Homework 9: Probabilities, Bayes

Problem 1 Left-over from Counting: Stack Permutations

A stack-permutation is a permutation obtainable by precisely $2n$ stack operations (n **push** and n **pop**) using the input order [123...n]. Each push puts the next item at top of the stack; each pop takes to output the item at top of the stack (possible only f the stack is not empty). For example if $n = 4$ the sequence (push, pop, push, push, pop...) after 5 operations has unused input =[4], the current stack [2] and output [13].

i. Is [2431576] a stack permutation for $n = 7$? Show corresponding stack operations until either it gets stuck or it finishes.

ii. Describe an one-to-one map between stack permutations and a Catalan problem from class. Conclude that the number of stack permutations is the Catalan number $C_n = \binom{2n}{n} - \binom{2n}{n+1}$

iii. Prove that $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$, then that $C_n < 4^n$, and use these to show that the stack permutations are a proportion arbitrary small (out of all permutations) as n gets large.

iv. Do stack permutations form a subgproup of the permutation group S_n ? Analyze each subgroup required condition: identity, inversion, multiplication.

Problem 2 Halloween Probabilities

It is November and so it is time for trees to change color and for probability questions in math courses to be based on picking Halloween candy. Alice is out trick-or-treating and she goes to her friend Bob's house. Bob's mom, Eve, has a bag of candy containing 10 Whoppers and 15 Skittles. Alice likes Skittles but she doesn't like Whoppers. Eve randomly draws four pieces of candy from the bag to give to Alice.

i. What is the probability that Alice will like all the four pieces of candy that she gets?

ii. What is the probability that Alice will dislike all four pieces?

iii. What is the probability that Alice will like exactly two of the four pieces of candy?

iv. What is the expected number of pieces of candy that Alice will like?

v. Suppose Eve hands Alice two pieces of candy and drops two on the ground by accident. Alice cannot see the dropped pieces since it is dark but she knows that she has two Whoppers in her hand. What is the probability that the two dropped pieces are of the same kind?

vi. Suppose Eve, instead of picking four pieces of candy randomly, first pulls out eight pieces of candy randomly to set aside for Bob, and then draws four additional pieces randomly to give to Alice. What is the probability that Alice will like all four pieces?

vii. Suppose Eve allows Alice one do-over, i.e. if Alice doesn't like the initial set of four she can return them to the bag and get a fresh random set of four. What is the expected number of Skittles that will be drawn on a complete do-over? Keeping that expectation in mind, what is the largest number of Skittles drawn that Alice should rationally request a do-over for? And finally, what is the overall expected payoff if Alice follows that strategy?

viii. Suppose Eve hands out two (instead of four) randomly drawn pieces of candy and suppose that it is the case that Alice is just as likely to get two of the same as two different ones then prove that the total number of pieces of candy in the bag is a square i.e. 4, 9, 16 etc.

Problem 3 High/Low Betting

Two six-sided dice will be rolled independently. First, you have to guess if the sum is high or low. The sum is low if the sum is at most 6. The sum is high if the sum is at least 7. Once you have guessed, the dice are rolled, each roll independent. If your guess is correct, you win. Otherwise, you lose.

i. Should you bet high or low?

ii. You think you can see a 4 on the face of the first die. Should you guess high or low? Explain.

iii. Suppose you guessed HIGH in part iii). The sum was 6 and you lost. You realize the die's face has faded over time, meaning it was equally likely you saw a 4 or a 5 (but no other). What is the probability the die was indeed a 4?

Problem 4 Markov Chains

The Faux Sox are a baseball team we have been tracking for quite some time. We wanted to predict their future wins based solely on last game played:

- When the Faux Sox won a game, they won their next game once, lost their next game 6 times, and drew 3 times.
- When the Faux Sox lost a game, they won their next game 3 times and lost their next game twice.

- When the Faux Sox got a draw, they won their next game 4 times, lost their next game twice, and got another draw 4 times.

We want to speculate the Faux Sox's performance in the Faux Series. The series ends when one team wins 4 games and that team is declared the winner.

i. Draw the Markov Chain.

ii. The Faux Sox lost the first three games. What is the probability of of a comeback victory (i.e. they win the next 4 games)?

iii. The Faux Sox won Game 1. What is the probability that the Faux Sox win the Faux Series in 5 games?

iv. Over time, how often should we expect the Faux Sox to win a game in the World Series (assuming World Series games are no different than normal games)?

Problem 5 Alien Leaders

You leave Earth to escape the election in the United States only to land on a planet that is currently in the middle of an election. This alien race forms its government leadership by selecting two members from each of the four providences: East, West, North, and South. They are much more advanced than the United States; they have a three party system. The party names as far as you can translate are Red, Blue, and Purple. On this alien world, they have tried many different ways of government and have realized that just randomly picking citizens is just as good as any other for getting good leaders.

The political affiliations in the providences are as follows:

In the East providence there are 12 Red affiliated citizens, 8 Purple affiliated citizens, and 20 Blue Affiliated citizens.

In the West providence there are 16 Red affiliated citizens, 18 Purple affiliated citizens, and 14 Blue affiliated citizens.

In the South providence there are 22 Red affiliated citizens, 12 Purple affiliated citizens, and 10 Blue affiliated citizens.

In the North providence there are 9 Red affiliated citizens, 14 Purple affiliated citizens, and 18 Blue affiliated citizens.

i. What is the probability Blue has exactly 6 members in the new government?

ii. You meet an alien affiliated with Purple. What is the probability they are from the South providence?

iii. ★ ★ 3-R.V. Bayes
Supose you meet an <u>elected leader</u> alien affiliated with Purple. What is the probability they are from the South providence?

Problem 6 ★: Bayes with a biased coin

Suppose that you are told a given coin is biased $\frac{2}{3} : \frac{1}{3}$, but you don't know which way: it might be biased 2/3-heads 1/3-tails, or it might be biased 1/3-heads 2/3 tails. Your a priori belief is that there is a 3/4 chance that the coin is heads-biased and a 1/4 chance the coin is tails-biased. You plan to flip the coin four times and update your initial subjective belief based on Bayes Theorem. You then guess the bias of the coin (heads-biased or tails-biased) according to which of these two hypotheses has the larger final subjective probability.

For each of the two possible situations (i.e., the coin is really heads-biased or it is really tails-biased), compute the probability that your guess will be wrong.

Problem 7 ★ : Expected Trials until first success with fixed prob

Say you playing the lottery every day, and the chance of winning is a constant probability $p = 0.04$.

(A) What is the probability that your first win comes on day k ?

(B) What is the expected number of days till you get the first win?

Problem 8 ★ : Zombie virus (optional, no credit)

A new Zombie virus is going around and 5% of the population is already infected. Everyone infected has a 50% chance to turn into Zombies eventually, and 50% to remain virus carriers forever. Coming in contact with a zombie or carrier gives 1% chance of getting infected.

1. Alice had dinner with Bob, and then Bob turned into a zombie. What is the probability that Alice is infected after the dinner? (Alice might have given it to Bob, or viceversa, or either of them might have had it before dinner.)

2. Alice takes a zombie test which works as follows: when applied to a carrier, the test is positive 99% of the time; when applied to a healthy person, the test returns negative results 95% of the time. Alice test comes back positive! What is the probability that she becomes a zombie? Should we take into account somehow the dinner evidence (on Bob), or should we ignore it?