The geometric distribution + mean of trials expected to the first succes with fixed probability.

You play lottery every day, fixed chance of winning p. Say X = R.V. = # days until first success. What is the distribution of X? $E[X]=$?

Solution $Pr[X=1]$ is the chance of success p

$Pr[X=2]$ is fail first day $(1-p)$ and succeed second : $p(1-p)$

$Pr[X=3]$ is fail 1^{st} day, fail 2^{nd} day ($\substack{both \\ 1-p}$) and succeed 3^{rd} $p(1-p)^2$

$\vdots$

$Pr[X=k]$ is fail the first $(k-1)$ days and succeed k^{th} : $p(1-p)^{k-1}$

"Geometric distribution"

Sanity check : $\sum_{k=1}^{\infty} P(X=k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p\sum_{k=1}^{\infty}(1-p)^{k-1} = p \cdot \frac{1}{1-(1-p)} = 1$ ✓

Now the $E[X] =$

$$= \sum_{k=0}^{\infty} k \cdot P[X=k] = \sum_{k=1}^{\infty} k \cdot p \cdot (1-p)^{k-1}$$

The plan here is to telescope V: Compare V vs $V\cdot(1-p)$

$$V = \sum_{k=1}^{\infty} k\cdot p\cdot(1-p)^{k-1} = p + \sum_{k=2}^{\infty} k\,p(1-p)^{k-1}$$

$$(1-p)V = \sum_{k=1}^{\infty} k\,p\,(1-p)^{k} = \sum_{k=2}^{\infty} (k-1)\,p\,(1-p)^{k-1}$$

$$V-(1-p)V = p+\sum_{k=2}^{\infty} kp(1-p)^{k-1} - \sum_{k=2}^{\infty} (k-1)\,p\,(1-p)^{k-1} =$$

$$= p + \sum_{k=2}^{\infty} \left[kp-(k-1)p\right]\cdot(1-p)^{k-1} =$$

$$= p + \sum_{k=2}^{\infty} p\cdot(1-p)^{k-1} = \sum_{k=1}^{\infty} p\cdot(1-p)^{k-1} = \sum_{k=1}^{\infty} P(X=k) = 1$$

(part A)

$$\Rightarrow V(1-(1-p)) = 1$$

$$V\cdot p = 1$$

$$V = \frac{1}{p} = E[X].$$

Note: if chance of winning lottery is $p=10^{-6}$

$\Rightarrow E[\#\text{ days to win}] = \frac{1}{p} = 10^{6} = 1000{,}000$ days!