Homework 8: Advanced Counting

Problem 1 Permutations cycle, sign, subgroup

For a permutation of a, an inversion is a pair $(i > j)$ that is not in the increasing order: i comes before j in the permutation. For example the permutation $a = [6, 4, 1, 7, 5, 2, 3]$ has inversions (6,4) ; (6,1) ; (6,5) ; (6,2) ; (6,3) ; (4,1) ; (4,2) ; (4,3) ; (7,5) ; (7,2) ; (7,3) ; (5,2) ; (5,3). We define the $sgn(a) = 1$ "even" if A has an even number of inversions, and $sgn(a) = -1$ "odd permutation" if A has an odd number of inversions.

Further recall (class notes) that any permutation is a product of cycles , ie $[6, 1, 4, 3, 7, 2, 5] = (162)(34)(57)$. Cycles of length 2 such as (34) and (57) are called "transpositions" or "2-cycles".

i. Show that all transpositions are odd , that is have $sgn() = -1$, by counting the inversions.

ii. Show that any cycle is the product of transpositions. Then conclude any permutation is a product of transpositions.

iii. Write $a = [6, 1, 4, 7, 5, 2, 3]$ as a product of transpositions.What is $sgn(a)$?

iv. ★ Prove that $sgn(ab) = sgn(a)\cdot sgn(b)$. Hint: prove this first on transpositions $a = (i, j); b = (u, v)$ by looking at three cases of how the i, j, u, v are positioned to each other.
The case $i < u < j < v$ is illustrated below

index	i	...	u	...	j	...	v
val	$\lvert j \rvert$	...	$\lvert v \rvert$	...	$\lvert i \rvert$	...	$\lvert u \rvert$

Conclude the Parity Theorem: that while the decomposition into transpositions is NOT unique, the number of transpositions ("parity") must be consistent: even permutations are product of even number of transpositions, and odd permutations are product of odd number of transpositions.

v. For two transpositions $t_1 = (i, j)$ and $t_2 = (u, v)$ show that the product $t_1t_2 = (ij)(uv)$ can be decomposed into three 3-cycles (product of cycles of length 3). Then conclude that every even permutation is a product of 3-cycles

vi. The identity permutation $id = id_n = [1, 2, 3...n]$ is the only permutation that has n cycles of length 1. The inverse of a permutation a is the permutation a^{-1} with property $a \cdot a^{-1} = id$
What is the inverse of a cycle $(c_1c_2...c_k)$?
Find the inverse of $a = [6, 1, 4, 7, 5, 2, 3]$ using the cycle decomposition

vii. Prove that in general every permutation a has an inverse a^{-1} and that $sgn(a) = sgn(a^{-1})$

viii. A subset G of the symmetric group of permutations S_n ($|S_n| = n!$) is "subgroup" if it satisfies the following properties:
- $id \in G$
- $a \in G \Rightarrow a^{-1} \in G$
- $a, b \in G \Rightarrow a \cdot b \in G$

Prove that the subset of even permutations A_n (the "alternating group") forms a subgroup: includes identity, includes all inverses, and all products, of even permutations.

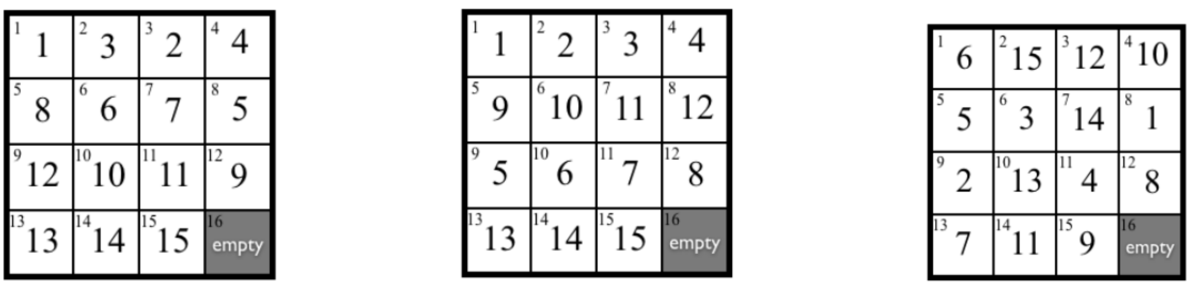
ix. ★ Let G be a subgroup of permutations. For an index $i = 1 : n$ define the **stabilizer of** i **in** G as the set of permutations in G with i fix point $stab_G(i) = \{b \in G | b(i) = i)\}$
Also define the **orbit of** i **in G** as the set of all values permutations in G have on i-th spot $orb_G(i) = \{a(i) | a \in G\}$

Prove the Orbit-Stabilizer Theorem:
$|G| = |orb_G(i)| \cdot |stab_G(i)|$

Hint:If $orb_G(i) = \{v_1, v_2, ..., v_k, ...v_m\}$, for each value choose a permutation in $a_k \in G$ that has that value in position i, $a_k(i) = v_k$ for $k = 1 : m$. Now we partition G subssets $T_k = stab_G(i) \cdot a_k = \{b \cdot a_k | b \in G, b(i) = i\}$

Problem 2 15puzzle - from "Permutation Puzzles" by Jamie Mulholland

To solve the 15Puzzle one has to run a sequence of "slide" tile-to-empty moves in order to arrange all 15 tile to their correct spot, and thus have the botton corner spot 16 empty. Below there are three initial puzzles configurations, also written as permutations decomposed into cycles.



(2 3)(5 8)(9 12) (a)

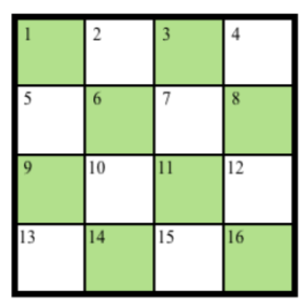
(5 9)(6 10)(7 11)(8 12) (b)

(1 8 12 3 6)(2 9 15)(4 11 14 7 13 10) (c)

Figure 9.1: Which of the positions are solvable?

(A) Define the parity of a box.

Color the 15-puzzle like a checker board as in Figure 9.2. We will call the shaded boxes even and the white boxes odd. This is because if the empty space is in a shaded box it takes an even number of moves to bring it to box 16, similarly if it is in a white box it takes an odd number of moves. Under this definition boxes 1, 3, 6, 8, 9, 11, 14, 16 are even, whereas the other boxes are odd.



Argue that if initially the empty is on spot 16, any completion of the puzzle must have an even number of moves.

(B) Prove that if an (initial) permutation a is solvable, then it must be even ($sgn() = +1$). Use the decomposition into cycles, that each move is a transposition, and that the number of moves to completion must be even.

(C) Starting from solved-state permutation (identity) show the exact moves that can produce the 3-cycle $c = (11, 12, 15)$, by focusing on the bottom right corner of the puzzle

(D)★ Now that we have one 3-cycle c , we will show that we can use c to construct any other 3-cycle we want. From a solved puzzle, pick any tile, say $i \in [15]$. Move tiles 12 and 11 to boxes 16 and 12, respectively, by the move sequence $a = (12, 16)(11, 12)$: Then using one of the two tours in Figure 9.4 we can move tile i to box 15, without disturbing the contents of boxes 12 and 16. Call this move sequence b

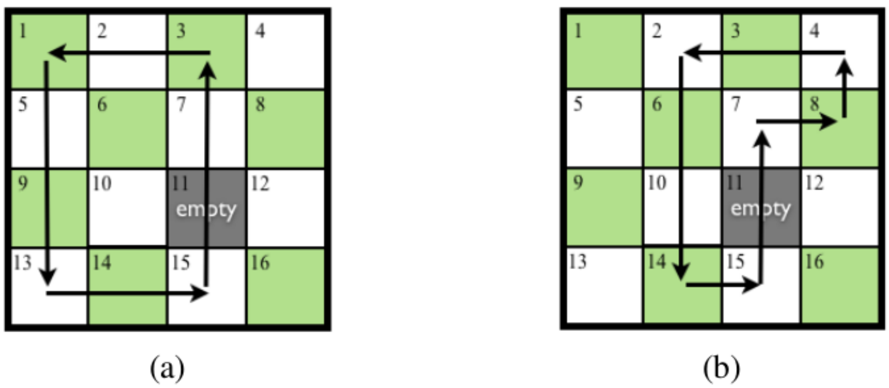


Figure 9.4: Tours for producing 3-cycles.

Show how to use the 3-cycle c to to obtain a cycle (11,12, i). Hint: Use the tours above and sequence of moves $(aba^{-1})^{-1} = ab^{-1}a^{-1}$

(E)★ Solvability Criteria for 15-Puzzle - Part 1: A permutation a of the 15-puzzle which fixes 16 that is even: i.e. $a \in A_{15}$, is solvable. Prove this in reverse: show every even permutation of the 15 tiles is obtainable through puzzle moves, starting from the solved-state (identity permutation), as a product of 3-cycles.

(F) ★ (optional , no credit) Solvability Criteria for 15-Puzzle - Part 2. A permutation of the 15-puzzle is solvable if and only if the parity of the permutation is the same as the parity of the location of the empty space.

Problem 3 Binomial identity of squares

(A) Prove the following, using a combinatorial argument:

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Hint: imagine making pizza with r toppings out of m vegetarian ones and n non-vegetrian ones. What are your options?

(B) Prove again the relation in part (A) by using the expansion $(1+x)^{m+n} = (1+x)^n(1+x)^m$ and checking the RHS coefficient for x^r

(C) Conclude as a particular case that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Problem 4

Compute in two ways $T_n = \sum_{k=0}^{n} k \cdot \binom{n}{k}$

(A) use the factorials fraction and simplify terms to get a summation corresponding to binomial expansion for degree $n-1$

(B) Consider the function $f(x) = (1+x)^n$ as binomial expansion and calculate $f'(1)$ where $f' =$ differential of f

Problem 5 Ways to buy paper plates

The convenience store sells paper plates in packages of 1, 5, or 20. In how many different ways can Jack Sparrow buy a total of 55 paper plates?

(A) solve the question using disjoint counting cases.

(B) solve using generative functions

Problem 6 Rolling 4 dice to get 19

How many different rolls of a red die, a green die, a blue die, and a yellow die total up to 19? Assume that these are 6-sided dice.

(A) Solve the question using cases or an integer equation. Hint use $(6 - r, 6 - g, 6 - b, 6 - y)$ as roll output with r, g, b, y integers between 0 and 5.

(B) Solve using generative functions.

Problem 7 ★★ (optional no credit) Number of surjective functions
(A)★★ If $n \geq m$ we denote $S(n,m)$ the number of surjective functions $f : \{1,2,3...,n\} \to \{1,2,3,...,m\}$ (surjective= covers the entire destination set). Use PIE to prove that

$$S(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

(B) Conclude as a particular case that

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n$$

Problem 8 ★★ (optional no credit)
Prove using algebra, induction etc (not the previous problem combinatorial argument):

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n$$

Problem 9 ★★ (optional, no credit) Prove that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} = \sum_{k=0}^{n} 2^k \binom{n}{k}^2$$

Problem 10 ★★ (optional, no credit) Prove that

$$\sum_{k=0}^{n} \binom{n+k}{k} \frac{1}{2^k} = 2^n$$