11/8

Induction

Consider this problem regarding change (coins)

Claim: for $n >= 8$, n cents can be made with

3¢ coins and 5¢ coins

We build from a base to make something true for All numbers

Is it true for 8 - yes 1 3¢
1 5¢

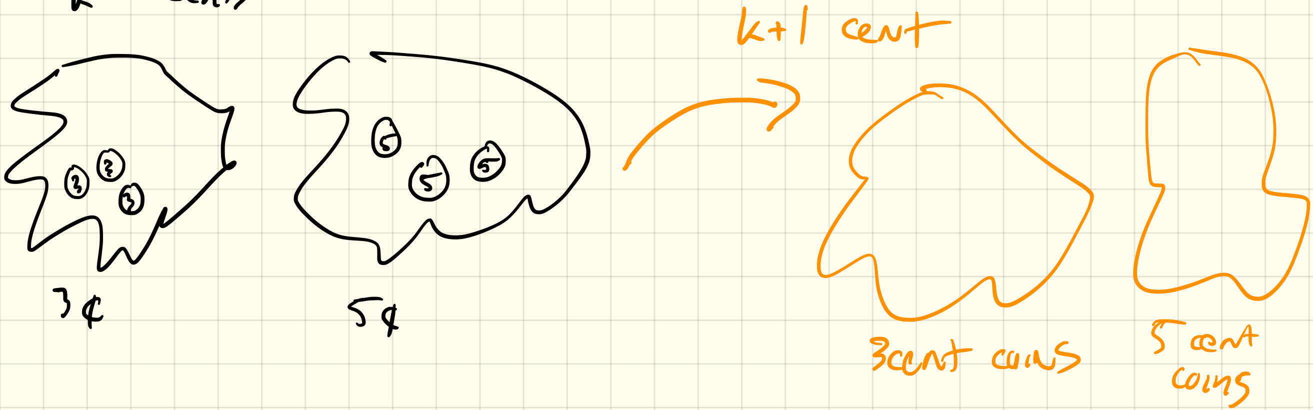
Is it true for 9 - yes 3 · 3¢

Is it true for 10 - yes 2 · 5¢

How might we prove for all n?

Let's say we know we have done all the work up to k for some k.

k cents



Can I use this to get $k+1$?

2 cases

$|5 \text{ coins}| = 0$ — take away 3 - 3cent coins ADD 2 · 5cent coins

$$k - 9 + 10 = k + 1$$

$|5 \text{ coins}| \neq 0$ — take away 1 5cent coin ADD 2 3¢ coins.

$$(k) \to$$

$$k - 5 + 6 = k + 1$$

$\boxed{P(n)}$

Mawolic - 1575

used by Fermat & Pascal

But

De Morgan formalized it & named it Mathematical induction

P is some property defined over the integers and a is some fixed integer AND

1. $P(a)$ is true

2. $\forall k \in \mathbb{Z}, k >= a, P(k) \rightarrow P(k+1)$

(For all integers k, k bigger or equal to a, if $P(k)$ is true then $P(k+1)$ is true)

This implies $\forall n \in \mathbb{Z}, n >= a, P(n)$ (Fr all For all bigger Than or equal to a $P(n)$ is true)

Proof Technique:

We often want to prove a statement (property)

$\forall n \in \mathbb{Z}, n \geq a, P(n)$ is true

We do this in two step!

Base case (Basis case) — show $P(a)$ is true

Inductive step (we want to satisfy $\forall k \geq a$ if $P(k)$ is true then $P(k+1)$ is true)

To do this we assume $P(k)$ and use it to prove $p(k+1)$

Coins V. 2.0

$\forall n \geq 8, \exists m_1, m_2 \in \mathbb{N}$, s.t
$n = 3m_1 + 5m_2$

$\forall n \geq 8$
$n = 3m_1 + 5m_2$
for some: m_1 & m_2

P(n) $\forall n >= 8$, n cents can be made with 3¢ coins and 5¢ coins

(we argue by mathematical induction.)

Proof by induction:

Consider the base case when $n = 8$. We know that 8 can be made with 1 5¢ coin and 1 3¢ coin. So 8 cents can be made with 3 and 5 cent coins.

Now consider an inductive hypothesis for k an arbitrary integer. $k \geq 8$ We know from our hypothesis that k can be made from 3 cent coins and 5 cent coins,

There are two cases:

Case 1. k is made up of at least one 5 cent coin.

then to show that $k+1$ can be made from 3 & 5 cent coins we use all of the coins from k leaving out 1 5 cent coin and adding in 2 3 cent coins. Since $k-5+6=k+1$ we know $k+1$ can be made from 3 and 5 cent coins.

Case 2: k is made up of only 3 cent coins.

We know from our assumption $k \geq 9$, so we have at least 3 · 3 cent coins.

We remove 3 · 3cents coins and add two 5 cent coins.

Since $k-9+10=k+1$, we know $k+1$ can be made from 3 and 5 cent coins.

So for all $n \geq 8$, n cents can be made using 3 and 5 cent coins.

$\boxtimes$ Q.E.D.

$$\underbrace{\sum_{i=1}^{n} i = 1+2+3+\dots+n = \frac{n(n+1)}{2}}_{a \;=\; b}$$

property is the statement itself.

(Sum of 1 to n is equal to $\frac{n(n+1)}{2}$)

Base Case: $a = 1$

1 ~ left hand side

$\frac{1(1+1)}{2} = \frac{2}{2} = 1$ → right hand side

So we're good.

Inductive Hypothesis $k >= 1$

$$\boxed{1+2+3 \dots + k = \frac{k(k+1)}{2}}$$

We assume k

From our Assumption we know $1+2+3+\ldots+k = \frac{k(k+1)}{2}$

Consider left hand side for $k+1$

$$1+2+3+\ldots+k+k+1$$

From our Assumption this is $\frac{k(k+1)}{2} + k+1$

But this is $\frac{k(k+1)}{2} + \frac{2(k+1)}{2}$

which simplifies to $\frac{(k+2)(k+1)}{2}$

which is the derived right hand side for $k+1$

So $1+2+3+\ldots+k+1 = \frac{(k+2)(k+1)}{2}$ which implies

$1+2+\ldots+n = \frac{n(n+1)}{2} \quad \forall n, n>=1$ ☒

$$\sum_{i=1}^{n} i^2 = 1+4+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by induction:

Consider the base case when $n=1$

we have 1 on left side

we have $\frac{1(1+1)(2+1)}{6} = \frac{6}{6} = 1$ as desired.

Now we assume by inductive hypothesis that for some $k \geq 1$

$$1+4+9+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

And we argue that $1+4+9+\dots k^2+(k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$

Left Hand Side

$$1+4+9+\dots+k^2+(k+1)^2$$

Right Hand Side

$$\frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

we know $1+4+9+\dots+k^2 = \dfrac{k(k+1)(2k+1)}{6}$

So

$$1+4+9+\dots+k^2+(k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)(k+1)$$

which is $\dfrac{k(k+1)(2k+1)}{6} + \dfrac{6(k+1)(k+1)}{6}$

which is $\dfrac{k(k+1)(2k+1)+6(k+1)(k+1)}{6} = \dfrac{k(2k+1)(k+1)+6(k+1)(k+1)}{6}$

$$\big(6(k+1)+k(2k+1)\big)(k+1)$$

$$\frac{(6k+6+2k^2+k)(k+1)}{6}$$

$$\frac{(2k^2+7k+6)(k+1)}{6}$$

$$\frac{(2k+3)(k+2)(k+1)}{6}$$

Now consider right hand side $\frac{(k+1)(k+2)(2(k+1)+1)}{6}$

But this is $(k+1)(k+2)(2k+3)/6$ - so LHS = RHS.