

Recitation 2: Logic, Quantifiers, Proofs

Instructions: Submit to gradescope by the deadline.

Problem 1 : Logical Equivalence

Determine if each pair is logically equivalent. If so, show the steps to transform one into the other. If not, show on what inputs they differ.

i. $\neg(a \vee T) \vee b, b$

ii. $(\neg a \vee b) \wedge a, a \wedge b$

iii. $(a \implies b) \wedge \neg b, \neg a$

iv. $\neg((a \wedge \neg b) \vee (\neg a \wedge b)), \neg b$

v. $(\neg a \wedge (\neg b \vee c)) \wedge (\neg a \vee c), \neg(a \vee (\neg b \wedge \neg c))$

Problem 2 Proofs by Existence, Contradiction and Contrapositive

Recall we said a proof is essentially an argument that would lead a reasonable reader to conclude that the statement in question must be true.

- i. Write as a predicate and then prove using the contrapositive: For all integers a , if a^2 is not divisible by 4 then a is odd.

ii. Prove the following statement by counterexample (convert the predicates first to simplify your work).

$$\neg(\forall x, \forall y \in \mathbb{Z}, x + y \neq x * y)$$

iii. ★ Prove $\forall a, \forall b, \forall c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even. Make sure you re-write the conjecture.

iv. Prove $\forall a, \forall b \in \mathbb{Z}: a \geq 2 \implies (a \nmid b \vee a \nmid (b+1))$.

In other words, if a is greater than 2 b is not divisible by a or $b+1$ is not divisible by a . What would a proof by contradiction assume?

Hint: If b is divisible by a , then you can write b as $a \cdot c$ where c is some integer.

Problem 3 Powers of 10 are sum of two squares

Prove that for any $n > 0$, $\exists a, b \in \mathbb{Z}, 10^n = a^2 + b^2$. Use construction by cases n odd/even, do not use induction.

Problem 4 Logic Puzzles

- i. You have two ropes that each take an hour to burn, but burn at inconsistent rates. How can you measure 45 minutes? (You can light one or both ropes at one or both ends at the same time.)

You're at a fork in the road in which one direction leads to the City of Lies (where everyone always lies) and the other to the City of Truth (where everyone always tells the truth). There's a person at the fork who lives in one of the cities, but you're not sure which one. What question could you ask the person to find out which road leads to the City of Truth?

- ii. A farmer wants to cross a river and take with him a wolf, a goat and a cabbage. He has a boat, but it can only fit himself plus either the wolf, the goat or the cabbage. If the wolf and the goat are alone on one shore, the wolf will eat the goat. If the goat and the cabbage are alone on the shore, the goat will eat the cabbage. How can the farmer bring the wolf, the goat and the cabbage across the river without anything being eaten?

iii. ★ (optional, no credit)

You meet three gods on a mountain top. One always tells the truth, one always lies, and one tells the truth or lies randomly. We can call them Truth, False and Random. They understand English but answer in their own language, with “ja” or “da” for “yes” and “no” —but you don’t know which is which. You can ask three questions to any of the gods (and you can ask the same god more than one question), and they will answer with ja or da. What three questions do you ask to figure out who’s who?