e as the limit of $(1 + 1/n)^n$

Math 122 Calculus III

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This is a small note to show that the number e is equal to a limit, specifically

$$\lim_{n\to\infty} \left(1 + \tfrac{1}{n}\right)^n = e.$$

Sometimes this is taken to be the definition of e, but I'll take e to be the base of the natural logarithms.

For a positive number x the natural logarithm of x is defined as the integral

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

Then e is the unique number such that $\ln e = 1$, that is,

$$1 = \int_1^e \frac{1}{t}\,dt.$$

The natural exponential function e^x is the function inverse to $\ln x$, and all the usual properties of logarithms and exponential functions follow.

Here's a synthetic proof that $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$. A synthetic proof is one that begins with statements that are already proved and progresses one step at a time until the goal is achieved. A defect of synthetic proofs is that they don't explain why any step is made.

Proof. Let t be any number in an interval $[1, 1+\frac{1}{n}]$. Then

$$\frac{1}{1+\frac{1}{n}} \le \frac{1}{t} \le 1.$$

Therefore

$$\int_1^{1+\frac{1}{n}} \frac{1}{1+\frac{1}{n}}\,dt \le \int_1^{1+\frac{1}{n}} \frac{1}{t}\,dt \le \int_1^{1+\frac{1}{n}} 1\,dt.$$

The first integral equals $\frac{1}{n+1}$, the second equals $\ln(1+\frac{1}{n})$, and the third equals $\frac{1}{n}$. Therefore,

$$\tfrac{1}{n+1} \le \ln\left(1 + \tfrac{1}{n}\right) \le \tfrac{1}{n}.$$

Exponentiating, we find that

$$e^{\frac{1}{n+1}} \le 1 + \tfrac{1}{n} \le e^{\frac{1}{n}}.$$

Taking the $(n+1)^{\text{st}}$ power of the left inequality gives us

$$e \le \left(1 + \tfrac{1}{n}\right)^{n+1}$$

while taking the n^{th} power of the right inequality gives us

$$\left(1 + \tfrac{1}{n}\right)^n \le e.$$

Together, they give us these important bounds on the value of e:

$$\left(1 + \tfrac{1}{n}\right)^n \le e \le \left(1 + \tfrac{1}{n}\right)^{n+1}.$$

Divide the right inequality by $1 + \frac{1}{n}$ to get

$$\frac{e}{1+\frac{1}{n}} \le \left(1 + \tfrac{1}{n}\right)^n$$

which we combine with the left inequality to get

$$\frac{e}{1+\frac{1}{n}} \le \left(1 + \tfrac{1}{n}\right)^n \le e.$$

But both $\dfrac{e}{1+\frac{1}{n}} \to e$ and $e \to e$, so by the pinching theorem

$$\left(1 + \tfrac{1}{n}\right)^n \to e,$$

also. Q.E.D.

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