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Why are the eigenvalues of a covariance matrix equal to the variance of its eigenvectors?

AlexMayleAlexMayle 10811 silver badge77 bronze badges

2-3 minutes

This assertion came up in a Deep Learning course I am taking. I understand intuitively that the eigenvector with the largest eigenvalue will be the direction in which the most variance occurs. I understand why we use the covariance matrix's eigenvectors for Principal Component Analysis.

However, I do not get why the eigenvectors' variance are equal to their respective eigenvalues. I would prefer a formal proof, but an intuitive explanation may be acceptable.

(Note: this is not a duplicate of this question.)

asked Feb 16 '17 at 14:07

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Here's a formal proof: suppose that v denotes a length-1 eigenvector of the covariance matrix, which is defined by

$$\Sigma = \mathbb{E}[XX^T]$$

Where $X = (X_1, X_2, \dots, X_n)$ is a column-vector of random variables with mean zero (which is to say that we've already absorbed the mean into the variable's definition). So, we have $\Sigma v = \lambda v$ (for some $\lambda \geq 0$), and $v^T v = 1$.

Now, what do we really mean by "the variance of v"? v is not a random variable. Really, what we mean is the variance of the associated component of X. That is, we're asking about the variance of $v^T X$ (the dot product of X with v). Note that, since the X_is have mean zero, so does $v^T X$. We then find

$$\mathbb{E}([v^T X]^2) = \mathbb{E}([v^T X][X^T v]) = \mathbb{E}[v^T (XX^T) v] = v^T \mathbb{E}(XX^T) v = v^T \Sigma v = v^T \lambda v = \lambda (v^T v) = \lambda$$

and this is what we wanted to show.

answered Feb 16 '17 at 14:53