Homework 6: Induction part B

Problem 1 Series by induction Prove the following statements by induction. Make sure you write first the induction step as an implication.

i. $S(n) = 1^4 + 2^4 + 3^4 + ... + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$

ii. $S(n) = 1^5 + 2^5 + 3^5 + ... + n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

iii. Before proving the following write the left side as a sum iterated by $k = 1 : n$

$$\frac{5}{1\cdot2\cdot3} + \frac{6}{2\cdot3\cdot4} + \frac{7}{3\cdot4\cdot5} + \ldots + \frac{n+4}{n(n+1)(n+2)} = \frac{n(3n+7)}{2(n+1)(n+2)}$$

Problem 2 More Fibonacci

The Fibonacci numbers $1, 1, 2, 3, 5, 8, \ldots$ is a sequence defined by the equation $F_n = F_{n-1} + F_{n-2}$ with base $F_0 = 0$ and $F_1 = 1$. Prove by induction the following statements.

i. $F_{n+1} < (\frac{7}{4})^n$ for all $n > 0$

ii. $F_1 + F_3 + F_5 + ... + F_{2n-3} + F_{2n-1} = F_{2n}$

iii. $F_1^2 + F_2^2 + ... + F_n^2 = F_n \cdot F_{n+1}$

iv. ★ $F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1}$, requires "double" induction over m and n, effectively induction over $sum = m + n$: assume as IH that the result holds for all previous $sum = i + j < m + n$, that is for $i \leq m$ and $j \leq n$ with one of these being strict.

v. F_{2n} is a multiple of F_n; use the previous point for a particular value of m

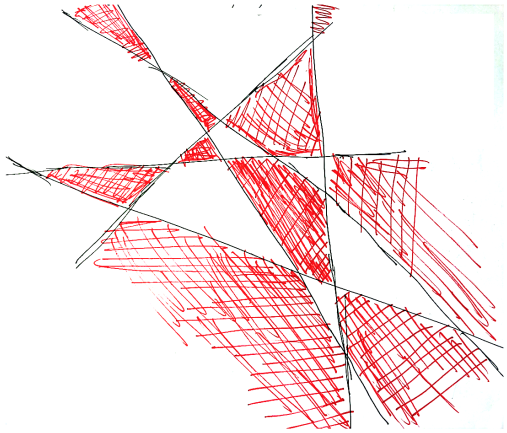
Problem 3 Induction simple proofs

i. Let a and n be positive integers, with a odd. Prove by induction over n that $2^{n+2} | (a^{2^n} - 1)$

ii. Prove by induction over n that any set of size n has 2^n subsets

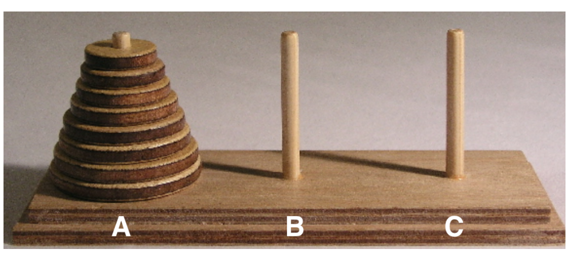
Problem 4 Structural induction : line intersecting regions

i. If n lines are drawn on a plane, and no two lines are parallel, and no 3 lines are concurrent. Show that they dive plane into $(n^2 + n + 2)/2$ regions. Use induction over n the number of lines



ii. ★ If n lines are drawn on a plane, and no two lines are parallel, and no 3 lines are concurrent (previous exercise). Show that it is possible to color the regions formed with only two colors so that no two adjacent regions (a common side) share the same color.Use induction over n the number of lines

Problem 5 ★ Structural Induction: Towers of Hanoi Lower Bound

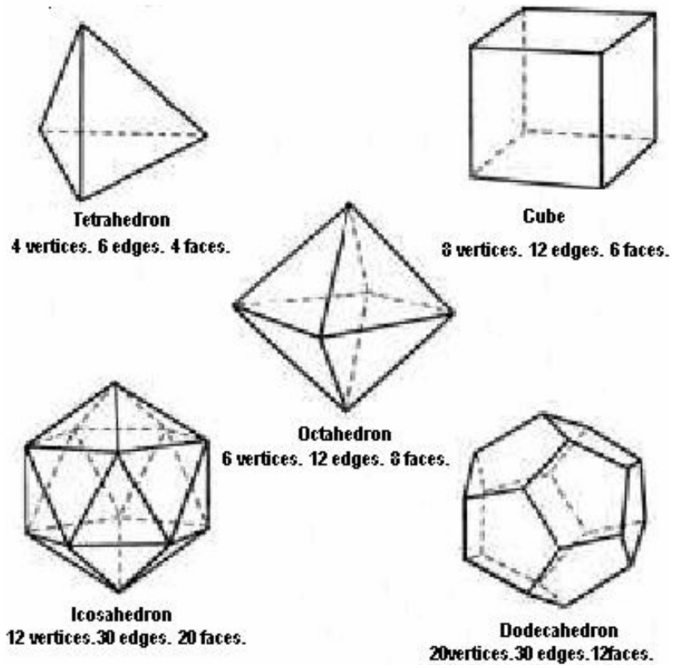


The "Towers of Hanoi" task consists on three towers, towers B and C empty, and tower A containing the disks of radius 1 to n on top of each other (see picture). BIGGER DISKS ARE NOT ALLOWED ON TOP OF SMALLER DISKS AT ANY TIME.
The task is to move all disks to tower B, one disk at a time, allowing any valid moves between towers. Prove that the minimum number of moves required to complete the task (on any strategy) is $2^n - 1$

part B ★ ★ (optional no credit) There are four towers A, B, C, D instead of three. What is the minimum number of moves required to move all disks to tower B?

Problem 6 Structural Induction: Euler's Theorem

Prove that a convex polyhedron with F faces, E edges, V vertices satisfies V+F = E +2. Use induction over number of vertices V.



See https://en.wikipedia.org/wiki/Euler_characteristic for a larger article.

Problem 7 ★ Cauchy–Schwarz inequality (optional no credit)

Let (x_k) and (y_k) be real numbers for $k = 1 : n$. You can think of X and Y as two real vectors in n-dimensions. Prove by induction over n that

$$(\sum_k x_k^2)(\sum_k y_k^2) \geq (\sum_k x_k y_k)^2$$

See `https://en.wikipedia.org/wiki/Cauchy-Schwarz_inequality` for a larger article. An easy way to think about Cauchy inequality (not a proof for this exercise) is to loook at cosine formula $cosine(X,Y) = \frac{<X \cdot Y>}{||X|| \cdot ||Y||} = \frac{\sum_k x_k y_k}{\sqrt{(\sum_k x_k^2)(\sum_k y_k^2)}}$, and realise that any cosine is a number between [-1,1].

Problem 8 ★★ (optional no credit)

If $x_1, x_2, ..., x_n$ are positive numbers then their "mean inequality" states that the order from big to small is : quadratic mean, arithmetic mean, geometric mean, harmonic mean. We proved the first inequality in class; this problem asks to prove the second and third:

$$\sqrt{\frac{1}{n}(\sum_k x_k^2)} \geq \frac{1}{n}(\sum_k x_k) \geq (\prod_k x_k)^{1/n} \geq \frac{n}{\sum_k \frac{1}{x_k}}$$

Hint for second inequality:

- first prove that the function $f(a) = (n+1)\log(1+a) - \ log(a)$ has the minimum at $a = 1/n$
- induction step consider the new value x_{n+1} and take logs on both sides
- use the above inequality (f minimum) for $a = x_{n+1}/(\sum_{k=1}^{n} x_k)$