Quantifiers and Negation

For all of you, there exists information about quantifiers below.

We often quantify a variable for a statement, or predicate, by claiming a statement holds **for all** values of the quantity or we say **there exists** a quantity for which the statement holds (at least one). Notationally, we can write this in shorthand as follows:

$\forall x \in A, P(x),$ which claims: for all x in the set A, the statement $P(x)$ is true.
$\exists x \in A, P(x),$ which claims: there exists at least one x in the set A such that the statement $P(x)$ is true.

There are many equivalent way to express these quantifiers in English. Here are a few examples:

Universal Quantifier: Here are a few ways to say $\forall x \in \mathbb{N}$:
"For all natural numbers x, ...", "For all $x \in \mathbb{N}$, ...", "For any $x \in \mathbb{N}$, ...", "For every $x \in \mathbb{N}$, ..."

Existential Quantifier: Here are a few ways to say $\exists x \in \mathbb{N}$:
"There exists a natural number x such that ...", "There exists $x \in \mathbb{N}$ such that ...", "For at least one $x \in \mathbb{N}$ ...", "For some $x \in \mathbb{N}$ ..."

Nested (or Compound) Quantifiers: If there is more than one quantity it is fairly common to see more than one quantifier in a statement. When we see nested quantifiers we must take special care in the order they appear (as this can effect the meaning).

RULE 1: If we are using the same quantifier, then the ordering doesn't matter.

Examples

- 'For all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$, $x + y = 4$.', is the same as
  'For all $y \in \mathbb{R}$ and for all $x \in \mathbb{R}$, $x + y = 4$.', which is the same as
  'For all $x, y \in \mathbb{R}$, $x + y = 4$.' (Note: You should be able to tell that this is a false statement.)
- 'There exists $x \in \mathbb{R}$ and there exist $y \in \mathbb{R}$ such that $x + y = 4$.', is the same as
  'There exists $y \in \mathbb{R}$ and there exists $x \in \mathbb{R}$ such that $x + y = 4$.', which is the same as
  'There exist $x, y \in \mathbb{R}$ such that $x + y = 4$.' (Note: You should be able to tell that is a true statement.)

RULE 2: If we are using mixed quantifiers, then the ordering DOES matter.

Examples

- 'For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x + y = 4$.'
  This statement says that the following in this exact order:
  1. The variable x can set as ANY real number.
  2. After x is set, we can find at AT LEAST ONE y based on x such that $x + y = 4$.
  In other words, the variable y is nested 'inside' in a sense, so we are saying that y can be found after we already select x. (Note: This is a true statement.)
- 'There exists $y \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x + y = 4$.'
  This statement says that the following in this exact order:
  1. AT LEAST ONE y can be found BEFORE any other variable is set. And this one special y will work within the rest of the statement no matter what the other variables are.
  2. After this special y is set, we can put ANY x in the equation $x + y = 4$ will be valid.
  In other words, the variable y is nested 'outside' in a sense, so we are saying that y can be found beforehand and it works for all x values. (Note: This is a false statement.)

Observe: By looking at the two examples above, you should note that $\forall x \in A, \exists y \in B, \ldots$ and $\exists y \in B, \forall x \in A, \ldots$ are very different quantifications. The second statement is much *stronger* in the sense that if you can find y ahead of time, then certainly you can find it after the fact. Thus, if the statement $\exists y \in B, \forall x \in A, P(x, y)$ is true, then automatically the statement $\forall x \in A, \exists y \in B, P(x, y)$ must be true (but in general it doesn't go the other way).

Note: Occasionally, you will see a nested quantifier at the end of a statement, in which case it is implied that the quantifier is the last in terms of order. For example, here is the definition of bounded:
"There exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$."
Note that we wrote a quantifier at the end just to make it sound nice. Putting all quantifiers at the beginning, in the right order, this is the same as
"There exist $M \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $|f(x)| \leq M$."
(The order is very important in this definition because you have to find a bound M ahead of time that works for all values of x).

Important Fact: Quantifiers give the 'range' and 'universe' over which the statement is being claimed. When we work with a quantified statement and start to talk about negations it is important to remember that the 'universe' does not change (if a statement is quantified for students in this classroom, the negation will still still be quantified over students in this classroom). Thus, we are only negating the quantifier. Also note that the negation of the quantifiers is a quick exercise that doesn't require the use of any sophisticated logic rules, you just 'flip' the quantifiers, then negate the statement (when you get to the statement then you will need logic rules to negate).

Negation Rules: When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negative the statement.

1. $\neg[\forall x \in A, P(x)] \Leftrightarrow \exists x \in A, \neg P(x)$.
2. $\neg[\exists x \in A, P(x)] \Leftrightarrow \forall x \in A, \neg P(x)$.
3. $\neg[\forall x \in A, \exists y \in B, P(x,y)] \Leftrightarrow \exists x \in A, \forall y \in B, \neg P(x,y)$.
4. $\neg[\exists x \in A, \forall y \in B, P(x,y)] \Leftrightarrow \forall x \in A, \exists y \in B, \neg P(x,y)$.

Unrelated, but important, when it comes time to negate the statement remember how to negate an implication:

$$\neg[\text{IF } P, \text{ THEN } Q] \Leftrightarrow P \text{ AND NOT } Q$$