3 Countable and Uncountable Sets

A set A is said to be *finite*, if A is empty or there is $n \in \mathbb{N}$ and there is a bijection $f : \{1, \dots, n\} \to A$. Otherwise the set A is called *infinite*. Two sets A and B are called *equinumerous*, written $A \sim B$, if there is a bijection $f : X \to Y$. A set A is called *countably infinite* if $A \sim \mathbb{N}$. We say that A is countable if $A \sim \mathbb{N}$ or A is finite.

Example 3.1. The sets $(0, \infty)$ and $\mathbb{R}$ are equinumerous. Indeed, the function $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = e^x$ is a bijection.

Example 3.2. The set $\mathbb{Z}$ of integers is countably infinite. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even;} \\ -(n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Then f is a bijection from $\mathbb{N}$ to $\mathbb{Z}$ so that $\mathbb{N} \sim \mathbb{Z}$.

If there is no bijection between $\mathbb{N}$ and A, then A is called *uncountable*.

Theorem 3.3. *There is no surjection from a set A to $\mathcal{P}(A)$.*

Proof. Consider any function $f : A \to \mathcal{P}(A)$ and let

$$B = \{a \in A |\, a \notin f(a)\}.$$

We claim that there is no $b \in A$ such that $f(b) = B$. Indeed, assume $f(b) = B$ for some $b \in A$. Then either $b \in B$ hence $b \notin f(b)$ which is a contradiction, or $b \notin B = f(b)$ implying that $b \in B$ which is again a contradiction. Hence the map f is not surjective as claimed. ■

As a corollary we have the following result.

Corollary 3.4. *The set $\mathcal{P}(\mathbb{N})$ is uncountable.*

Proposition 3.5. *Any subset of a countable set is countable.*

Proof. Without loss of generality we may assume that A is an infinite subset of $\mathbb{N}$. We define $h : \mathbb{N} \to A$ as follows. Let $h(1) = \min A$. Since A is infinite, A is nonempty and so $h()$ is well-defined. Having defined $h(n-1)$, we define $h(n) = \min(A \setminus \{h(1), \dots, h(n-1)\})$. Again since A is infinite the set $(A \setminus \{h(1), \dots, h(n-1)\})$ is nonempty, $h(n)$ is well-defined. We claim that h is a bijection. We first show that h is an injection. To see this we prove that $h(n+k) > h(n)$ for all $n, k \in \mathbb{N}$. By construction $h(n+1) > h(n)$

for all $n \in \mathbb{N}$. Then setting $B = \{k \in \mathbb{N} | \, h(n+k) > h(n)\}$ we see that $1 \in B$ and if $h(n+(k-1)) > h(n)$, then $h(n+k) > h(n+(k-1)) > h(n)$. Consequently, $B = \mathbb{N}$. Since n was arbitrary, $h(n+k) > h(n)$ for all $n, k \in \mathbb{N}$. Now taking distinct $n, m \in \mathbb{N}$ we may assume that $m > n$ so that $m = n + k$. By the above $h(m) = h(n+k) > h(n)$ proving that h is an injection. Next we show that h is a surjection. To do this we first show that $h(n) \geq n$. Let $C = \{n \in \mathbb{N} | \, h(n) \geq n\}$. Clearly, $1 \in C$. If $k \in C$, then $h(k+1) > h(k) \geq n$ so that $h(k+1) \geq k+1$. Hence $k+1 \in C$ and by the principle of mathematical induction $C = \mathbb{N}$. Now take $n_0 \in A$. We have to show that $h(m_0) = n_0$ for some $m_0 \in \mathbb{N}$. If $n_0 = 1$, then $m_0 = 1$. So assume that $n_0 \geq 2$. Consider the set $D = \{n \in A | \, h(n) \geq n_0\}$. Since $h(n_0) \geq n_0$, the set D is nonempty and by the well-ordering principle D has a minimum. Let $m_0 = \min D$. If $m_0 = 1$, then $h(m_0) = \min A \leq n_0 \leq h(m_0)$ and hence $h(m_0) = n_0$. So we may also assume that $n_{>} \min A$. Then $h(m_0) \geq n_0 > h(m_0 - 1) > \ldots > h(1)$ in view of definitions of m_0 and h. Since $h(m_0) = \min(A \setminus \{h(1), \ldots, h(m_0-1)\})$ and $n_0 \in A \setminus \{h(1), \ldots, h(m_0-1)\}$ and $h(m_0) \geq n_0$, it follows that $h(m_0) = n_0$. This proves that h is also a surjection. ∎

Proposition 3.6. *Let A be a non-empty set. Then the following are equivalent.*

(a) A is countable.

(b) There exists a surjection $f : \mathbb{N} \to A$.

(c) There exists an injection $g : A \to \mathbb{N}$.

Proof. $(a) \implies (b)$ If A is countably infinite, then there exists a bijection $f : \mathbb{N} \to A$ and then (b) follows. If A is finite, then there is bijection $h : \{1, \ldots, n\} \to A$ for some n. Then the function $f : \mathbb{N} \to A$ defined by

$$f(i) = \begin{cases} h(i) & 1 \leq i \leq n, \\ h(n) & i > n. \end{cases}$$

is a surjection.
$(b) \implies (c)$. Assume that $f : \mathbb{N} \to A$ is a surjection. We claim that there is an injection $g; A \to \mathbb{N}$. To define g note that if $a \in A$, then $f^{-1}(\{a\}) \neq \emptyset$. Hence we set $g(a) = \min f^{-1}((a)\})$. Now note that if $a \neq a'$, then the sets $f^{-1}(\{a\}) \cap f^{-1}(\{a'\}) = \emptyset$ which implies $\min^{-1}(\{a\}) \neq \min^{-1}(\{a'\})$. Hence $g(a) \neq g(a')$and $g : A \to \mathbb{N}$ is an injective.
$(c) \Rightarrow (a)$. Assume that $g : A \to \mathbb{N}$ is an injection. We want to show that A

is countable. Since $g : A \to g(A)$ is a bijection and $g(A) \subset \mathbb{N}$, Proposition 3.5 implies that A is countable. ■

Corollary 3.7. *The set* $\mathbb{N} \times \mathbb{N}$ *is countable.*

Proof. By Proposition 3.6 it suffices to construct an injective function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f(n, m) = 2^n 3^m$. Assume that $2^n 3^m = 2^k 3^l$. If $n < k$, then $3^m = 2^{k-n} 3^l$. The left side of this equality is an odd number whereas the right is an even number implying $n = k$ and $3^m = 3^l$. Then also $m = l$. Hence f is injective. ■

Proposition 3.8. *If* A *and* B *are countable, then* $A \times B$ *is countable.*

Proof. Since A and B are countable, there exist surjective functions $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$. Define $h : \mathbb{N} \times \mathbb{N} \to A \times B$ by $F(n, m) = (f(n), g(m))$. The function F is surjective. Since $\mathbb{N} \times \mathbb{N}$ is countably infinite, there is a bijection $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Then $G : \mathbb{N} \times A \times B$ defined by $G = F \circ h$ is a surjection. By part (c) of Proposition 3.6, the set $A \times B$ $A \times B$ is countable. ■

Corollary 3.9. *The set* $\mathbb{Q}$ *of all rational numbers is countable.*

Proposition 3.10. *Assume that the set* I *is countable and* A_i *is countable for every* $i \in I$. *Then* $\bigcup_{i \in I} A_i$ *is countable.*

Proof. For each $i \in I$, there exists a surjection $f_i : \mathbb{N} \to A_i$. Moreover, since I is countable, there exists a surjection $g : \mathbb{N} \to I$. Now define $h : \mathbb{N} \times \mathbb{N} \to \bigcup_{i \in I} A_i$ by $F(n, m) = f_{g(n)}(m)$ and let $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection. Then F is a surjection and the composition $G = F \circ h : \mathbb{N} \to \bigcup_{i \in I} A_i$ is a surjection. By Proposition 3.6, $\bigcup_{i \in i} A_i$ is countable. ■

Proposition 3.11. *The set of real numbers* $\mathbb{R}$ *is uncountable.*

The proof will be a consequence of the following result about nested intervals.

Proposition 3.12. *Assume that* $(I_n)_{n \in \mathbb{N}}$ *is a countable collection of closed and bounded intervals* $I_n = a_n, b_n]$ *satisfying* $I_{n+1} \subset I_n$ *for all* $n \in \mathbb{N}$. *Then* $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Proof. Since $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n, it follows that $a_n \leq b_k$ for all $n, k \in \mathbb{N}$. So, the set $A = \{a_n | \, n \in \mathbb{N}\}$ is bounded above by every b_k and consequently $a := \sup A \leq b_k$ for all $k \in \mathbb{N}$. But this implies that the set $B = \{b_k | \, k \in \mathbb{N}\}$ is bounded below by a so that $a \leq b := \inf B$. Hence $\bigcap_{n \in \mathbb{N}} I_n = [a, b]$. ■

Proof of Proposition 3.11. Arguing by contradiction assume that $\mathbb{R}$ is countable. Let $x_1, x_2, x_3, \dots$ be enumeration of $\mathbb{R}$. Choose a closed bounded interval I_1 such that $x_1 \notin I_1$. Having chosen the closed intervals $I_1, I_2, \dots, I_{n-1}$, we choose the closed interval I_n to be a subset of I_{n-1} such that $x_n \notin I_n$. Consequently, we have a countable collection of closed bounded intervals (I_n) such that $I_{n+1} \subset I_n$ and $x_n \notin I_n$. Then by the above proposition, $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. Observe that if x belongs to this intersection, then x is not on the list $x_1, x_2, \dots$, contradiction.

■