Recitation 8: Sets, Basic counting

Problem 1 Set Counting, Indexing

Let $A = \{x | x \in \mathbb{N} \wedge x < 15\}$, $B = \{x | x \in \mathbb{Z} \wedge x < 20 \wedge x > -4 \wedge x/2 \in \mathbb{Z}\}$, $C = \{x | x \in \mathbb{N} \wedge x^2 \in A\}$. Calculate the following expressions using the counting tools we have learned as appropriate. Show your work.

i. $|A \cup B \cup C|$

ii. $|A \times (B \cup C)|$

iii. $|A - B| + |C - B| + |B| - |(A \cap C) - B|$

iv. How many positive integers less than 2500 are divisible by 15 but not by 20 ?

Problem 2 Counting Fashion

i. A math professor wears for each class a combination of : one of their three hats (red, blue, green); one of their pants (black, blue, white, green); one of their shirts (white, red, black); and one of their pair of shoes (black, red, blue, white). They never wear combinations with three items of the same color. How many different combinations are wearable?

ii. The professor is teaching 70 lectures for the term. Can they wear a different combination for each lecture during one term?

iii. If they do wear a different combination each lecture, show that they must use either the red or the blue shoes that term.

iv. There are 4 terms for the year. Show that at least one combination will be used at least 3 times during the year.

v. How many valid combinations have blue hat or shoes ?

Problem 3: Reasoning with set sizes

i. Is there any set S such that $|S \times S| = 30$?

ii. Let A, B be arbitrary sets. Can $|(A \times B) \cap (B \times A)| = 30$? Give an example or explain why not.

iii. When is $A \triangle B = \emptyset$?
Recap : Symmetric Difference of sets A, B is $A \triangle B = (A \cup B) - (A \cap B)$

iv. Do sets $A \neq B$ exist such that $A \times B = B \times A$?

v. Are there sets A and B such that $A \times B = \{(1,2),(1,5),(2,3),(2,5),(3,3),(3,5)\}$?

vi. True or False: $(A - C) - B = A - (C - B)$

vii. True or False: $(A - C) \cup (A - B) = A - (B \cup C)$

viii. Using the set equivalence laws, show that $\overline{(\overline{A} \cup \overline{B}) \cap A} = \overline{A} \cup B$

ix. Prove that $|A| + |B| + |A \cap B \cap C| \geq |A \cap B| + |A \cap C| + |B \cap C|$

Problem 4: Who owns the fish? There are 5 houses in 5 different colors in a row. In each house lives a person with a different nationality. Each of the 5 owners drinks a certain type of beverage, smokes a certain brand of cigar, and keeps a certain pet. No two owners have the same pet, smoke the same brand of cigar, or drink the same beverage. One of them owns a fish.

Other facts:

- The Brit lives in the red house.
- The Swede keeps dogs as pets.
- The Dane drinks tea.
- The green house is on the immediate left of the white house.
- The green house's owner drinks coffee.
- The owner who smokes Pall Mall rears birds.
- The owner of the yellow house smokes Dunhill.
- The owner living in the center house drinks milk.
- The Norwegian lives in the leftmost house.
- The owner who smokes Blends lives next to the one who keeps cats.
- The owner who keeps the horse lives next to the one who smokes Dunhill.
- The owner who smokes Bluemasters drinks beer.
- The German smokes Prince.
- The Norwegian lives next to the blue house.
- The owner who smokes Blends lives next to the one who drinks water.

The question is.... Who owns the fish???

Problem 5 Religion Around the Table

i. In an effort to address religious differences, 8 Christians ($c_1, .., c_8$), and 8 Muslims ($m_1, m_2, .., m_8$) sit at a round discussion table. In how many different ways can they sit such that every two adjacent people have different religions? In here "different ways" refers to at least one person having different left or right neighbors.

ii. (difficulty ★) Same problem with 8 Christians ($c_1, .., c_8$), 6 Hindus ($h_1, .., h_6$), and 3 Muslims (m_1, m_2, m_3).

Problem 6 (Intro to Project 3): How many valid dates?

10 men who are pairs of brothers ($a_1b_1, a_2b_2, a_3b_3, a_4b_4, a_5b_5$) are to blind-date 10 women who are pairs of sisters ($x_1y_1, x_2y_2, x_3y_3, x_4y_4, x_5y_5$) such that any two brothers do not date two corresponding sisters, that is for example if (a_2, x_4) is a date then b_2 cannot date y_4.
In how many ways can the dates be arranged? Explain why these solutions are wrong:

i. Jack Sparrow 's solution: There are 10! ways to arrange the dates without restrictions. There are $5! * 2^5$ ways to arrange dates that violates the restriction since it comes down permuting the pairs and then choosing for each pair which brother dates which sister (2 possibilites per pair). So the answer is $10! - 5! * 2^5$. Why is this wrong?

ii. (difficulty ★) Virgil has a solution, also wrong: We need the number of derangements = permutations without fix point for n=5. Examples: 21453, 41253. Not a derangement : 52134 because 2 is in original position. For n=5 there are $D_5 = 44$ derangements which can be counted by brute force or by Inclusion-Exclusion (next exercise). Then the answer is 5! (choose a permutation of the 5 men $a_1..a_5$) * 2^5 (choose which sister to date) * D_5 (choose a derangement for brothers $b_1..b_5$).
Why is Virgil's solution wrong?

Problem 7 Derangements ★ (optional)

A *derangement* of 1 2 3 $\ldots n$ is a permutation that leaves none of these numbers in place. By inspection, the derangements of 123 are 312 and 231. Find the number of derangements of 1 2 3 4 5 using Inclusion-Exclusion

Problem 8 Counting Schedules (optional)

Students are picking their schedules for next semester and need to figure out their NUPath requirements. The overall course counts for each of the attributes are as follows:

- Societies/Institutions 271
- Interpreting Culture 243
- Creative Expression 157
- Difference/Diversity 182
- Natural/Designed World 136
- Ethical Reasoning 100
- Formal/Quantitative Reasoning 50
- Analyzing/Using Data 135

Two schedules are the same if they have the same courses.

i. Sarah is planning to take four courses next semester. She is planning to take 2 Formal/Quantitative Reasoning courses. She needs Societies/Institutions, Interpreting Culture, Creative Expression, and Difference/Diversity. Assuming no course counts for more than one NUPath requirement and that she wants to do as many NUPath requirements as possible. How many schedules can she build?

ii. A course can have 1 or 2 predefined NUPath attributes. Rick is planning to take 4 courses. He wants to maximize the number of possible NUPath requirements he can complete this semester. Rick wants to know how many schedules he can build. How many *cases* does Rick need to consider to do so?

iii. Sam needs to take 2 courses for their major. These two courses each have one of the following attributes: Formal/Quantitative Reasoning, Natural/Designed World, or Analyzing/Using Data. They want to maximize the other attributes for the two courses, but they don't want to take any Creative Expression courses if they take an Interpreting Culture course. Sam wants to know how many schedules they can build. How many *cases* does Sam need to consider to do so?

Problem 9: Help Jack Sparrow count sequence chopping (optional)

In how many ways one can arrange symbols (a,b,c,d,e,f,g) into 5 bins preserving relative symbols order ? Solution: It comes down to choosing 4 spots for "bin separators" is a sequence of 7 symbols + 4 separators = 11. So the answer is $\binom{11}{4}$

Jack Sparrow sees this as balls in bins, sort of, but he uses partition/sum rule to break the problem into several disjoint cases, count them separately, and add up.

A case corresponds to exactly how many bins are non-empty; for each case, there are exactly 3(non-empty bins)-1 separators which can be anywhere in between the symbols, that is in 8 possible spots: $|a|b|c|d|e|f|g|$. Here are the cases:

- all symbols to 1 bin, thus 0 separators. Choosing the bin $\binom{5}{1}$; choosing the separators $\binom{8}{0}$
- all symbols to 2 bins thus 1 separators. Choosing the bins $\binom{5}{2}$; choosing the separators $\binom{8}{1}$
- all symbols to 3 bins so 2 separators. Choosing the bins $\binom{5}{3}$; choosing the separators$\binom{8}{2}$
- all symbols to 4 bins so 3 separators. Choosing the bins $\binom{5}{4}$; choosing the separators$\binom{8}{3}$
- all symbols to 5 bins so 4 separator. Choosing the bins $\binom{5}{5}$; choosing the separators$\binom{8}{4}$

Applying product rule for each case then partition rule across cases gives $\sum_{k=1}^{5}\binom{5}{k}\binom{8}{k-1}$. This is incorrect, why ?

How can it be fixed following initial idea to break into cases by number of non-empty bins?

Problem 10 (optional, no credit) Cutting parts from a large sheet

Can one cut three 3x3 squares and six 2x3 rectangles from a sheet of paper 8x8 square?

Can one cut four 3x3 squares and eight 2x4 rectangles from a sheet of paper 10x10 square?

Can one cut four 4x4 squares and and six 2x4 rectangles from a sheet of paper 11x11 square?