

Induction problems

Induction problems can be hard to find. Most texts only have a small number, not enough to give a student good practice at the method. Here are a collection of statements which can be proved by induction. Some are easy. A few are quite difficult. The difficult ones are marked with an asterisk. I would not ask you to do a problem this hard in a test or exam.

1. $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n+1)$
2. $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$
3. $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n+1)^2$
- 4* $1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$
- 5* $1^5 + 2^5 + 3^5 + \dots + n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$

Is there a pattern in this? Consider the following.

6. $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$
7. $\sum_{r=1}^n r(r+1) = \frac{1}{3}n(n+1)(n+2)$
8. $\sum_{r=1}^n r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$

Can you see how the results from numbers 6-8 could be used to obtain the results mentioned in 1-3. Numbers 6-8 suggest a general pattern. This too could be proved by induction.

$$9^* \sum_{r=1}^n r(r+1)(r+2)\dots(r+p-1) = \frac{1}{p+1}n(n+1)(n+2)\dots(n+p)$$

Now some more series problems, including some of a different type.

10. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$
11. $\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$
12. $\frac{1}{1 \times 2 \times 2} + \frac{1}{2 \times 3 \times 2^2} + \frac{1}{3 \times 4 \times 2^3} + \dots + \frac{1}{n \times (n+1) \times 2^n} = 1 - \frac{1}{(n+1)2^n}$
13. $1 \times 3 + 2 \times 3^2 + 3 \times 3^3 + \dots + n \times 3^n = \frac{3}{4}[(2n-1)3^n + 1]$
14. $\frac{1}{2 \times 5} + \frac{1}{5 \times 8} + \frac{1}{8 \times 11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{6n+4}$
15. $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$
16. $\frac{1}{1 \times 2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{3 \times 4 \times 5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(3n+7)}{2(n+1)(n+2)}$
17. $1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{4n^3 - n}{3}$

Problems involving divisibility are also quite common.

18. Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all $n \geq 0$.
19. Prove that $a^2 - 1$ is divisible by 8 for all odd integers a .
20. Prove that $a^4 - 1$ is divisible by 16 for all odd integers a .
- 21* Prove that $a^{2^n} - 1$ is divisible by 4×2^n for all odd integers a , and for all integers n .
22. Prove that $n^3 + 2n$ is divisible by 3 for all integers n .
23. Prove that $17n^3 + 103n$ is divisible by 6 for all integers n .
24. Prove that $2^n + 1$ is divisible by 3 for all odd integers n .

There are a lot of neat properties of the Fibonacci numbers that can be proved by induction. Recall that the Fibonacci numbers are defined by $f_0 = 0$, $f_1 = f_2 = 1$ and the recursion relation $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. All of the following can be proved by induction (we proved number 28 in class). These exercises tend to be more challenging.

25. f_n and f_{n+1} are ‘relatively prime’ for all $n \geq 1$. That is, they share no factor in common other than the number 1.
26. $f_{n+1} < \left(\frac{7}{4}\right)^n$ for all $n \geq 1$.
27. $f_1 + f_2 + f_3 + \dots + f_n = f_{n+2} - 1$ for all $n \geq 1$.
28. If $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$ then $f_n = (a^n - b^n)/\sqrt{5}$.
29. $f_1 + f_3 + f_5 + \dots + f_{2n-1} = f_{2n}$ for all $n \geq 1$
30. $f_2 + f_4 + f_6 + \dots + f_{2n} = f_{2n+1} - 1$ for all $n \geq 1$
31. $f_1 - f_2 + f_3 - \dots + (-1)^n f_{n+1} = (-1)^n f_n + 1$ for all $n \geq 1$
32. $f_1^2 + f_2^2 + f_3^2 + \dots + f_n^2 = f_n f_{n+1}$ for all $n \geq 1$
33. $f_1 f_2 + f_2 f_3 + f_3 f_4 + \dots + f_{2n-1} f_{2n} = f_{2n}^2$ for all $n \geq 1$
- 34* $f_{m+n+1} = f_m f_n + f_{m+1} f_{n+1}$ for all $m, n \geq 0$. This requires a ‘double’ induction. Assume here that the result holds true for all values of m and n with $m \leq M$ and $n \leq N$, with one of these inequalities being strict. Then you must prove it holds for $m = M$ and $n = N$. Not for beginners!
35. f_{2n} is divisible by f_n for all $n \geq 1$.
36. f_{kn} is divisible by f_n for all $n \geq 1$, where k is any fixed integer.

Now we have an eclectic collection of miscellaneous things which can be proved by induction.

37. Give a formal inductive proof that the sum of the interior angles of a convex polygon with n sides is $(n-2)\pi$. You may assume that the result is true for a triangle. Note - a *convex* polygon is one where all the interior angles are smaller than π radians. hint: note that you can cut a convex n -gon into a convex $n-1$ -gon and a triangle.
38. A convex n -gon has $n(n-3)/2$ diagonals.
39. Prove that, if $x > 0$ is any fixed real number, then $(1+x)^n > 1+nx$ for all $n \geq 2$.
40. Prove that, for all $n \geq 2$, $\sqrt[n]{n} < 2 - \frac{1}{n}$
41. Give a proof of De-Moivre’s theorem using induction. You will need the addition of angle formulae for sine and cosine.
42. Consider the game which in class we called ‘the tower of Hanoi’. If all the tiles are initially stacked on the left peg, and we desire to move them eventually to the right peg, to which peg should we move the first tile in order to do this in the smallest number of moves. (experiment). Prove your claim by induction on n , the number of tiles.

Finally, here are some identities involving the binomial coefficients, which can be proved by induction. Recall (from secondary school) the definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ and the recursion relation } \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

For appropriate values of n and k . It is a useful exercise to prove the recursion relation (you don’t need induction).

43. Prove, using induction, that all binomial coefficients are integers. This is not obvious from the definition.
44. Show that $\binom{2n}{n} < 2^{2n-2}$ for all $n \geq 5$.
- 45* Prove the binomial theorem using induction. This states that for all $n \geq 1$,

$$(x+y)^n = \sum_{r=0}^n \binom{n}{r} x^{n-r} y^r$$

There is nothing fancy about the induction, however unless you are careful, things can get messy with the summation, which is why I’ve marked it as difficult.