The Inclusion-Exclusion Principle

1. The probability that at least one of two events happens

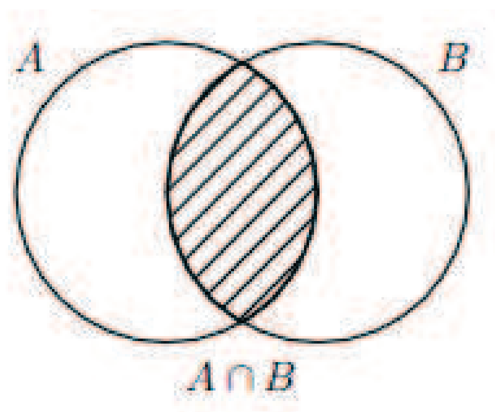
Consider a discrete sample space Ω. We define an event A to be any subset of Ω, which in set notation is written as $A \subset \Omega$. Then, Boas asserts in eq. (3.6) on p. 732 that[1]

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)\,, \tag{1}$$

for any two events $A, B \subset \Omega$. This is equivalent to the set theory result,

$$|A \cup B| = |A| + |B| - |A \cap B|\,, \tag{2}$$

where the notation $|A|$ means the number of elements contained in the set A, etc. In writing eq. (2), we have assumed that A and B are two finite discrete sets, so the number of elements in A and B are finite.



The proof of eq. (2) is immediate after considering the Venn diagram shown above. In particular, adding the number of elements of A and B overcounts the number of elements in $A \cup B$, since the events in $A \cap B$ have been double counted. Thus, we correct this double counting by subtracting the number of elements in $A \cap B$, which yields eq. (2). The corresponding result in probability theory is given by eq. (1).

[1]Boas uses a nonstandard notation by writing $A + B$ for $A \cup B$. The latter is standard in set theory and we shall use it in these notes. $A \cup B$ means the union of the sets A and B and is equivalent to the "inclusive or," i.e. "either A or B or both." Likewise, Boas uses a nonstandard notation by writing AB for $A \cap B$. Again, the latter is standard in set theory and we shall use it in these notes. $A \cap B$ means the intersection of the sets A and B, or equivalently "both A *and* B."

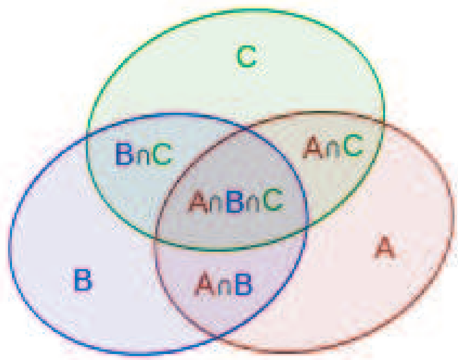
2. The probability that at least one of three events happens

It is straightforward to generalize the result of eq. (1) to the case of three events.[2]

$$P(A \cup B \cup C) = P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)\,, \quad (3)$$

for any three events $A, B, C \subset \Omega$. This is equivalent to the set theory result,

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|\,. \quad (4)$$



Once again, the proof of eq. (4) is immediate after considering the Venn diagram shown above.[3] In particular, adding the number of elements of A, B and C counts elements in $A\cap B\cap C$ three times, and counts elements of $A\cap B$, $A\cap C$ and $B\cap C$ not contained in $A\cap B\cap C$ twice. Thus, $|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$ will include all events in A, B and C once except for the events in $A\cap B\cap C$, which were all subtracted off. Thus, to include all events in $A\cup B\cup C$ exactly once, we must add back the number of events in $A\cap B\cap C$. Thus, eq. (4) is established. The corresponding result in probability theory is given by eq. (3).

3. The Inclusion-Exclusion principle

The inclusion-exclusion principle is the generalization of eqs. (1) and (2) to n sets. Let A_1, $A_2, \ldots, A_n$ be a sequence of n events. Then,

$$P(A_1\cup A_2\cup\cdots\cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i\cap A_j) + \sum_{i<j<k} P(A_i\cap A_j\cap A_k)$$
$$- \sum_{i<j<k<\ell} P(A_i\cap A_j\cap A_k\cap A_\ell) + \ldots + (-1)^{n+1}P(A_1\cap A_2\cap\cdots\cap A_n)\,, \quad (5)$$

[2]This is problem 15–3.8 on p. 734 of Boas.

[3]The Venn diagram above is taken from the Wikipedia webpage on the inclusion-exclusion principle. Check it out at `http://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle`.

where $A_1, A_2, \ldots A_n \subset \Omega$. This is equivalent to the set theory result,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$- \sum_{i<j<k<\ell} |A_i \cap A_j \cap A_k \cap A_\ell| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \,. \quad (6)$$

The proof of eq. (6) is an exercise in counting. Suppose a point is contained in exactly m of the sets, A_1, $A_2, \ldots A_n$, where m is a number between 1 and n. Then, the point is counted m times in $\sum_{i=1}^{n} |A_i|$, it is counted $C(m,2)$ times in $\sum_{i<j} |A_i \cap A_j|$, it is counted $C(m,3)$ times in $\sum_{i<j<k} |A_i \cap A_j \cap A_k|$, etc., where $C(m,k)$ is the number of combinations of m objects taken k at a time. After reaching $\sum_{i_1<i_2<\cdots i_m} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$, where the point is counted once [since $C(m,m) = 1$], one finds that the point is not counted at all in any of the terms that involve the intersection of more than m sets. The net result is that a point that is contained in exactly m of the sets will be counted S times in $|A_1 \cup A_2 \cup \cdots \cup A_n|$ given by eq. (6), where

$$S \equiv C(m,1) - C(m,2) + C(m,3) - C(m,4) + \cdots + (-1)^{m+1} C(m,m) \,, \quad (7)$$

after noting that $C(m,1) = m$.

To compute S, we recall the binomial theorem,

$$(x+y)^m = \sum_{k=0}^{m} C(m,k) x^k y^{m-k} \,, \quad (8)$$

where

$$C(m,k) \equiv \binom{m}{k} \equiv \frac{m!}{k!(m-k)!}$$

is the number of combinations of m objects taken k at a time. Setting $x = 1$ and $y = -1$ in eq. (8) yields,

$$\sum_{k=0}^{m} (-1)^k C(m,k) = 0 \,.$$

Using $C(m,0) = 1$, it follows that

$$1 - C(m,1) + C(m,2) - C(m,3) + \ldots + (-1)^m C(m,m) = 0 \,,$$

which implies that $S = 1$ [cf. eq. (7)]. Thus, we have shown that there is no multiple counting of points in eq. (6). That is, every point contained in the union of $A_1, A_2, \ldots A_n$ is counted exactly one time. Thus, eq. (6) is established. The corresponding result in probability theory is given by eq. (5). We have therefore verified the inclusion-exclusion principle.

There are numerous applications of the inclusion-exclusion principle, both in set theory and in probability theory. In particular, it provides a powerful tool for certain types of counting problems. An example is provided in the next section of these notes.

4. Derangements

Starting with n objects, how many different permutations are there such that none of the objects end up in their original positions? Such permutations are called *derangements* or permutations with no fixed points. In general, there are $n!$ possible permutations of n objects. In this section, we shall count the number of possible derangements of n objects, which we shall denote by the symbol D_n. The derivation of D_n will be based on the inclusion-exclusion principle.

Let A_i be the subset of the set of permutations of n objects such that the ith object alone ends up in its original position under the permutation. Then $|A_1 \cup A_2 \cup \cdots \cup A_n|$ counts the number of permutations in which at least one of the n objects ends up in its original position. Since there are $n!$ possible permutations of n objects, it follows that the number of permutations such that *none* of the objects end up in their original positions, i.e. the total number of derangements of n objects, is given by

$$D_n = n! - |A_1 \cup A_2 \cup \cdots \cup A_n| \,. \tag{9}$$

One can compute $|A_1 \cup A_2 \cup \cdots \cup A_n|$ using the inclusive-exclusive principle [cf. eq. (6)]. First,

$$|A_i| = (n-1)!\,,$$

since if exactly one of the n objects ends up in its original position, that leaves the other $n-1$ objects to be freely permuted in $(n-1)!$ possible ways. Hence,

$$\sum_{i=1}^{n} |A_i| = n \cdot (n-1)! = n!\,,$$

since there are n terms in the sum. Second,

$$|A_i \cap A_j| = (n-2)!\,,$$

since if exactly two of the n objects end up in their original positions, that leaves the other $n-2$ objects to be freely permuted in $(n-2)!$ possible ways. Hence,

$$\sum_{i<j} |A_i \cap A_j| = (n-2)!\,C(n,2) = (n-2)! \cdot \frac{n(n-1)}{2!} = \frac{n!}{2!}\,,$$

since there are $C(n,2)$ terms in the sum above. Third,

$$|A_i \cap A_j \cap A_k| = (n-3)!\,,$$

since if exactly three of the n objects end up in their original positions, that leaves the other $n-3$ objects to be freely permuted in $(n-3)!$ possible ways. Hence,

$$\sum_{i<j<k} P(A_i \cap A_j \cap A_k) = (n-3)!\,C(n,3) = (n-3)! \cdot \frac{n(n-1)(n-2)}{3!} = \frac{n!}{3!}\,,$$

since there are $C(n,3)$ terms in the sum above.

The pattern should be clear. When we reach the final term in eq. (6), we have

$$|A_1 \cap A_2 \cap \cdots \cap A_n| = 1\,,$$

which corresponds to all objects ending up in their original position.[4] Hence, eq. (6) yields

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = n!\left[1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n+1}\frac{1}{n!}\right] = n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}\,.$$

Inserting this result into eq. (9) yields the number of derangements of n objects,

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}\,. \qquad (10)$$

The probability that a permutation of n objects is a derangement is given by $D_n/n!$ since there are D_n possible derangements and $n!$ possible permutations. It is amusing to note that as $n \to \infty$, the probability that a permutation of n objects is a derangement is given by

$$\lim_{n\to\infty} P(\text{derangement}) = \lim_{n\to\infty}\frac{D_n}{n!} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = \frac{1}{e}\,.$$

Moreover, if n is large (in practical applications, any n greater than about 10 can be considered to be large), then the probability that the permutation of the n objects is a derangement is approximately $1/e$ almost independently of the precise value of n.

An example of derangements arises in a very famous problem called the hat check problem in which n hats are checked by customers at a restaurant. Unfortunately the hat checkers fail to do their jobs, and the hats are hopelessly scrambled during storage. What is the probability that no customer gets his or her own hat back? This is equivalent to asking for the probability of a derangement of n objects. If the number of customers involved is large, then the probability is approximately $1/e \simeq 0.367879$.

References

1. Charles M. Grinstead and J. Laurie Snell, *Introduction to Probability*, 2nd edition (American Mathematical Society, Providence, RI, 1997). A free copy of this textbook is available from `http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/amsbook.mac.pdf` under the terms of the GNU Free Documentation License.

2. Richard A. Brualdi, *Introductory Combinatorics*, 5th edition (Pearson Education, Inc., Upper Saddle River, NJ, 2010). A free copy of this textbook is available from `http://filetosi.files.wordpress.com/2010/12/combiatoric.pdf` courtesy of the China Machine Press.

[4] This is the so-called trivial permutation of the original n objects in which none of the objects are permuted, which is counted as one of the $n!$ possible permutations of n objects.