Binary Search Trees

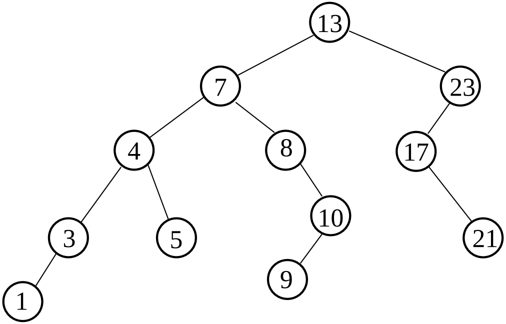
Chuck Cusack

Hope College

Spring Semester 2015

Binary Search Trees

- A **Binary Search Tree** is a binary tree with the following properties: Given a node x in the tree
  - if y is a node in the left subtree of x, then $key[y] \leq key[x]$.
  - if y is a node in the right subtree of x, then $key[x] \leq key[y]$.



- For simplicity, we will assume that all keys are distinct.

Binary Search Tree Operations

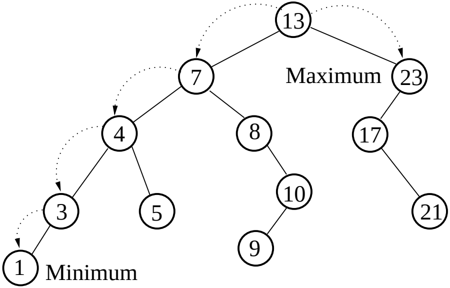
- Given a binary search tree, there are several operations we want to perform.
  - **Insert** an element
  - **Delete** an element
  - **Search** for an element
  - Find the **minimum/maximum** element
  - Find the **successor/predecessor** of a node.
- Once we see how these are done, it will be apparent that the complexity of each of these is $O(h)$, where h is the height of the tree.
- The insert and delete operations are the hardest to implement.
- Finding the minimum/maximum and searching are the easiest, so we will start with these.

BST: Minimum/Maximum

- The minimum element is the left-most node.
- The maximum element is the right-most node of the tree.
- Here are implementations of these methods:

```
node Find_Min(x) {
   while(x.left!=null)
       x=x.left;
   return x;
}
```

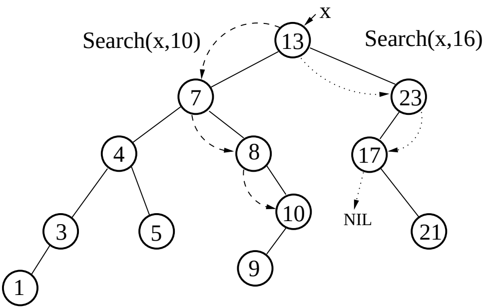
```
node Find_Max(x) {
    while(x.right!=null)
       x=x.right;
    return x;
 }
```



BST: Searching

- Search finds the node with value *k* in the tree rooted at *x*.

```
node Search(x,k) {
    while(x!=null && k !=x.key) {
         if(k<x.key)
           x=x.left;
         else
           x=x.right;
    }
    return x;
}
```



BST: Successor/Predecessor

- Finding the Successor/Predecessor of a node is harder.
- To find the successor y of a node x (if it exists)
  - If x has a nonempty right subtree, then y is the smallest element in the tree rooted at $x.right$. Why?
  - If x has an empty right subtree, then y is the lowest ancestor of x whose left child is also an ancestor of x. Clearly.

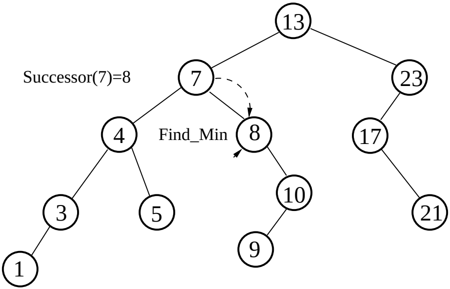
```
node Successor(x) {
     if(x.right!=null)
          return Find_Min(x.right);
     y=p[x];
     while(y!=null && x=y.right) {
          x=y;
          y=y.parent;
     }
     return y;
}
```

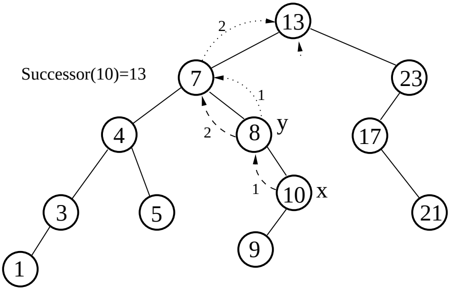
- The predecessor operation is symmetric to successor.

BST: Successor Argument

- So, why is it that if x has an empty right subtree, then y is the lowest ancestor of x whose left child is also an ancestor of x?
- Let's look at it the other way.
- Let y be the lowest ancestor of x whose left child is also an ancestor of x.
- What is the predecessor of y?
- Since y has a left child, it must be the largest element in the tree rooted at $y.left$
- If x is not the largest element in the subtree rooted at $y.left$, then some ancestor of x (in the subtree) is the left child of its parent.
- But y, which is not in this subtree, is the lowest such node.
- Thus x is the predecessor of y, and y is the successor of x.

BST: Successor Examples



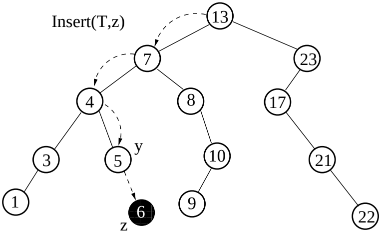


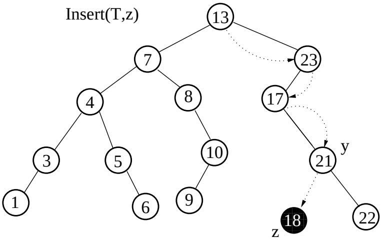
BST: Insertion

- First, search the tree until we find a node whose appropriate child is *null*. Then insert the new node there.
- Below, T is the tree, and z the node we wish to insert.

```
Insert(T,z) {
    node y=null;
    x=T.root;
    while(x!=null) {
        y=x;
        if(z.key<x.key)
          x=x.left;
        else
          x=x.right;
    }
    z.parent=y;
    if(y==null)
        T.root=z;
    else
        if(z.key<y.key)
           y.left=z;
        else
           y.right=z;
}
```

BST: Insertion Example





BST: Deletion

- Deleting a node z is by far the most difficult operation.
- There are 3 cases to consider:
  - If z has no children, just delete it.
  - If z has one child, splice out z. That is, link z's parent and child.
  - If z has two children, splice out z's successor y, and replace the contents of z with the contents of y.
- The last case works because if z has 2 children, then its successor has no left child. Why?
- Deletion is made worse by the fact that we have to worry about boundary conditions
- To simplify things, we will first define a function called **SpliceOut**.

BST: Splice Out

- Any node with at most 1 child can be “spliced out”.
- Splicing out a node involves linking the parent and child of a node.

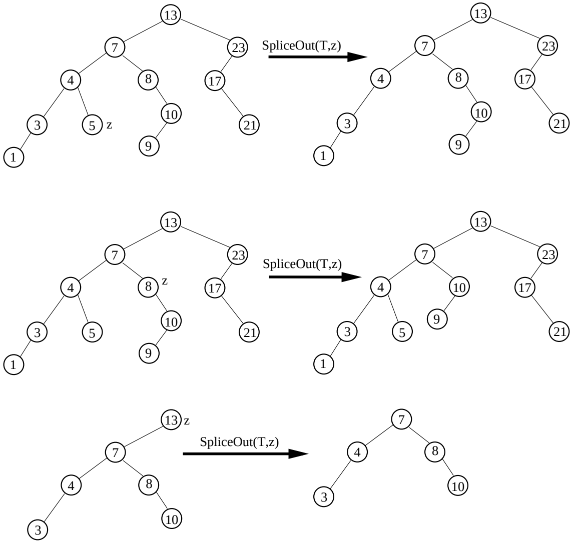
```
SpliceOut(T,y) {
   //Two children--can’t splice out.
   if(y.left!=null && y.right!=null) return;

   if(y.left!=null)          x=y.left;
   else if (y.right!=null)   x=y.right;
   else                       x=null;

   if(x!=null)                x.parent=y.parent;

   //Set y’s parent’s child to y’s child
   if(y.parent==null)  x=T.root;
   else {
      if(y==y.parent.left)  y.parent.left=x;
      else                    y.parent.right=x;
   }
}
```

BST: SpliceOut Examples

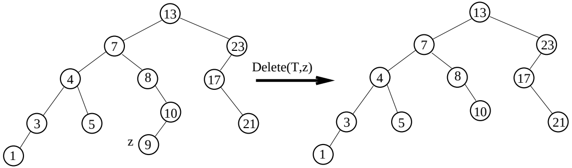


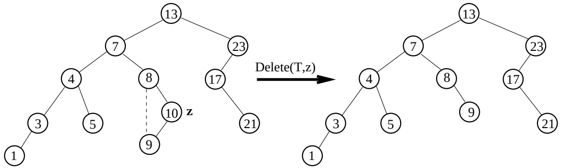
BST: Deletion Algorithm

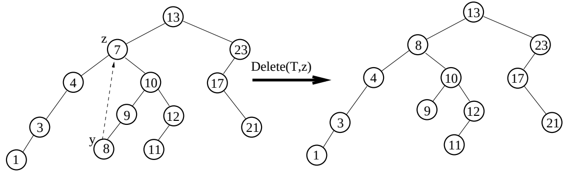
- Once we have defined the function SpliceOut, deletion looks simple.
- Here is the algorithm to delete z from tree T.

```
Delete(T,z) {
    if(z.left==null || z.right==null)
         SpliceOut(T,z);
    else {
         y=Successor(z);
         z.key=y.key;
         SpliceOut(T,y);
    }
}
```

BST: Deletion Examples





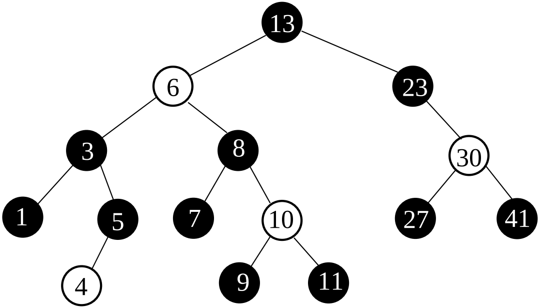


BST: Time Complexity

- We stated earlier, and have now seen, that all of the BST operations have time complexity $O(h)$, where h is the height of the tree.
- However, in the worst-case, the height of a BST is $O(n)$, where n is the number of nodes.
- In this case, the BST has gained us nothing.
- To prevent this worst-case behavior, we need to develop a method which ensures that the height of a BST is kept to a minimum.
- **Red-Black Trees** are binary search trees which have height $\Theta(\log n)$.

Red-Black Trees

- A **red-black tree** is a BST with the following properties:
  - Each node is colored either **red** or **black**.
  - If a node is red, both its children are black.
  - Every simple path from a node to a descendent leaf has the same number of black nodes.

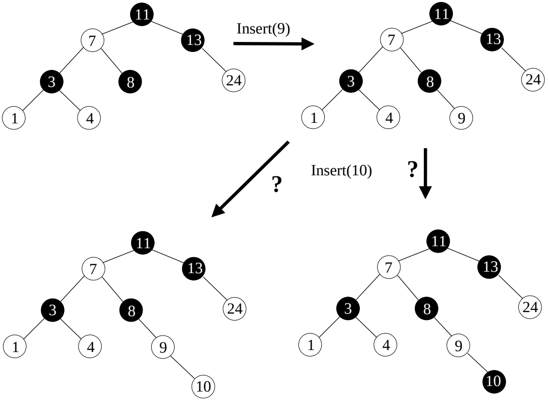


Red-Black Trees Fact and Terms

- The **black-height** of a node x is the number of black nodes, not including x, on a path to any leaf.
- A red-black tree with n nodes has height at most $2\log(n+1)$.
- Since **red-black trees** are binary search trees, all of the operations that can be performed on binary search trees can be performed on them.
- Furthermore, the time complexity will be the same–$O(h)$–where h is the height.
- Unfortunately, insertion and deletion as defined for regular binary search trees will not work for red-black trees. Why not?
- Fortunately, insertion and deletion can both be modified so that they work, and still have time complexity $O(h)$.

Insert and Delete in RB Trees

- Inserting a node into a red-black tree is not trivial.



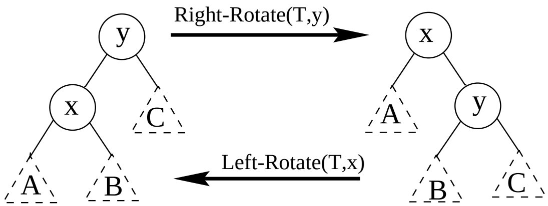
- Similar things happen when we try to delete nodes.
- We will not discuss in depth these operations.
- We will discuss some of the concepts, however.

Red-Black Tree Insertion: Method

- To insert a node x into a red-black tree, we do the following:
  - Insert x with the standard BST *Insert*.
  - Color x red.
  - If x's parent is red, fix the tree.
- Notice that x's children, *null*, are black.
- Since we colored x red, we have not changed the black height.
- The only problem we have is (possibly) having a red node with a red child.
- Fixing the tree involves re-coloring some of the nodes and performing rotations.

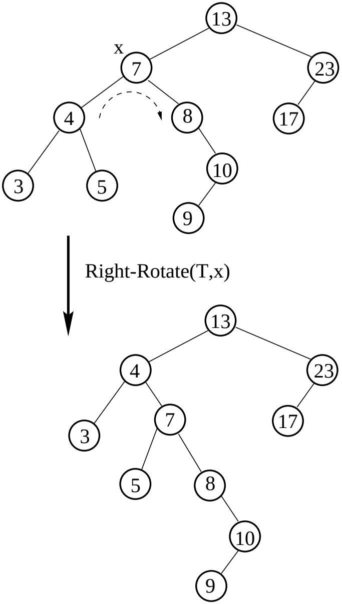
Left- and Right-Rotations

- Rotations are best defined by an illustration:

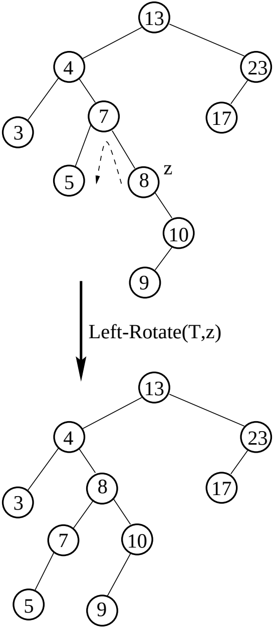


- Here, the letters *A*, *B*, and *C* represent arbitrary subtrees. They could even be empty.
- It is not too hard to see that the binary search tree property will still hold after a rotation.

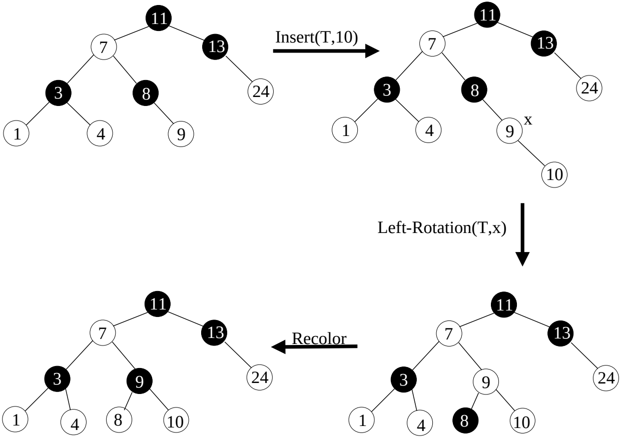
Rotation Example



Rotation Example



Insertion Example



Red Black Tree Summary

- Red-black trees are binary search trees which have height $\Theta(\log n)$ guaranteed.
- The basic operations can all be implemented in time $O(\log n)$.
- Although inserting and deleting nodes only requires time $O(\log n)$, they are nonetheless not trivial to implement.
- A regular binary search tree does not guarantee time complexity of $O(\log n)$, only $O(h)$, where h can be as large as n.
- Thus red-black trees are useful if one wants to guarantee that the basic operations will take $O(\log n)$ time.