EC: Counting with one-to-one functions between sets

A function $f : A \to B$ is "one-to-one" or "bijective" if it has two properties:

$\bullet$**injective:** different inputs result in different outputs:
$\forall x \neq y \in A \Rightarrow f(x) \neq f(y)$

$\bullet$**surjective:** covers the entire destination set:
$\forall z \in B, \exists x \in A, f(x) = z$

A one-to-one function guarantees that A and B have the same number of elements (sometimes infinite), so if we know the size of A it gives us the size of B or viceversa.

EXAMPLE $A = \{0, 1, 2, 3, 4, 5\}, B = \{x \text{ prime}; 10 < x < 43; x \notin \{19, 29, 37\}\}$
$f : A \to B, f(x) = x^2 + x + 11$ is a bijection (verify that). Then the size of right side set, $|B|$, is the same as $|A| = 6$.

For the following particular sets A, B show a one-to-one function from A to B, and conclude the size of B. You are asked for a bijective function f written a math expression (like $f(x) = 2x - 1$), not an enumeration of (input,output) pairs.

EC 1 : $A = \{1, 2, 3..., 10\}; B = \{x \in N; 2 \leq x \leq 72; 7 \mid x\}$

EC 2. $A = \mathbb{Z}_{77}$ and $B = \mathbb{Z}_7 \times \mathbb{Z}_{11}$

EC 3. $A = \mathbb{Z}_{240}$ and $B = \mathbb{Z}_{12} \times \mathbb{Z}_{20}$

EC 4. A={reminders coprime with 60}
in other words $A = \{x \in \mathbb{N}; x < 60; gcd(x, 60) = 1\}$.
B={reminders coprime with 12} × {reminders coprime with 5}, or
$B = \{x \in \mathbb{N}; x < 12; gcd(x, 12) = 1\} \times \{x \in \mathbb{N}; x < 5; gcd(x, 5) = 1\}$.
Conclude that $\phi(60) = \phi(12)\phi(5)$ where ϕ is Euler's totient.

EC 5. Assume finite set X includes element a. Take A={all subsets of X including a} and B={all subsets of X not including a}

EC 6. Assume finite set X includes elements $a \neq b$. Take A={all subsets of X including a} and B={all subsets of X including b}. Hint: make sure you've got a bijection from A to B, as the most obvious function is not one!

EC 7. $A = \mathbb{N}$ (naturals) and $B = \mathbb{Z}$ (integers). For the conclusion we call the size of natural numbers set the "countable infinite cardinal" N_0.

EC 8 $A = \mathbb{Z}$ and $B = \{$ multiples of 5 $\} = \{x \in Z; 5 \mid x\}$.

EC 9 :difficulty ★★. $A = \mathbb{N}$ and $B = \mathbb{Q}_+$ (non-negative rationals/fractions). The conclusion is that the set of positive rational numbers $\mathbb{Q}_+$ has the cardinality of N_0 as $\mathbb{N}$, i.e. $\mathbb{Q}_+$ is "countable".

EC 10. EXTRA CREDIT, difficulty ★. $A = \mathbb{Q}^+$(positive rationals) and $B = \mathbb{Q}$ (all rationals/fractions). Using the previous results that $\mathbb{Q}_+$ is countable, and that $\mathbb{Z}$ is countable, show that $\mathbb{Q}$ "countable".

EC 11difficulty ★★. $A = \mathbb{N}$ and $B = \mathbb{R}$ (reals). Show that no bijection is possible, because any function $f : \mathbb{N} \to \mathbb{R}$ cannot cover the entire destination set $\mathbb{R}$, thus $\mathbb{R}$ has "more elements" than $\mathbb{N}$. They are both infinite, but the cardinality of $\mathbb{R}$ is bigger, certainly not "countable"!

EC 12difficulty ★★★. $A = 2^{\mathbb{N}}$ and $B = \mathbb{R}$. Here A is the powerset of $\mathbb{N}$. The function f you are looking for is a one-to-one between subsets of natural numbers (input) and real numbers (output).