CS1802 Week 7 Honors: Sequences, Series, Induction

1 Series and sequences

1. Recognize the following sequence and write it in concise form. Compute a close-form summation of the first n terms.
   $5, 7, 9, 11, 13, 15, 17, ...$

   Solution: Arithmetic progression $2x + 3$ for $x = 1, 2, 3, 4, 5...$

$$\sum_{k=1}^{n}(2k+3) = 2\sum_{k=1}^{n} k + 3n = 2\frac{n(n+1)}{2} + 3n = n(n+1) + 3n = n(n+4)$$

2. Recognize the following sequence and write it in concise form. Compute a close-form summation of the first n terms.
   $3, 9, 19, 33, 51, 73, 99, ...$

   Solution: Quadratic progression $2x^2 + 1$ for $x = 1, 2, 3, 4, 5...$

$$\sum_{k=1}^{n}(2k^2+1) = 2\sum_{k=1}^{n} k^2 + n = 2\frac{n(n+1)(2n+1)}{6} + n =$$

$$= \frac{n(n+1)(2n+1)+3n}{3} = \frac{n(2n^2+3n+1)+3n}{3} = \frac{n(2n^2+3n+4)}{3}$$

3. Recognize the following sequence and write it in concise form. Compute a close-form summation of the first n terms.
$48, 96, 192, 384, 768, 1536, ...$

Solution: Geometric progression $24 * 2^x$ for $x = 1, 2, 3, 4, 5...$

$$\sum_{k=1}^{n}(24 * 2^k) = 24\sum_{k=1}^{n} 2^k = 24(\sum_{k=0}^{n} 2^k - 1) = 24(\frac{2^{n+1} - 1}{2 - 1} - 1) = 24(2^{n+1} - 2) = 48(2^n - 1)$$

4. (More challenging) Prove that $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ... + \frac{1}{n^2} < 1$ for any natural number n.

Solution:

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} ... + \frac{1}{n^2} < \frac{1}{1 * 2} + \frac{1}{2 * 3} + \frac{1}{3 * 4} ... + \frac{1}{(n-1) * n} < \sum_{k=1}^{n}(\frac{1}{k} - \frac{1}{k+1}) = 1 - \frac{1}{n+1}$$

2 Series Formula by Induction

1. $S_n = \sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$

   Solution: Base case $n = 1 : S_1 = 1^3 = (1)^2$
   Base case $n = 2 : S_2 = 1^3 + 2^3 = (1+2)^2$ Induction Step: Given S_n formula, we want to prove $S_{n+1} = \sum_{k=1}^{n+1} k^3 = (\sum_{k=1}^{n+1})^2$

$$S_{n+1} = S_n + (n+1)^3 = (\sum_{k=1}^{n})^2 + (n+1)^3 = (\frac{n(n+1)}{2})^2 + (n+1)^3 =$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2}{4}(n^2+4n+4)) = \frac{(n+1)^2}{4}(n+2)^2 =$$

$$= (\frac{(n+1)(n+2)}{2})^2 = (\sum_{k=1}^{n+1})^2$$

2. $S_n = (1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})....(1-\frac{1}{n^2}) = \frac{n+1}{2n}$

   Solution: Base case $n = 2 : S_2 = (1-\frac{1}{4}) = \frac{2+1}{2*2}$
   Inductive Step : Given S_n formula, we want to prove $S_n = \frac{n+1+1}{2(n+1)}$

$$S_{n+1} = S_n * (1 - \frac{1}{(n+1)^2}) = \frac{n+1}{2n} * \frac{(n+1)^2-1}{(n+1)^2} = \frac{(n+1-1)(n+1+1)}{2n(n+1)} = \frac{(n+1+1)}{2(n+1)}$$

3. $S_n = \sum_{k=1}^{n} (3n-2)^2 = n(6n^2 - 3n - 1)/2$

<u>*Solution:*</u> Base case $n = 1 : (3-2)^2 = 1(6-3-1)/2$
Inductive Step: Given S_n formula, we want to prove $S_{n+1} = (n+1)(6(n+1)^2 - 3(n+1) - 1)/2$

$$S_{n+1} = S_n + (3(n+1)-2)^2 = n(6n^2-3n-1)/2 + (9n^2+1+6n) = \frac{1}{2}(6n^3 - 3n^2 - n + 18n^2 + 2 + 12n) =$$

$$= \frac{1}{2}(6n^3 + 6n^2 + 9n^2 + 9n + 2n + 2) = \frac{1}{2}(n+1)(6n^2 + 9n + 2) =$$

$$= \frac{1}{2}(n+1)(6n^2 + 12n + 6 - 3n - 3 - 1) = \frac{1}{2}(n+1)(6(n+1)^2 - 3(n+1) - 1)$$

4. $S_n = 1 * 2 + 2 * 2^2 + 3 * 2^3 + 4 * 2^4 + ... + n * 2^n = 2 + (n-1)2^{n+1}$.

<u>*Solution:*</u> Base case $n = 1 : S_1 = 1 * 2 = 2 + (1-1)2^{1+1}$
Inductive Step Given S_n formula, we want to prove $S_{n+1} = 2 + n * 2^{n+2}$

$$S_{n+1} = S_n + (n+1)2^{n+1} = 2 + (n-1)2^{n+1} + (n+1)2^{n+1} = 2 + 2^{n+1}(n-1+n+1) = 2 + 2n * 2^{n+1} = 2 + n * 2^{n+2}$$

5. $S_n = \sum_{k=1}^{n} k * k! = (n+1)! - 1$

   Solution: Base case $n = 1 : S_1 = 1 * 1! = 2! - 1$
   Inductive Step Given S_n formula, we want to prove $S_{n+1} = (n+2)! - 1$

   $S_{n+1} = S_n + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)! = (n+1)!(n+1+1) - 1 = (n+2)! - 1$

6. $S_n = \sum_{k=1}^{n} (2k-1)^2 = n(4n^2-1)/3$

   Solution: Base case $n = 1 : S_1 = (2-1)^2 = 1 = 1(4 * 1^2 - 1)/3$

   Induction step: give S_n formula, we want to prove that $S_{n+1} = (n+1)(4(n+1)^2 - 1)/3$

$$S_{n+1} = S_n + (2(n+1)-1)^2 = n(4n^2-1)/3 + (2n+1)^2 = n(2n-1)(2n+1)/3 + 3(2n+1)^2/3 =$$

$$= \frac{1}{3}(2n+1)(n(2n-1)+3(2n+1)) = \frac{1}{3}(2n+1)(2n^2-n+6n+3) = \frac{1}{3}(2n+1)(2n^2+2n+3n+3) =$$

$$= \frac{1}{3}(2n+1)(2n(n+1)+3(n+1)) = \frac{1}{3}(n+1)(2n+1)(2n+3) = \frac{1}{3}(n+1)(4n^2+8n+3) =$$

$$= \frac{1}{3}(n+1)(4n^2+4n+4-1) = \frac{1}{3}(n+1)(4(n+1)^2-1)$$

7. $S_n = \sum_{i=1}^{n} (-1)^i * i^2 = (-1)^n \frac{1}{2} n(n+1)$

Solution: Base case $n = 1 : (-1)1^2 = (-1)\frac{1*(1+1)}{2}$

Induction step: assuming S_n true, we want to prove $S_{n+1} = (-1)^{n+1}\frac{1}{2}(n+1)(n+2)$

$$S_{n+1} = S_n + (-1)^{n+1}(n+1)^2 = (-1)^n \frac{1}{2} n(n+1) + (-1)^{n+1}(n+1)^2 =$$
$$= (-1)^n \frac{1}{2}(n+1)(n-2(n+1)) = (-1)^n \frac{1}{2}(n+1)(-n-2) = (-1)^{n+1}\frac{1}{2}(n+1)(n+2)$$

8. Prove that $n! > 3^n > 2^n > n^2 > n\log_2(n) > n > \log_2(n)$ for $n \geq 7$

Solution: Base case $n = 7$:
$7! = 5040 > 3^7 = 2187 > 2^7 = 128 > 7^2 = 49 > 7 * \log_2(7) = 19.65 > 7 > \log_2(7) = 2.81$

Induction step: using inequalities for n, we want to prove that
$(n+1)! > 3^{n+1} > 2^{n+1} > (n+1)^2 > (n+1)\log_2(n+1) > n+1 > \log_2(n+1)$
We start from the left side:

$$(n+1)! = n!(n+1) > 3^n(n+1) > 3^n * 3 = 3^{n+1} = 3^n * 3 > 2^n * 3 > 2^n * 2 = 2^{n+1}$$
$$2^{n+1} = 2^n * 2 > n^2 * 2 = n^2 + n^2 \geq n^2 + 7n = n^2 + 2n + 5n > n^2 + 2n + 1 = (n+1)^2$$

We have proved so far the first three inequalities. Here is the proof for the last one:
$2^{n+1} > n^2 \Rightarrow n+1 > \log_2(n^2) \geq \log_2(7n) = \log_2(n+6n) > \log_2(n+1)$

Finally the inequalities fourth and fifth:
$n+1 > \log_2(n+1) \Rightarrow (n+1)^2 > (n+1)\log_2(n+1) > (n+1)\log_2(7) > n+1$

3 Induction proofs

PB 1 Show that 5 divides $8^n - 3^n$ for any natural number n.

Solution: Base case $n = 0 : 8^0 - 3^0 = 1 - 1 = 0$ is a multiple of 5
Induction Step : Assuming $8^n - 3^n = 5k$ we want to prove that $5|(8^{n+1} - 3^{n+1})$
$8^{n+1} - 3^{n+1} = 8 * 8^n - 3 * 3^n = 5 * 8^n + 3(8^n - 3^n) = 5 * 8^n + 5k = 5(8^n + k)$ thus multiple of 5

Solution without induction:
$8^{n+1} - 3^{n+1} = (8-3)\sum_{k=0}^{n} 8^k * 3^{n-k}$
which is a multiple of 5 due to the first factor.

PB 2 Binary trees height Prove that depth (height) of a binary tree with n nodes is at least $\lfloor \log_2(n) \rfloor$ (depth is the max number of edges on a path from root to a leaf).

<u>*Solution:*</u> Base case $n = 1, depth = 0 \geq log(1) = 0$
Base case $n = 2, depth = 1 \geq log(2) = 1$

Strong Induction Step: Will assume the property is true for any $k < n$, and will prove it for n. In particular the k-s for which we are going to need it are the number of nodes in the Left and Right subtrees.

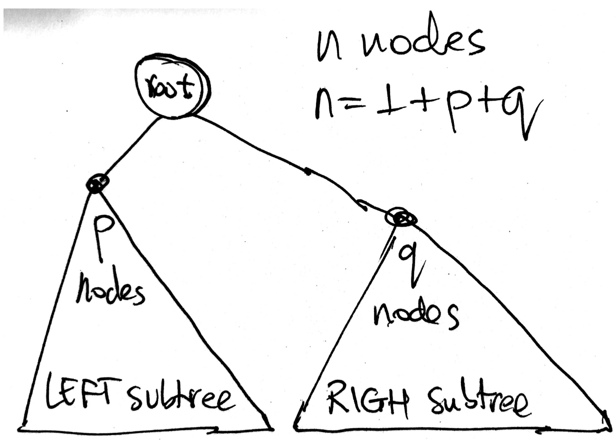
Lets say the root of the binary tree has a left subtree with p nodes and a right subtree with q nodes. Then $n = 1 + p + q$. Lets assume (without loss of generality) that $p \geq q$.
$p < n$ so by induction hypothesis we know $depth_L \geq \lfloor \log_2(p) \rfloor$

$depth = 1 + \max(depth_L, depth_R) \geq 1 + depth_L \geq$

$1 + \lfloor \log_2(p) \rfloor = 1 + \lfloor \log_2(\frac{2p}{2}) \rfloor = 1 + \lfloor \log_2(2p) - 1 \rfloor = \lfloor \log_2(2p) \rfloor$

If n is even then $p \geq q = n - p - 1 \Rightarrow 2p > n \Rightarrow \lfloor \log_2(2p) \rfloor \geq \lfloor \log_2(n) \rfloor$

If n is odd then $\lfloor \log_2(n-1) \rfloor = \lfloor \log_2(n) \rfloor$
and $p \geq q = n - p - 1 \Rightarrow 2p \geq n - 1 \Rightarrow \lfloor \log_2(2p) \rfloor \geq \lfloor \log_2(n-1) \rfloor = \lfloor \log_2(n) \rfloor$

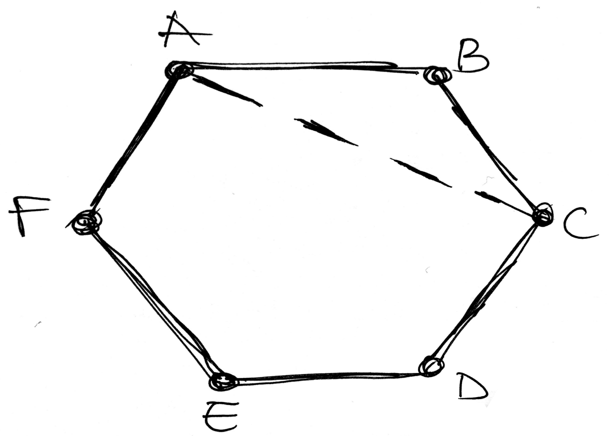


PB 3 Polygon sum of angles Prove that the sum of the interior angles of a convex polygon with n sides is $(n-2)\pi$. You can assume known that the sum of angles of any triangle is π

Solution: Base case is the triangle $n = 3$ for which the property is given as known : sum of triangle angles is $(3-2)\pi = \pi$.

Induction step : given the property holds for a convex polygon with n sides, we want to prove it for a convex polygon with $n+1$ sides. If P=ABDC... is the polygon with $n+1$ sides, cutting a triangle ABC out of it results in a convex polygon with n sides.

So we can sum the angles of P as the sum of angels in triangle ABC plus the sum of angles of n-sides-polygon ACDE.... Applying induction hypothesis we get
$\pi + (n-1-2)\pi = (n-2)\pi$



PB 5 Will everyone get the same grade? Jimmy found a proof that every student in CS1800 will get the same exact grade, by induction over the number n of students in the class:

- Define the statement P_n="in a class of size n, all students get the same grade"
- Base step $n = 1, P_n$ is true, since there is only one student
- Inductive step: If a the class has size n, consider all n subsets of size $n-1$. Since P_{n-1} is true, then all these subsets of students will get the same grade (per subset). But the subsets intersect, so it means all students will get the same grade, or P_n is true.

Where is Jimmy wrong?

Solution: Jimmy is wrong in Inductive Step, specifically $P_1 \Rightarrow P_2$ (the other implications, and the best case do hold true). For $n = 2$ there are two subsets of $n - 1 = 1$ each; they DO NOT OVERLAP (INTERSECT) so the argument is flawed.

PB 6 Sum of Perfect Squares. Prove that for all natural number n, 10^n can be written as a sum of two perfect squares ($10^n = a^2 + b^2$ for some a, b positive integers).