1 = (p+(1-p))^n = \sum_{j=0}^{n} \underbrace{\binom{n}{j} p^{n-j}(1-p)^j}_{\text{prob(exactly } j \text{ successes)}}$$ Binomial distribution

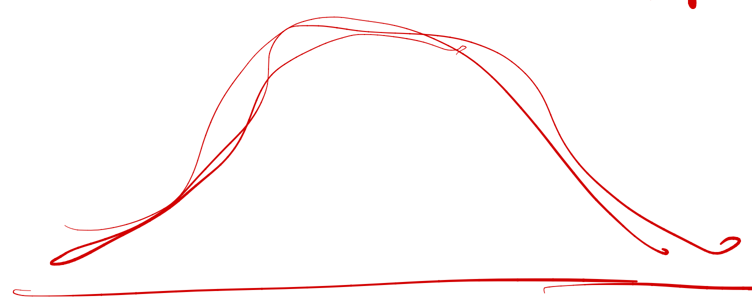
$p = \frac{1}{3}$ chance of sum

$n = 100$ coin flips

$p = \frac{1}{2}$:

$$\frac{\sum_{j=0}^{n} \binom{n}{j}}{2^n}$$

dist on piazza ; **Binomial** $n=100, p=\frac{1}{2}$



gauss approxim (mean $n/2$, var $n/4$)

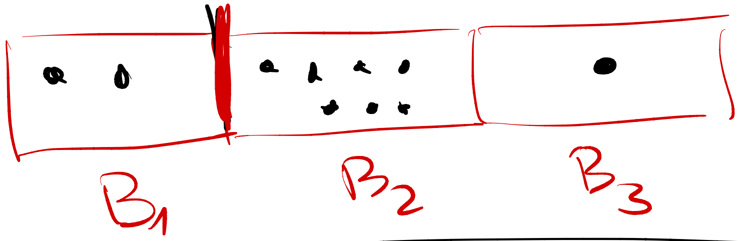
good approx for $n \geq 9$

BALLS INTO BINS (STARS AND BARS)

#ways to place n identical balls in K bins
(not distinguishable) (distinguishable)

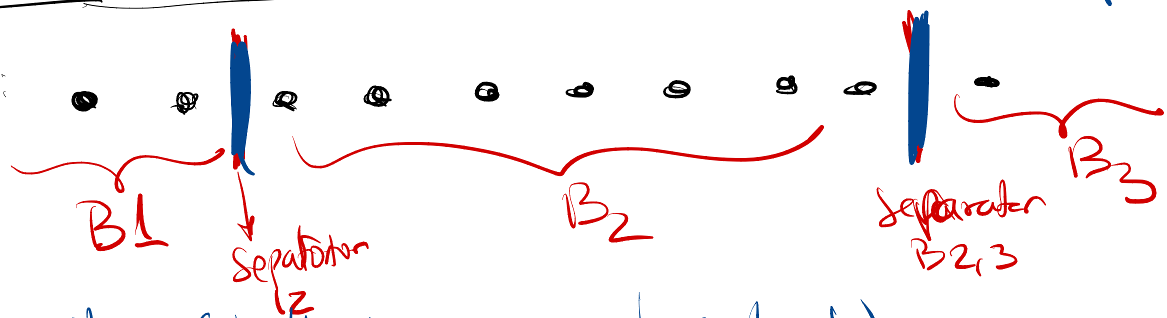
ex: $n=10$ candies distribute to $K=3$ children
(identical) $c_1, c_2, \dots c_k$

Throw balls at random



$K-1$ separators

Solution:



$n+k-1$ items (balls n, separators $k-1$)

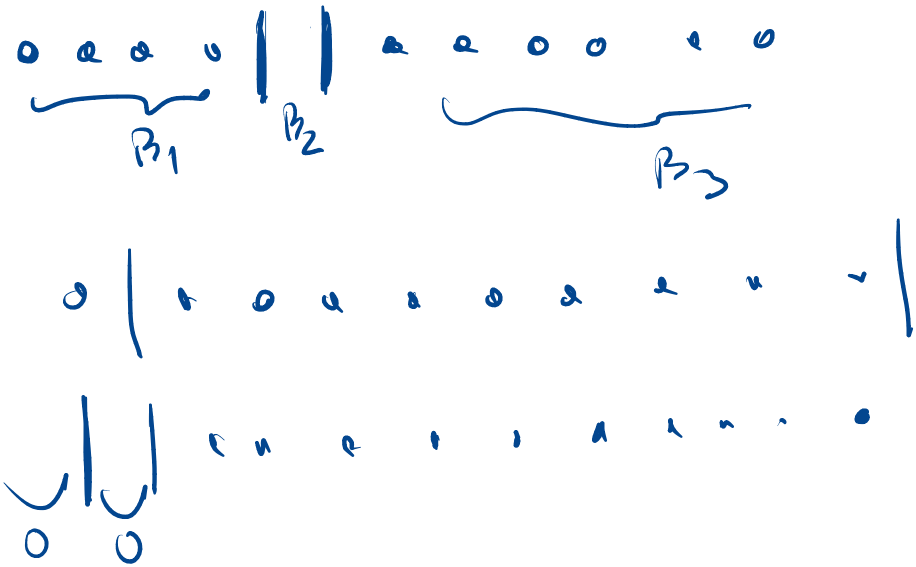
2,7,1 • • | • • • • • • • | • • → #ways choose $k-1$ spots "|"

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

4,0,6

1,9,0

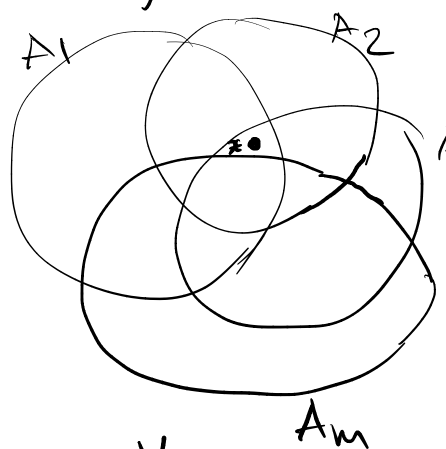
0,0,10



Rules for proper counting

- items are distinguishable / NOT
- REPETITIONS / NOT (REP)
- ORDER / NOT ORDER

PIE general proof $|A_1 \cup A_2 \cup A_3 \cup \dots \cup A_m| =$



$= |A_1| + |A_2| \dots + |A_m|$ // |one set|

$- |A_1 \cap A_2| \dots - |A_i \cap A_j| \dots - |A_{m-1} \cap A_m|$ $-|\cap$ of $2|$

$+ |A_1 \cap A_2 \cap A_3| \dots + |A_i \cap A_j \cap A_k| + \dots + |A_{m-2} A_{m-1} A_m|$ $+ |\cap$ of $3|$

$-$

$+$

$\vdots$

$+ (-1)^{m+1} |A_1 \cap A_2 \cap \dots \cap A_m|$ $\dots$ $|\cap$ of $m|$

Select $\forall x$ in $A_1 \cup A_2 \cup \dots A_m$. Its going to part of some sets

without loss of generality assume $x \in A_1 \cap A_2 \dots \cap A_n$ $(n \leq m)$

$x \notin A_{n+1} \cup A_{n+2} \dots A_m$

plan $\boxed{\text{count } x}$ on RHS

$+ \binom{n}{1}$ $|A_1|$ $|A_2|$ $\dots$ $|A_n|$

$- \binom{n}{2}$ $|A_1 \cap A_2|$, $|A_1 \cap A_3|$ $\dots$ $|A_{n-1} \cap A_n|$

$+ \binom{n}{3}$ $|A_1 \cap A_2 \cap A_3|$ $\dots$ $|A_{n-2} \cap A_{n-1} \cap A_n|$

$\vdots$

$$(-1)^{n+1}\binom{n}{n}|A_1 \cap A_2 \cap \dots A_n|$$

$$\text{count}(x) = \binom{n}{1} - \binom{n}{2} + \binom{n}{3} \dots + (-1)^{n+1}\binom{n}{n}$$

Binomial Th: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} \dots + (-1)^n\binom{n}{n} = 0$

$$1 - \text{count}(x) = \overset{1\ \parallel}{\binom{n}{0}} - \binom{n}{1} + \binom{n}{2} \dots - (-1)^n\binom{n}{n} = 0$$

$$1 - \text{count}(x) = 0 \implies \text{count}(x) = 1 \quad \checkmark$$

PIE application: Derangement = permutation without fix points (index sits on its own spot)

n=5

2 3 1 5 4 Derang...

pos 1 2 3 4 5

3 $\boxed{2}$ 4 5 1 NOT DER (pos(2)=2)

#Derangement(5) = ?

all perm − all perm fixed points

A_1 = {permut (1 fixed)} $\boxed{1}\ \underbrace{\_\ \_\ \_\ \_}_{4!}$

A_2 = {perm (2 fixed)} _ 2 _ _ _

A_3 = {perm (3 fixed)} _ _ 3 _ _

A_4 = {perm (4 fixed)} _ _ _ 4 _

A_5 = {perm (5 fixed)} _ _ _ _ 5

$$= n! - \underbrace{|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|}$$

$$= \underbrace{|A_1|}_{4!} + \underbrace{|A_2|}_{4!} \ldots$$

$$- \underbrace{|A_1 \cap A_2|}_{3!} \ldots$$

$$+ \underbrace{|A_1 \cap A_2 \cap A_3|}_{2!} \ldots$$

$A_1 \cap A_2$ = {1 2 $\underbrace{\_\ \_\ \_}_{3!}$}

$A_1 \cap A_2 \cap A_3$ = {1 2 3 $\underbrace{\_\ \_}_{2!}$}