Graph Theory, Part 2

7   Coloring

Suppose that you are responsible for scheduling times for lectures in a university. You want to make sure that any two lectures with a common student occur at different times to avoid a conflict. We could put the various lectures on a chart and mark with an "X" any pair that has students in common:

Lecture	A	C	G	H	I	L	M	P	S
Astronomy		X	X	X			X		
Chemistry	X								X
Greek	X			X		X	X	X	
History	X		X			X			X
Italian						X	X		X
Latin			X	X	X		X	X	X
Music	X		X		X	X			
Philosophy			X			X			
Spanish		X		X	X	X			

A more convenient representation of this information is a graph with one vertex for each lecture and in which two vertices are joined if there is a conflict between them:
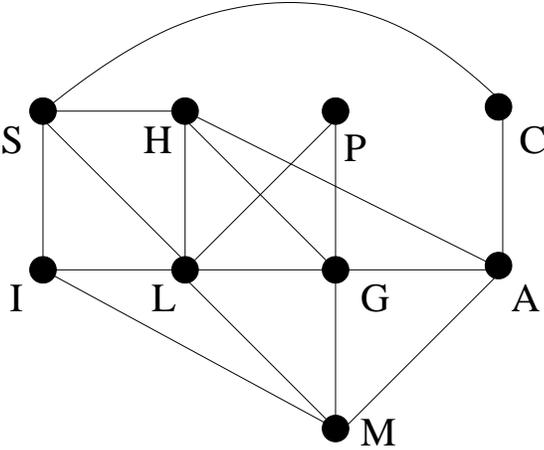
Now, we cannot schedule two lectures at the same time if there is a conflict, but we would like to use as few separate times as possible, subject to this constraint. How many different times are necessary?

We can code each time with a color, for example 11:00-12:00 might be given the color green, and those lectures that meet at this time will be colored green. The no-conflict rule then means that we need to color the vertices of our graph in such a way that no two adjacent vertices (representing courses which conflict with each other) have the same color. This motivates us to make a definition:

Definition 15 (Proper Coloring, k-Coloring, k-Colorable). A *proper coloring* is an assignment of colors to the vertices of a graph so that no two adjacent vertices have the same

color. A *k-coloring* of a graph is a proper coloring involving a total of k colors. A graph that has a k-coloring is said to be *k-colorable*.

So what we seek is a k-coloring of our graph with k as small as possible.

Here is a 4-coloring of the graph:
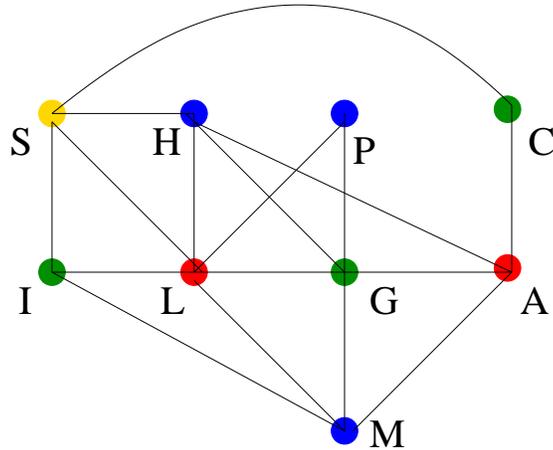
Question: Is there a proper coloring that uses less than four colors? Clearly we cannot get by with less than three colors, since G, H, and L are all adjacent to each other. So the question becomes: Can we find a 3-coloring? The answer is negative. We shall suppose that we can properly color the graph with only three colors, and show that this leads to a contradiction. We start by coloring G, H, and L three different colors, as we must. We might as well assume they are the colors as shown above. Then M must be the same color as H, since M is adjacent to G and L and there are only three colors available. So we color M blue as shown above. In a like manner, I must be colored green since it is adjacent to L and M, which we have determined to be red and blue. Now S is adjacent to H, L, and I, and so it must be a different color from them all. So we cannot make do with only three colors. So four colors are needed to properly color the graph. This means that we need to have at least four different times for lectures in our school. The following is now a very natural concept:

Definition 16 (Chromatic Number). The *chromatic number* of a graph is the minimum number of colors in a proper coloring of that graph.

How do we determine the chromatic number of a graph? In the last example, we did it by first finding a 4-coloring, and then making an intricate argument that a 3-coloring would be impossible. It may be quite difficult to compute the chromatic number of more complicated graphs. Can we at least make an *upper bound* on the number of colors we need, even if we cannot find the minimum number? There is an algorithm (procedure) for properly coloring vertices that does not always use as few colors as possible, but at least gives us an upper bound on the number of colors needed.

7.1   The Greedy Algorithm For Coloring Vertices

The algorithm is called *greedy* because it is a rather short-sighted way of trying to make a proper coloring with as few colors as possible. It does not always succeed in finding the minimum

number (the chromatic number), but at least provides some proper coloring. The procedure requires us to number consecutively the colors that we use, so each time we introduce a new color, we number it also. Here is the procedure:

1. Color a vertex with color 1.

2. Pick an uncolored vertex v. Color it with the lowest-numbered color that has not been used on any previously-colored vertices adjacent to v. (If all previously-used colors appear on vertices adjacent to v, this means that we must introduce a new color and number it.)

3. Repeat the previous step until all vertices are colored.

Clearly, this produces a proper coloring, since we are careful to avoid conflicts each time we color a new vertex. How many colors will be used? It is hard to say in advance, and it depends on what order we choose to color the vertices.

For example, suppose we decide to color the course conflict graph using the Greedy Coloring Algorithm, and we decide to color the vertices in order G, L, H, P, M, A, I, S, C. Then we would color G with color 1 (green), L with color 2 (red) since adjacency with G prevents it from receiving color 1 (green), and we color H with color 3 (blue) since adjacency with G and L prevents it from receiving colors 1 and 2 (green and red). So we have
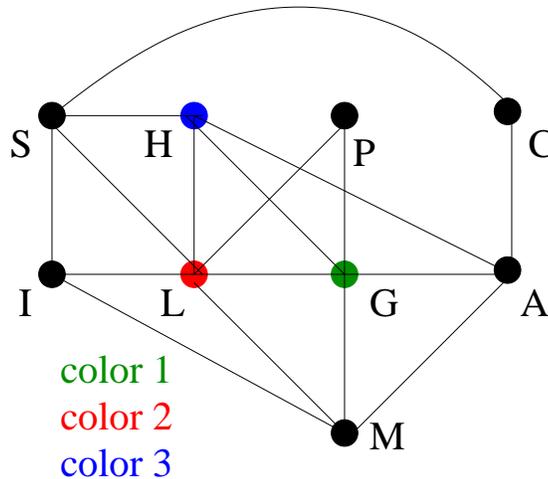
P and M also cannot receive colors 1 and 2 (green and red), so they are given color 3 (blue):
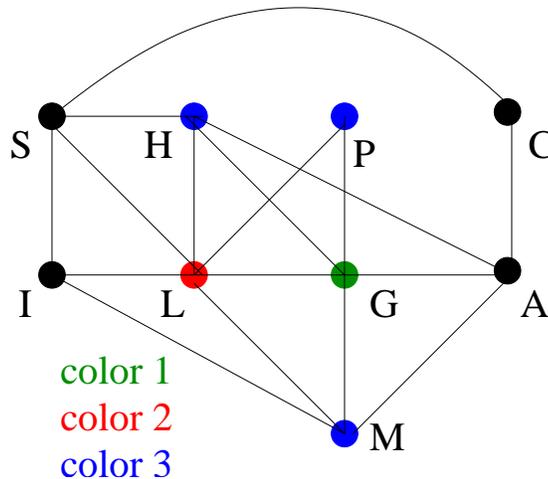
Then A cannot receive colors 1 and 3 (green and blue), so we give it color 2 (red), while I cannot receive colors 2 and 3 (red and blue), so we give it color 1 (green).

color 1
color 2
color 3
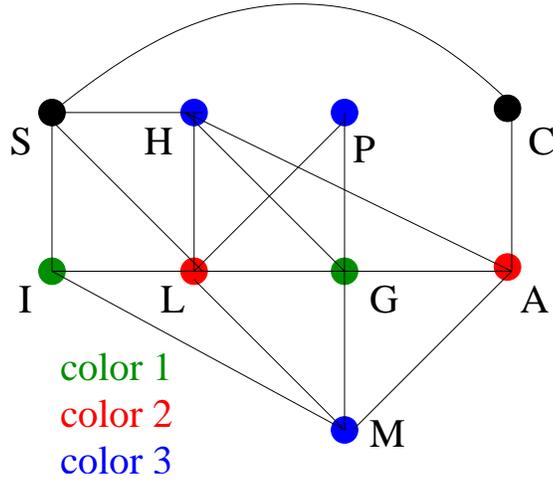
Vertex S cannot receive color 1, 2, or 3, and so we give it color 4 (say, yellow). Vertex C cannot receive color 2 or 4 (red or yellow), so we give it color 1 (green). We obtain the same coloring we had proposed earlier:

color 1
color 2
color 3
color 4
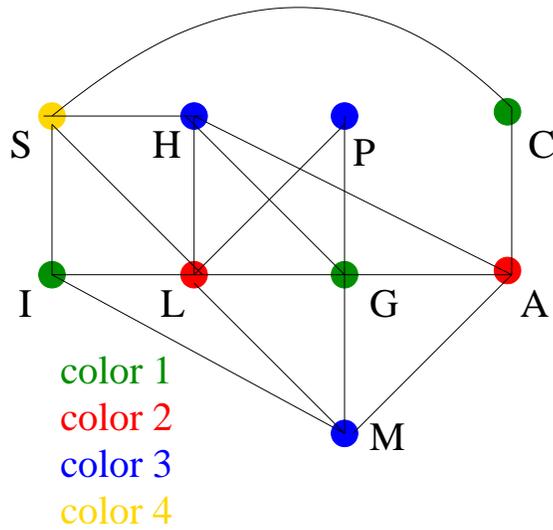
On the other hand, we could imagine choosing a different order. Suppose we chose to color the vertices in the order A, I, P, M, S, C, H, L, G. First we color A with color 1 (green) and also I and P color 1.
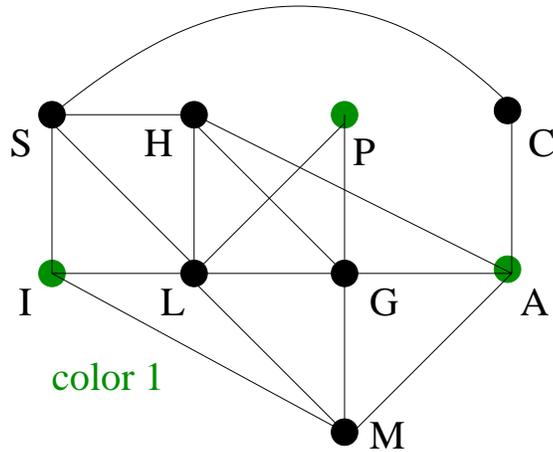
color 1

Then we color M and S with color 2 (red), because they cannot be colored green.
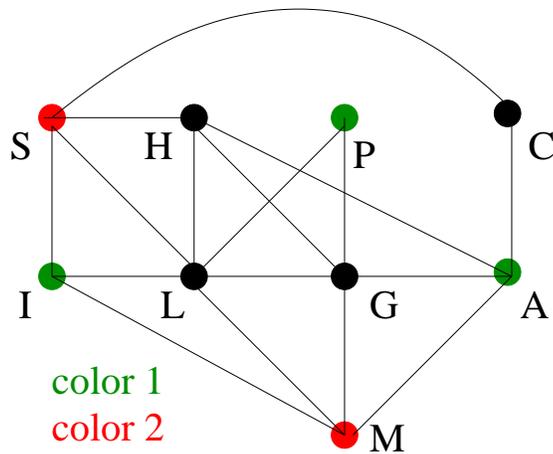
color 1
color 2

Then we color C and H with color 3 (blue), because they cannot be colored green or red.

color 1
color 2
color 3
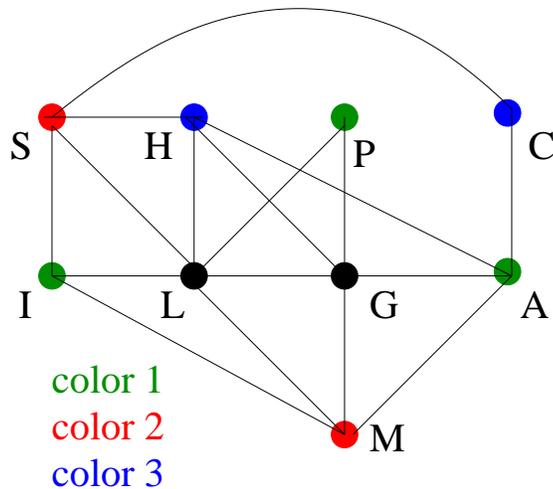
We color L with color 4 (yellow), because it cannot be colored with green, red, or blue.
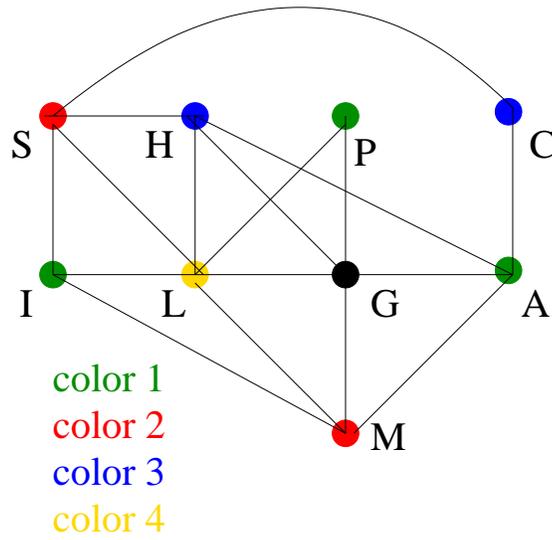
We color G with color 5 (light blue), because it cannot be colored with any of the four other colors.
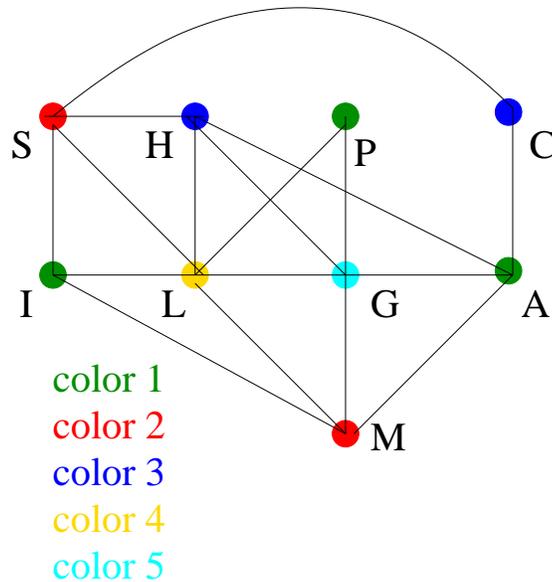
So we see that the quality of our coloring (with fewer colors being considered better) from the Greedy Coloring Algorithm is dependent on the order in which we color the vertices.

Nonetheless, there is a certain minimum quality we get, which we can determine by the following theoretical argument: Suppose that d is the largest degree of any vertex in our graph, i.e., all vertices have d or fewer edges attached, and at least one vertex has precisely d edges attached. As we go about coloring, when we color any particular vertex v, it is attached to at most d other vertices, of which some may already be colored. Then there are at most d colors that we must avoid using. We use the lowest-numbered color not prohibited. That means that we use some color numbered $d + 1$ or lower, because at least one of the colors 1, 2, ..., $d + 1$ is NOT prohibited. So we never need to use any color numbered higher than $d + 1$.

This gives us the following theorem:

Theorem 4 (Greedy Coloring Theorem). *If d is the largest of the degrees of the vertices in a graph G, then G has a proper coloring with $d + 1$ or fewer colors, i.e., the chromatic number of G is at most $d + 1$.*

This gives an upper bound on the chromatic number, but the real chromatic number may be below this upper bound. For example, in our course conflict graph above, the highest degree is $d = 6$ (vertex L has this degree), so the Greedy Coloring Theorem states that the chromatic number is no more than 7. In fact, the chromatic number is 4, and a 4-coloring can be obtained by employing the Greedy Coloring Algorithm (as we saw above) if one is not too unlucky in picking the order to color vertices.

8 Coloring Maps

A difficult problem that was addressed by graph theorists is the answer to the following question:

Question 1. What is the maximum number of colors you need to color a map so that adjacent territories are given different colors?

For us, a territory is a connected mass of land. Note that not all countries are like this. For example, the Kaliningrad exclave is part of Russia, although it is not connected to the main part of Russia. (Kaliningrad is the modern name given to the city once known as Königsberg.) Although we pose this problem in terms of coloring maps, real cartographers are seldom very interested in knowing the minimum number of colors they need. Nonetheless, this problem, like other coloring problems, has ramifications for computer science and engineering disciplines. For example, suppose we want to minimize the number of radio frequencies we use while not using the same frequencies in nearby regions (to prevent interference). Then colors could represent frequencies, vertices could represent regions, and edges connect the vertices representing neighboring regions. Then we want to assign frequencies (i.e., color the vertices) with no conflicts (i.e., no adjacent vertices have the same color) using as few frequencies (i.e., colors) as possible.

By now we should be used to abstracting relations into graphical form. We can make each territory a vertex, and join two vertices with an edge when the territories they represent are adjacent. Then coloring the vertices of the graph so that no two adjacent vertices receive the same color is equivalent to coloring the original map so that no two adjacent territories receive the same color.

For example, if we want to abstract a portion of the following map of Europe[1]

we might produce the following graph:

[1]All the geographical maps in these notes are from the Perry-Castañeda Library Map Collection at the University of Texas, http://www.lib.utexas.edu/maps/.

Taking the graph away from the map and flexing it a little to make it easier to look at, we obtain
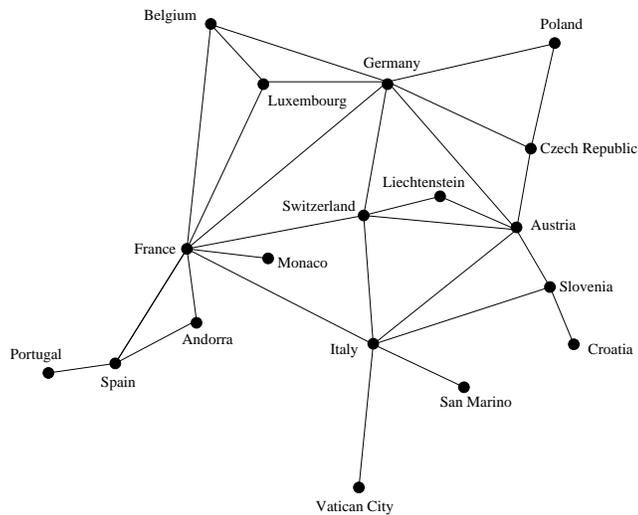
We note that this graph cannot be colored with less than four colors: Belgium, Luxembourg, France, and Germany all touch each other, so they need four distinct colors. However, four colors are sufficient to color the graph, as shown below.
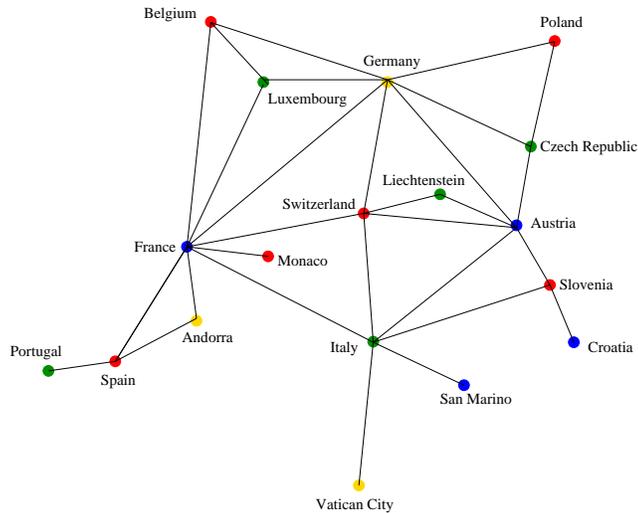
If you try coloring other maps, and make shrewd choices, you will find that it seems that four colors are always sufficient.

This is not an easy problem. What we shall show in these notes is the following:

Theorem 5 (Six Color Theorem). *Any map can be colored with six or fewer colors in such a way that no adjacent territories receive the same color.*

This will require us to develop some notions and tools. With some further insights which would require more time and care than we can devote to this problem in the lecture, one can show that the sixth color isn't really necessary.

Theorem 6 (Five Color Theorem, P. J. Heawood, 1890). *Any map can be colored with five or fewer colors in such a way that no adjacent territories receive the same color.*

But one never seems to really need the fifth color if one is careful. Mathematicians worked on this problem for a long time, sometimes producing flawed proofs that four colors suffice. Finally, a solution was announced:

Claim 1 (Appel and Haken, 1976). *Any map can be colored with four or fewer colors in such a way that no adjacent territories receive the same color.*

Appel and Haken used mathematical theory to reduce the problem from a problem about all possible maps (and there are infinitely many of these) to checking a large but finite number of cases. The written proof that reduces to cases is hundreds of pages long, and the case-checking is not written out (it is too long for a person to work out), but had to be carried out with a computer. Programs can have bugs, so some mathematicians do not accept it as a proof. Other methods have also used computers, and they very strongly confirm the claim of Appel and Haken. Still, it somewhat unsatisfactory that there is no proof that an individual person can digest.

8.1 Planar Graphs

We intend prove the Six Color Theorem, i.e., that all maps can be colored with six colors. We convert maps into graphs and then try to color their vertices with six colors, such that no adjacent vertices have the same color. But wait a minute! We need to think about this: some graphs cannot be colored with six colors. For example, the graph below has seven vertices and each one is connected to all the others.
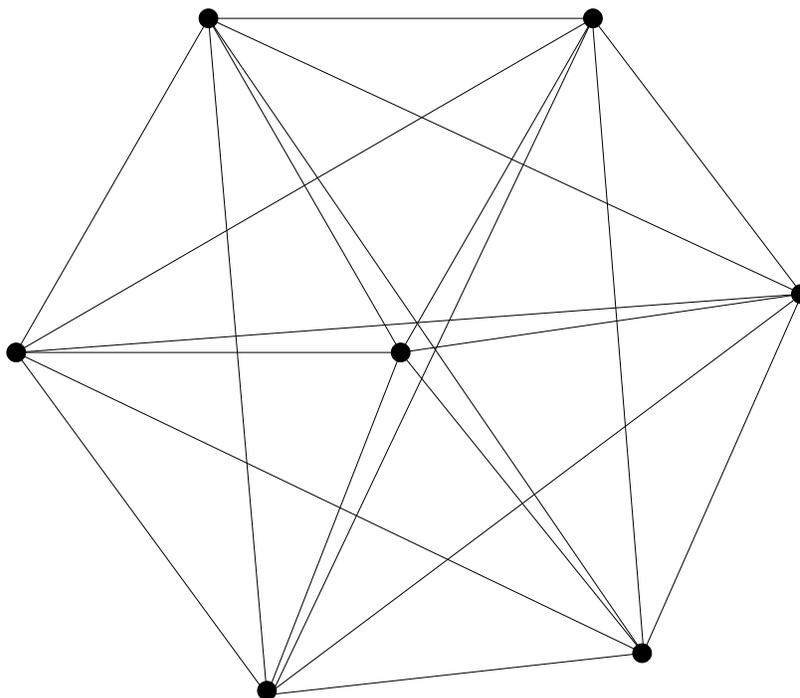
So we would need seven colors to properly color it! What went wrong? In fact, now that we mention it, we can always draw a graph with N vertices and join each vertex to every other vertex, so that N different colors are needed to properly color it. And N could be as big as you want.

This seems to contradict all the theorems mentioned above. But the contradiction is only apparent. In fact, graphs like the one that requires seven colors *cannot arise* as abstractions of geographical maps. A graph that comes from a map has some constraints on it that make it 6-colorable.

But how can we figure out the intellectual content of these constraints? The key fact is the following:

Principle: A graph arising from a map can be drawn in the plane so that no two edges cross over each other.

We offer the following intuitive justification. Suppose we want to start making a graph for Europe. First we place a dot inside each country:

Then, country by country, we draw lines from the dot inside the country to each of the international borders. For example, we start with Germany here:

These rays are like half-edges.

We then pick another country, say, the Czech Republic. When we draw rays for the Czech Republic, and we join the half-edges that strike the Czech-German border.

Then we pick another country, Poland, and join its half-edges to those of Germany and the Czech Republic.

Continuing in this fashion, we would get a graph for this portion of the map of Europe, and no edges would cross! No edges cross because we are simply drawing half-edges inside each country, which cannot touch half-edges from any other country, except where they meet at common borders.

So the graphs we get from maps are special in that they can be drawn in a plane without edges crossing. This is unlike the graph shown above that needs seven colors to be properly colored. This motivates us to make the following definition:

Definition 17. A *planar graph* is one that can be drawn in the plane so that none of the edges cross over each other.

So all graphs arising from geographical maps are planar graphs. Our goal is to prove that all maps can be colored with six or fewer colors. With our special terminology, this is tantamount to proving the following:

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Theorem 7 (Six Color Theorem, restated). *Any planar graph has a chromatic number less than or equal to* 6.

8.2 Euler Characteristic

We now introduce an fascinating discovery of Euler which will make our proof of the Six Color Theorem possible. Let just forget about trying to color graphs for a while and go on a mathematical jaunt. Consider our graph representing a portion of Europe:
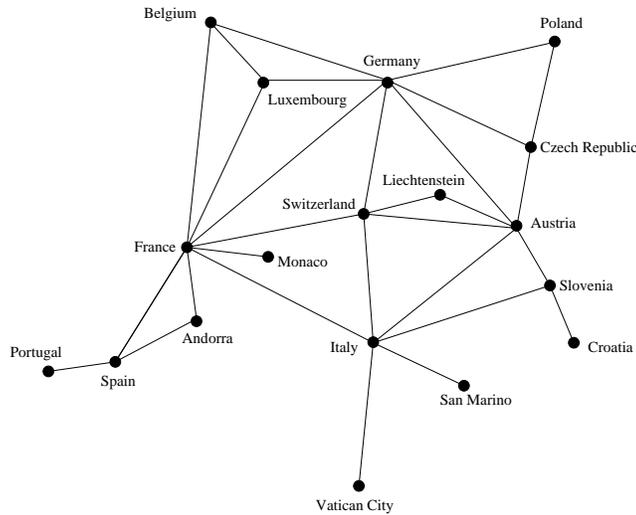
We count that it has 18 vertices and 29 edges; these are numbered in black (vertices, i.e., countries) and red (edges, i.e., borders) below:
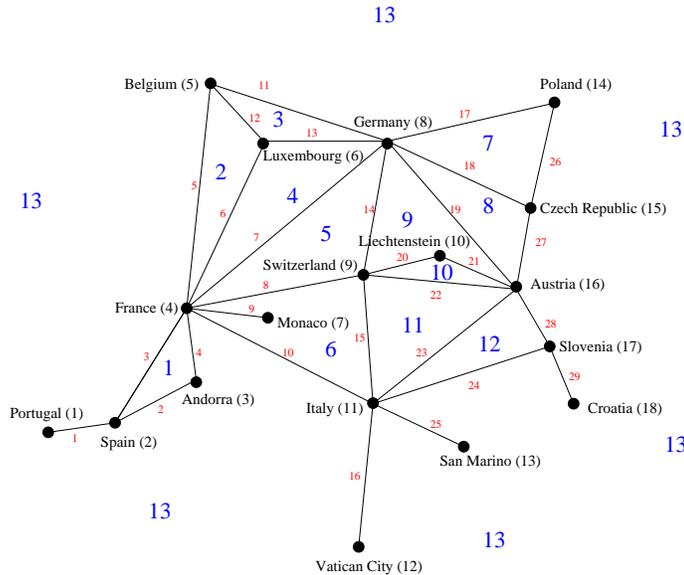
But what are the blue numbers? Note that the graph partitions the plane into 13 panels or facets, which we call *faces*. For example, the edges Spain-Andorra-France-Spain fence in a triangular face. The edges Italy-Austria-Slovenia-Italy fence in another triangular face. There are 12 such compartments, and we also count the unbounded portion of the plane that lies outside as a

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face (its boundary is Portugal-Spain-Andorra-France-Italy-Vatican City-Italy-San Marino-Italy-Slovenia-Croatia-Slovenia-Austria-Czech Republic-Poland-Germany-Belgium-France-Spain-Portugal). So we have $V = 18$ vertices, $E = 29$ edges, and $F = 13$ faces.

We could do this for another map. For example, let's try South America:
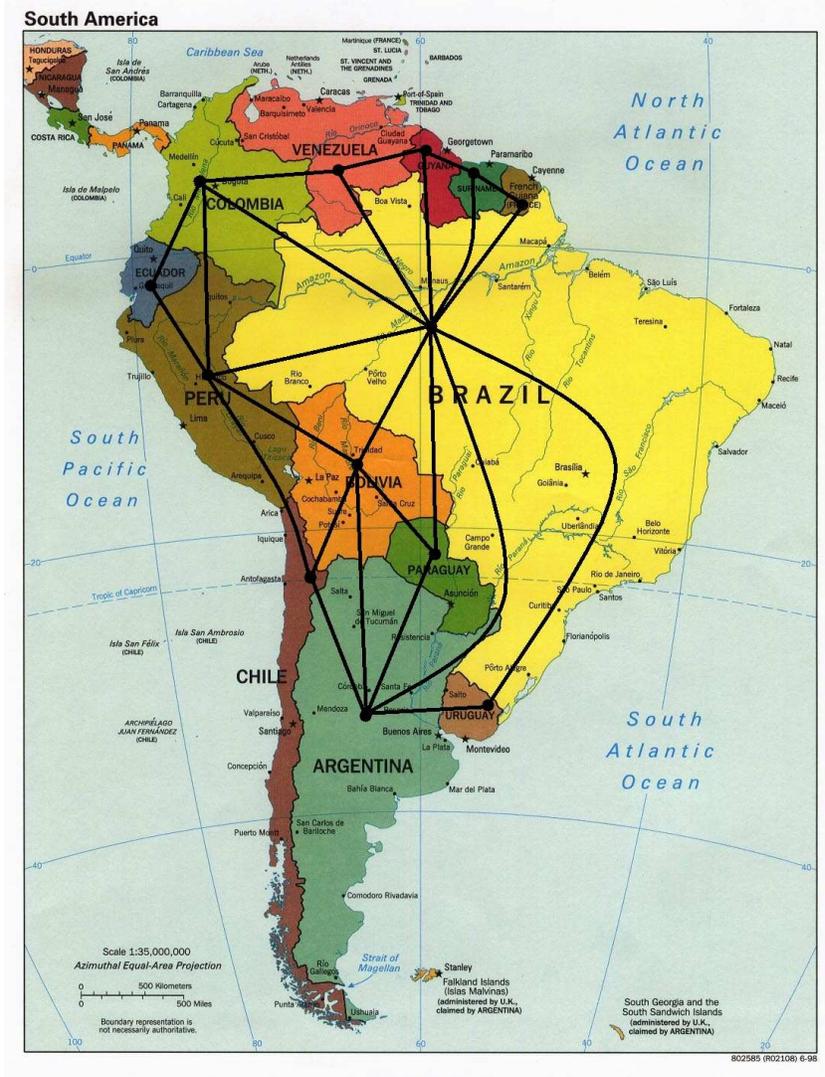
There are $V = 13$ vertices, $E = 25$ edges, and $F = 14$ faces.

The remarkable thing that $V - E + F = 2$ in both our cases, whether in the portion of Europe ($V - E + F = 18 - 29 + 13 = 2$) or in South America ($V - E + F = 13 - 25 + 14 = 2$). Euler discovered this phenomenon in a somewhat different context. He proved that the number of vertices minus the number of edges plus the number of faces of any reasonable polyhedron is always equal to two. We can check this for the cube and the regular tetrahedron:
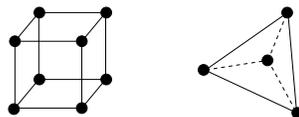
For the cube, $V - E + F = 8 - 12 + 6 = 2$, and for the tetrahedron, $V - E + F = 4 - 6 + 4 = 2$. We name this quantity in honor of Euler.

Definition 18. The *Euler characteristic* of a planar graph or polyhedron with V vertices, E edges, and F faces is the quantity $\chi = V - E + F$.

Our observations lead us to surmise that the following claim is always true:

Theorem 8. *Any planar graph has Euler characteristic equal to 2.*

A rigorous proof of this is beyond the scope of this course, but we have the following rationalization. We build up our graph vertex-by-vertex. First we start with one vertex, no edges, and one unbounded face.

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So $V - E + F = 1 - 0 + 1 = 2$. Now add a vertex. Since we only consider connected graphs, we need to add an edge to attach this to the first vertex.

Since we added one vertex and one edge (and the number of faces didn't change), we still have $V - E + F = 2$. So we keep adding vertices, with one added edge per vertex (just to keep things connected). Since we always add one edge with each vertex, $V - E + F$ stays the same, for we add the vertex and subtract the edge. (The number of faces stays at 1 unbounded face, since we do not "fence off" and regions when we add a vertex with an edge. So we might have something like this tree:
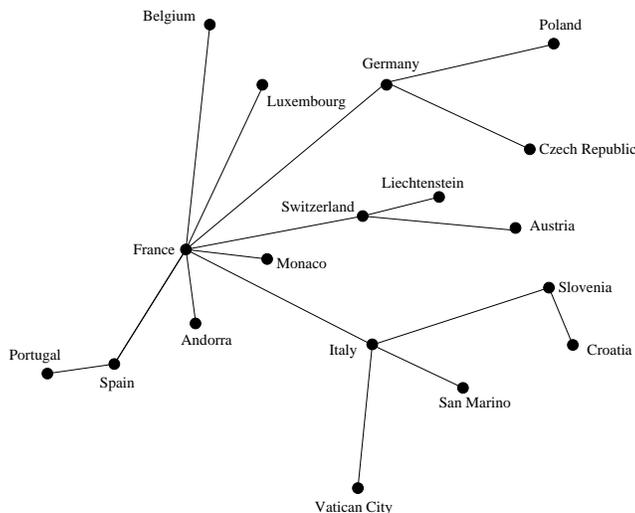
This could be obtained by starting with Portugal, then adding Spain, then France, then Belgium, Luxembourg, Germany, Switzerland, Monaco, Italy, Andorra, then Poland and the Czech Republic, then Liechtenstein and Austria, then Slovenia, San Marino, and Vatican City, then Croatia. There are 18 vertices, 17 edges, and 1 unbounded face, so $V - E + F = 2$.

We have all the vertices that belong in the final map, but not all the edges. Now when we add an edge (say, the one from Spain to Andorra), it also "fences off" an area to form a new face:
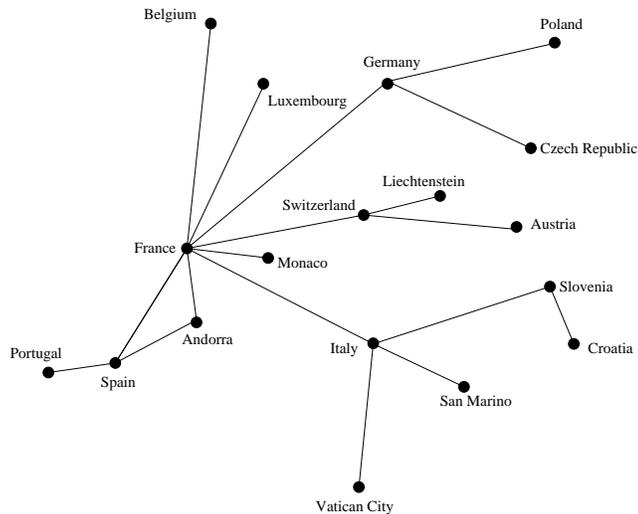
So we gain one face with our one edge, and so $V - E + F$ remains unchanged because the edge is subtracted while the face is added. Then we keep putting in extra edges until the graph is complete.
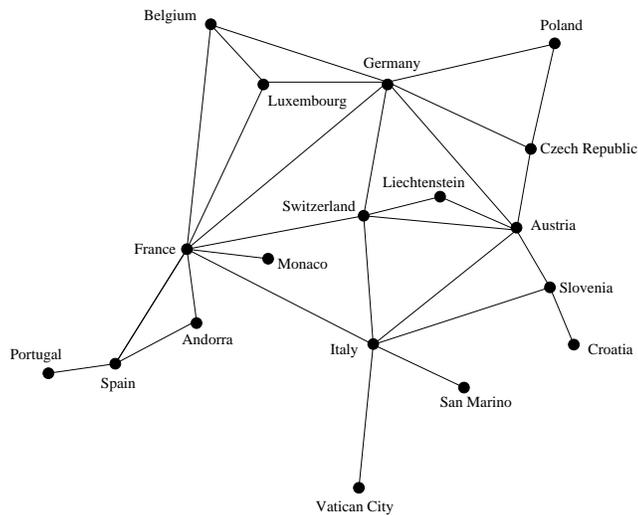
With each added edge we gain a face, and the new edge and new face cancel each other out as far as the Euler characteristic is concerned. So $V - E + F$ remains equal to 2 during the whole building process

In summary, we can build up our graph

1. A graph with a single vertex has $V - E + F = 1 - 0 + 1 = 2$.

2. Attach additional vertices, using one edge for each vertex. So $V - E + F$ remains equal to 2.

3. Draw in additional edges, forming a new face for every new edge added. So $V - E + F$ remains equal to 2.

We can build up any planar graph in this way. So all planar graphs have Euler characteristic equal to 2.

8.3 Every Planar Graph Has a Low-Degree Vertex

We'd like to use Euler characteristic to prove that every graph is 6-colorable. The proof will be devious—we need to go on a second side-trip and prove the following theorem:

Theorem 9. *Every planar graph has at least one vertex of degree 5 or less.*

Why will this help us? Because low-degree vertices, i.e., countries that do not border too many other countries, are easier to color. In fact, suppose that a planar graph G has a vertex v with the degree of v equal to 5 or less. If we can manage to properly color all the other vertices of G using six colors, and we are left to pick a color for v, we won't need to introduce a seventh color. Because v is only connected to 5 (or fewer) other vertices, so there are at most 5 colors that we are not allowed to use for v. So one of our six colors is allowable for v.

Let's try to prove the Theorem 9. We proceed by *reductio ad absurdum* or *proof by contradiction*. We assume that the theorem is false, and show that this leads to a contradiction, so that the theorem must, in fact, be true.

So we assume that there is some planar graph all of whose vertices have degree 6 or greater, and we try to show that this leads us to absurdity.

1. Suppose that there are V vertices, E edges, and F faces in our planar graph.

2. Since every vertex has 6 or more edges sticking out of it, we claim that there are at least three times as many edges as vertices. Why? Because each edge connects two vertices, and so we can say that half of the edge belongs to one vertex, and half of the edge belongs to the other. So each vertex "owns" one-half of each edges coming out of it, i.e., each vertex "owns" a number of edges equal to one-half of its degree. Since all edges have degree 6 or higher, by our assumption, each vertex owns at least $\frac{6}{2} = 3$ edges. So $E \geq 3V$, or equivalently, $V \leq \frac{E}{3}$.

3. Some edges separate one face from another, like the France-Switzerland edge or the France-Italy edge in our graph of a portion of Europe. We will reckon that half of the edge belongs to one face, and half the edge belongs to the other.

4. Other edges have the same face on both sides, for example, the France-Monaco edge or the Slovenia-Croatia edge. We reckon that such edges belong entirely to the face that surrounds them.

5. A bounded face must have at least three edges touching it, because the only way to bound a face with two or one edges is with loops or multiple edges, which do not occur in planar graphs arising from maps:

(a) The same is true of the unbounded face if there is at least one other (bounded) face in the planar graph, because one cannot separate the unbounded face from the other face(s) with fewer than 3 edges unless loops or multiple edges are used.

(b) If the unbounded face is the only face (i.e., if our planar graph is a tree), then this face touches *all* the edges, and there must be at least six edges, since every vertex has degree six or higher by our original assumption right before Step 1.

6. So every face has three or more edges touching it, and it "owns" at least half of each edge touching it, so it owns at least $\frac{3}{2}$ edges. So the number of edges is at least $\frac{3}{2}F$, i.e., $E \geq \frac{3F}{2}$. Equivalently, $F \leq \frac{2E}{3}$.

7. Now $V - E + F = 2$ since we have a planar graph. But using the inequalities $V \leq \frac{E}{3}$ and $F \leq \frac{2E}{3}$ from steps 2 and 6, we have

$$V - E + F \leq \frac{E}{3} - E + \frac{2E}{3}$$
$$= 0.$$

So $V + E - F \leq 0$ but $V - E + F = 2$. That's a contradiction!

So it is *impossible* to have a planar graph with all vertices of degree 6 or higher. That is, at least one vertex has degree 5 or less, so Theorem 9 is proved.

8.4   Proof of the Six-Color Theorem

Now we can use Theorem 9 to prove the Six Color Theorem. We recall the theorem:

Theorem 10 (Six Color Theorem, restated). *Any planar graph has a chromatic number less than or equal to 6.*

That is, every planar graph is 6-colorable.

Let's prove it. It is definitely true of planar graphs with 6 or fewer vertices! (Just give each a different color.)

What if the planar graph has more than 6 vertices? We will prove this result using a principle called *mathematical induction*. Let's be humble and just try to see if we can prove that the next biggest size of planar graph, that is, a planar graph with 7 vertices, is 6-colorable. Here is our strategy:

1. I know that planar graphs with 6 vertices can be 6-colored.

2. I want to show that any planar graph G with 7 vertices can be 6-colored.

3. By Theorem 9, G has a vertex v with degree 5 or less.

4. If I ignore v and the edges attached to v (say erase them temporarily), I have a planar graph with 6 vertices, and that can be properly colored with six colors, since we already know that graphs with 6 colors are 6-colorable.

5. Now I restore v and the edges attached to it, and I need to find a way to color v with a color different than the colors of adjacent vertices.

6. Since v has degree 5 or less, there are 5 or fewer vertices attached to it, so there are 5 or fewer colors I must avoid when coloring v. But since we are trying to color with six colors, that means that at least one color is not prohibited. Color v with this, and now the whole graph G is 6-colored.

7. So all planar graphs with 7 vertices are 6-colorable.

That's good. So now we know that graphs with 7 or fewer vertices are 6-colorable. What about graphs with 8 vertices? We go through the same steps:

1. We have shown: planar graphs with 7 vertices can be 6-colored.

2. Want to show: a planar graph G with 8 vertices can be 6-colored.

3. By Theorem 9, G has a vertex v with degree 5 or less.

4. Ignoring v, we can 6-color the other 7 vertices of G

5. Now restore v and try to color it without a conflict.

6. Since v has degree 5 or less, 5 or fewer colors are adjacent to it, leaving at least one color usable. So we can color it without a conflict.

7. So all planar graphs with 8 vertices are 6-colorable.

And we keep going on like this. In general, suppose that N is a number (say 57, if you want to be concrete), and suppose that we have shown that planar graphs with N or fewer (say, 57 or fewer) vertices can be 6-colored. Then

1. We have shown: planar graphs with N (say, 57) vertices can be 6-colored.

2. Want to show: a planar graph G with $N + 1$ (say, 58) vertices can be 6-colored.

3. By Theorem 9, G has a vertex v with degree 5 or less.

4. Ignoring v, we can 6-color the other N (say, 57) vertices of G

5. Now restore v and try to color it without a conflict.

6. Since v has degree 5 or less, 5 or fewer colors are adjacent to it, leaving at least one color usable. So we can color it without a conflict.

7. So all planar graphs with $N + 1$ (say, 58) vertices are 6-colorable.

So we can prove that a planar graph with *any* number of vertices is 6-colorable by this "chain of induction":

a planar graph with 6 vertices is 6–colorable

a planar graph with 7 vertices is 6–colorable

a planar graph with 8 vertices is 6–colorable

a planar graph with 9 vertices is 6–colorable

a planar graph with 10 vertices is 6–colorable

a planar graph with 11 vertices is 6–colorable
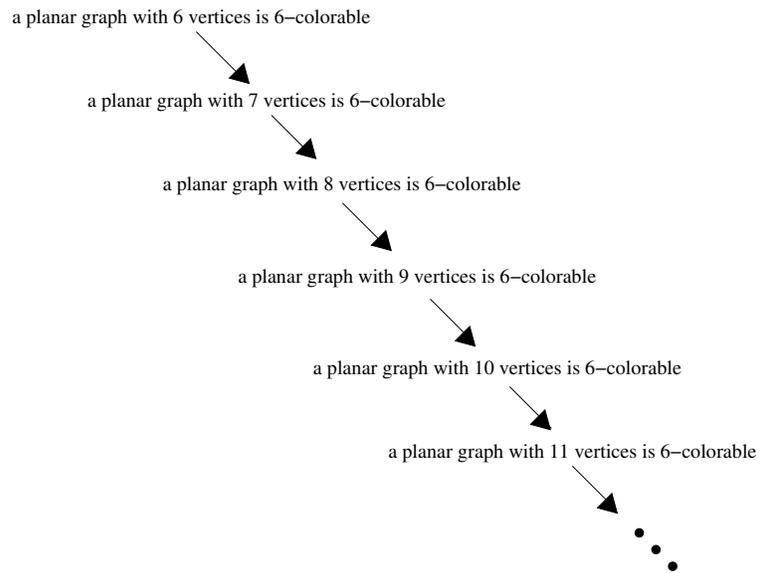
This chain proves Theorem 10 for any planar graph. So every map can be colored with six or fewer colors!