Intro to graphs

Minimum Spanning Trees

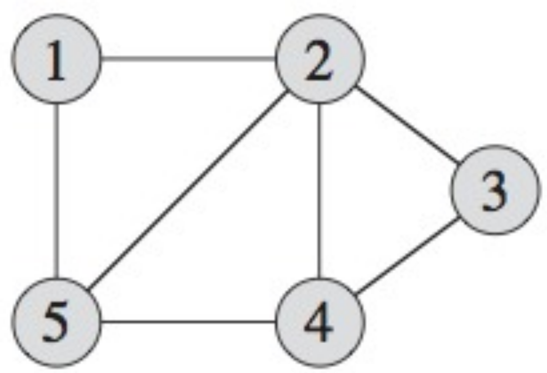
Graphs

- nodes/vertices and edges between vertices
  - set V for vertices, set E for edges
  - we write graph G = (V,E)
- example : cities on a map (nodes) and roads (edges)

Adjacency matrix

stores $|V|^2$

- a_{ij} =1 if there is an edge from vertex i to vertex j
- if graph is undirected, edges go both ways, and the adj. matrix is symmetric



$|V|$

$n = |V|$

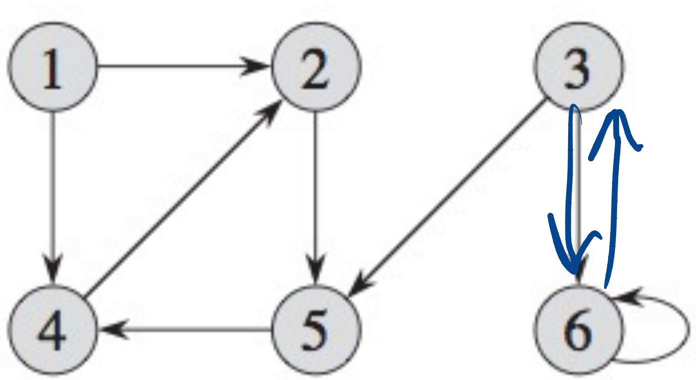
#vertices

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

edge (2→4)

edge (4→2)

- if the graph is directed, the adj. matrix is not necessarily symmetric



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

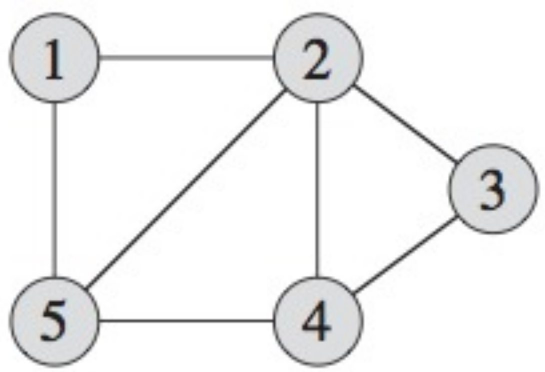
edge 2→5

edge 5→2

missing

Adjacency lists

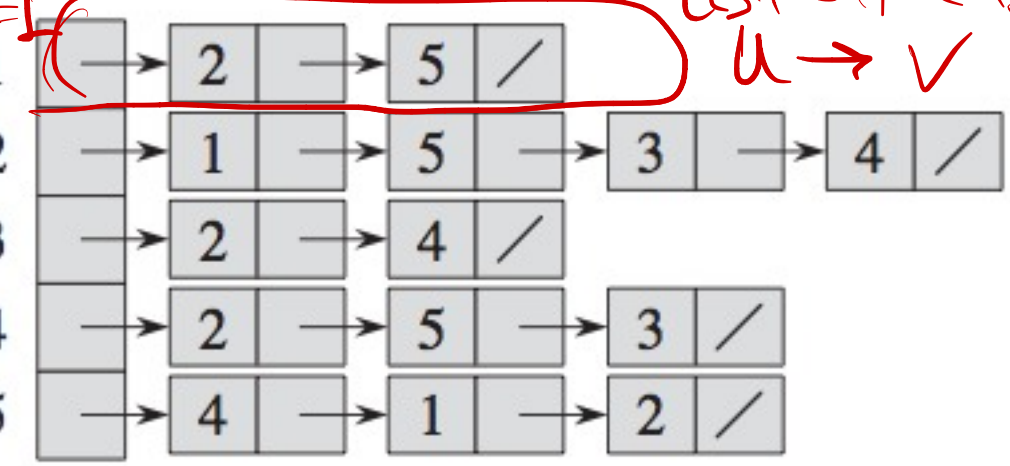
stores |E|



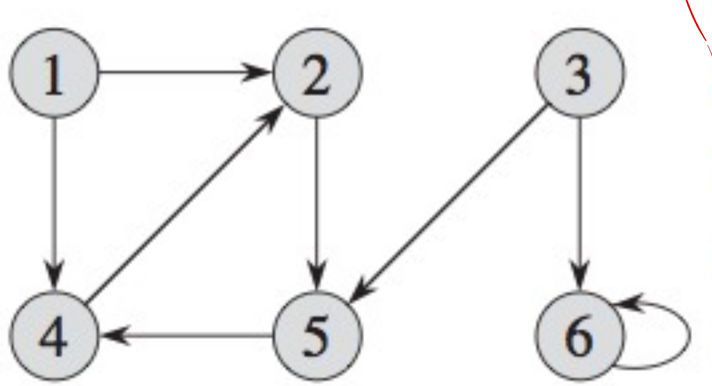
	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

∀v list of edges u → v

u=1



- linked list marks all edges starting off a given vertex



	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
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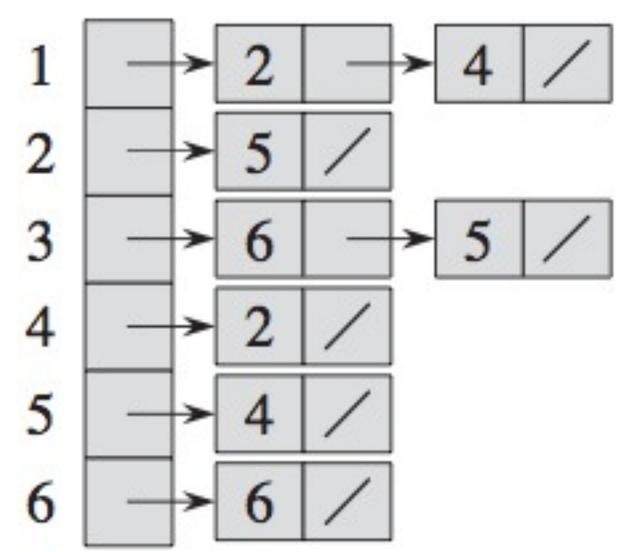
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1



reversed Adj List

1

2 ← 1 ← 4

3

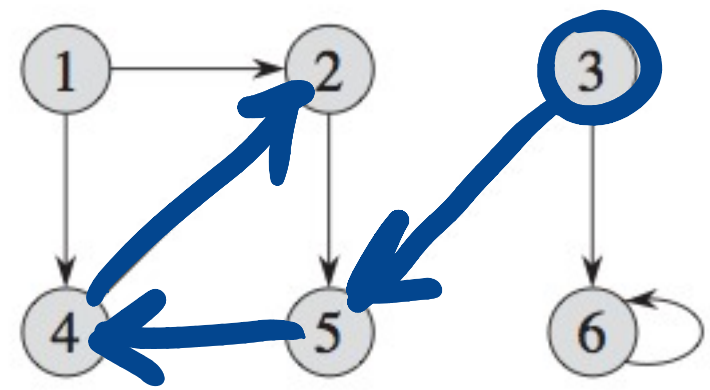
4 ← 1 ← 5

5 ← 2 ← 3

6 ← 3 ← 6

paths and cycles

- path: a sequence of vertices $(v_1,v_2,v_3,\ldots,v_k)$ such that all (v_i,v_{i+1}) are edges in the graph



$i \to j$ edge

$i \rightsquigarrow j$ path

- edges can form a cycle = a path that ends in the same vertex it started

- paths and cycles are defined for both directed and undirected graphs

paths and cycles

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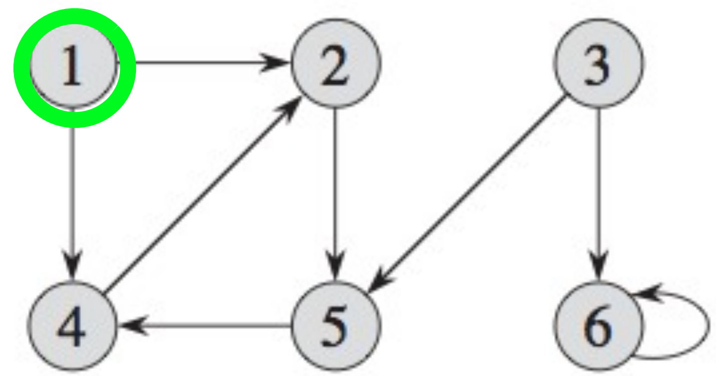

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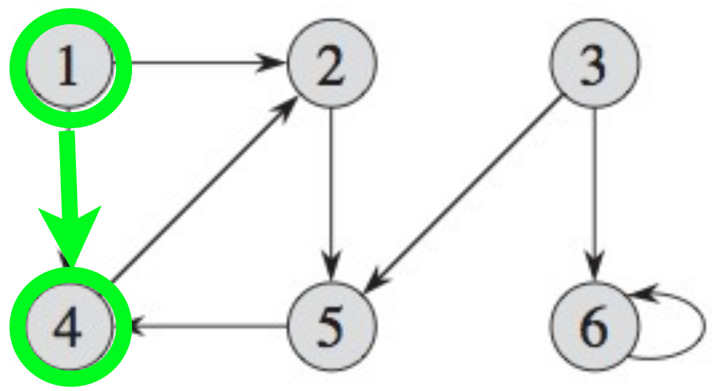

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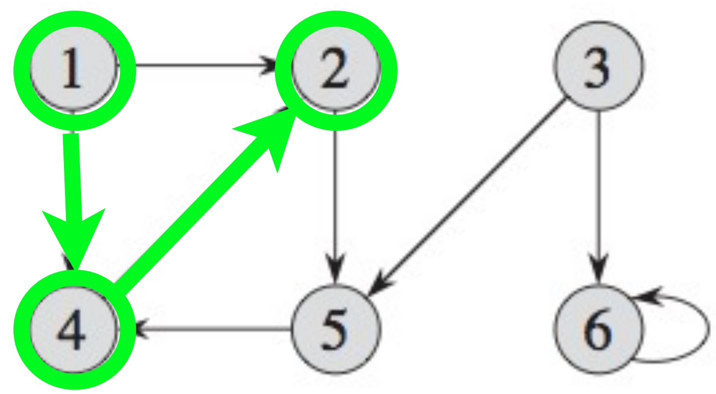

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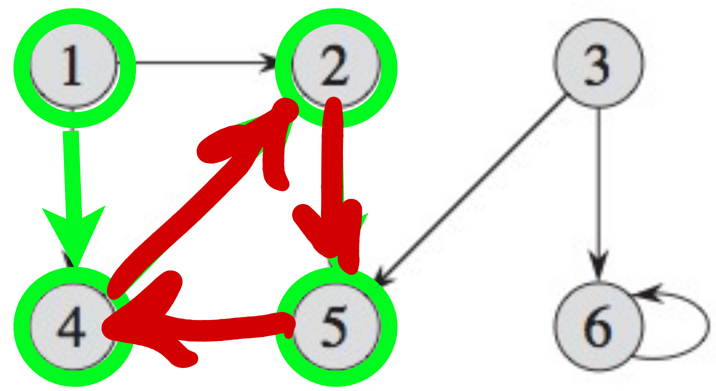

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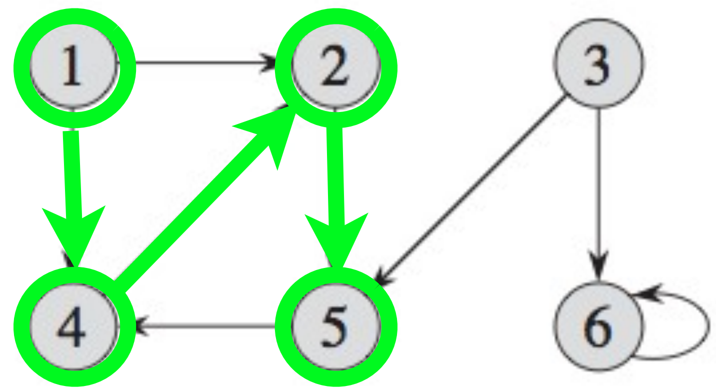

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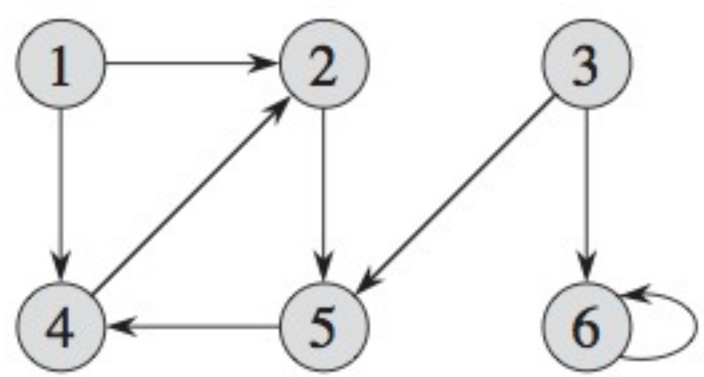

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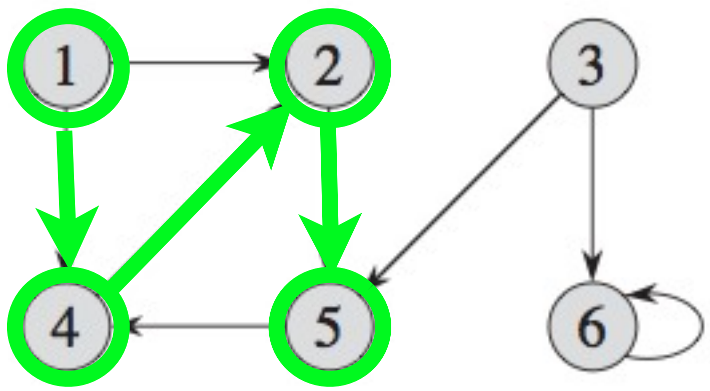

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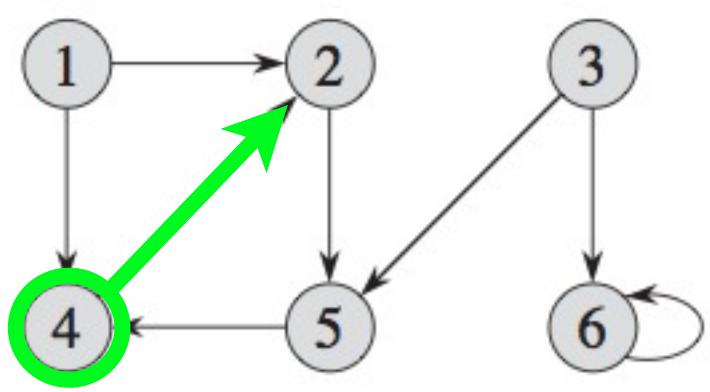

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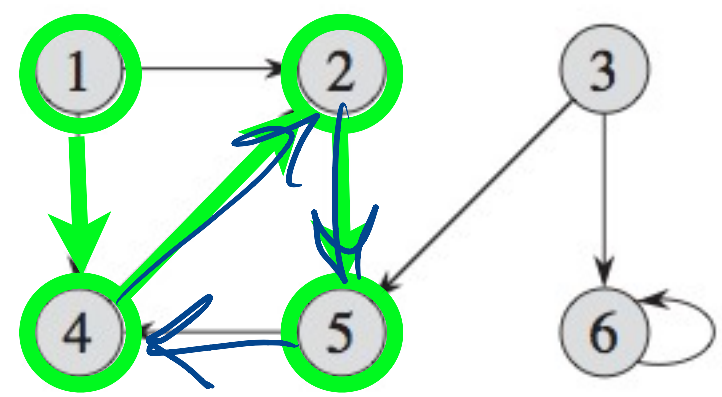

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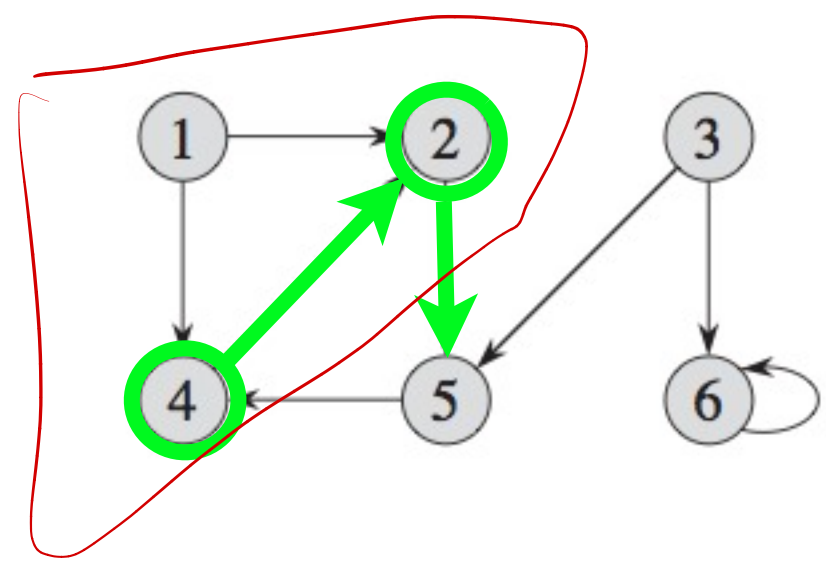

Directed

path, cycles, edges

: must follow edge direction

- paths and cycles are defined for both directed and undirected graphs

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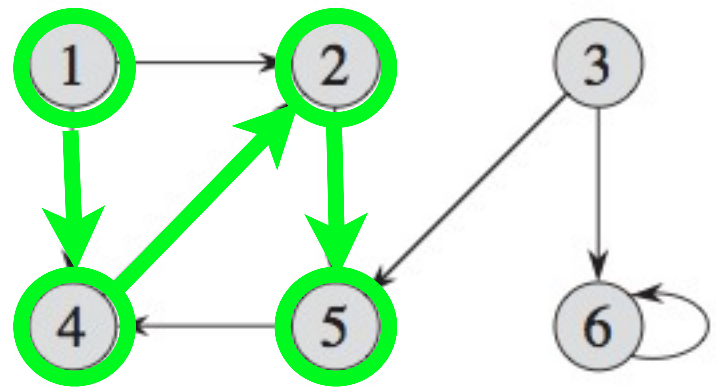

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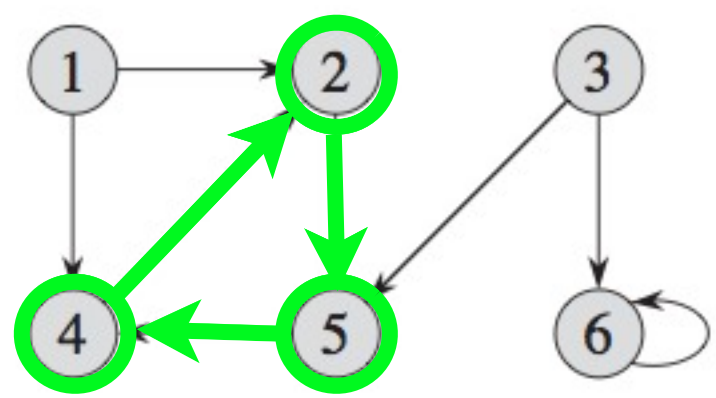

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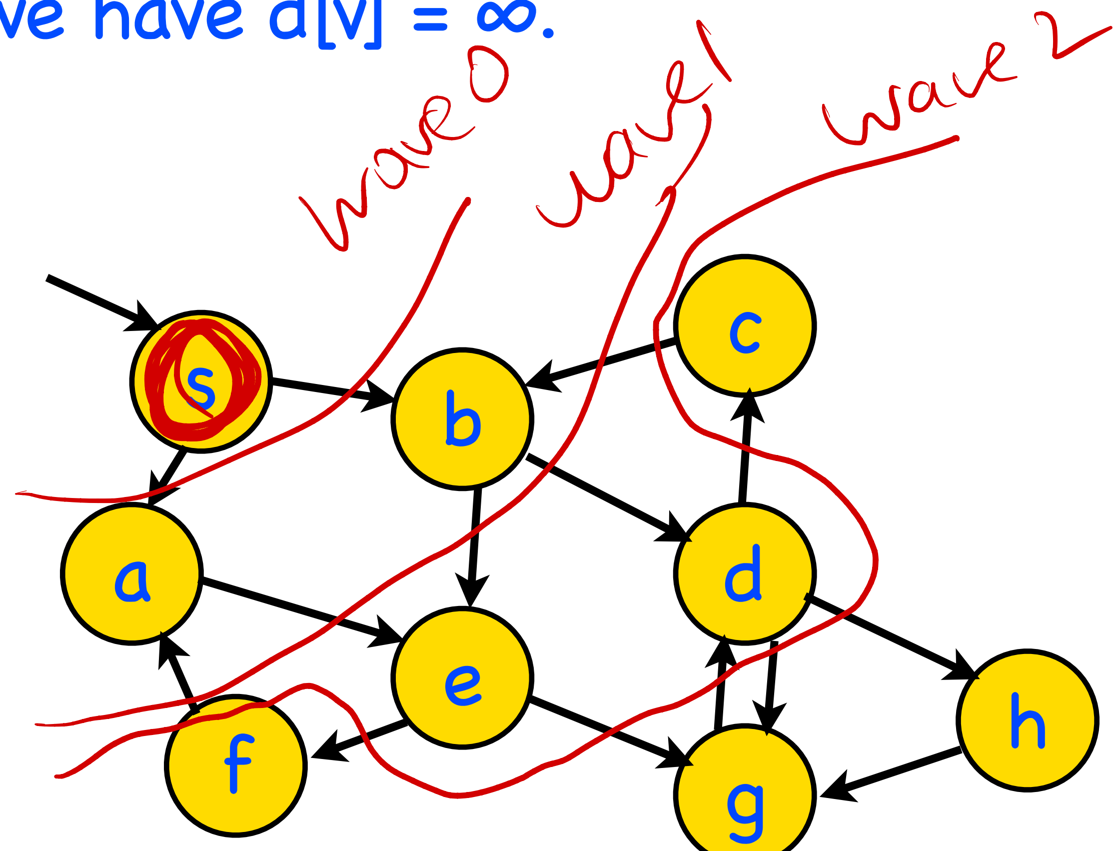
Traverse/search graphs : BFS

- BFS = breadth-first search.
- Start in a given vertex s, find all reachable vertices from s
  - proceed in waves
  - computes d[v] = number of edges from s to v. If v not reachable from s, we have d[v] = ∞.



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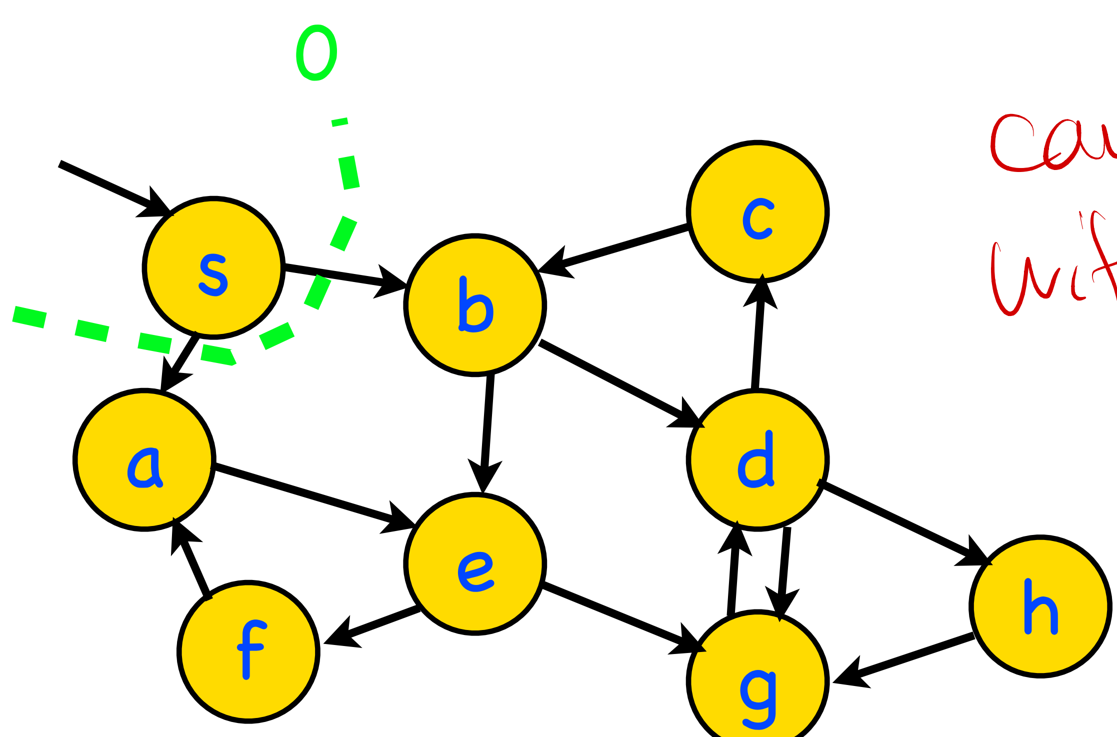

exercise :
can I start BFS with initial queue more than 1 node?

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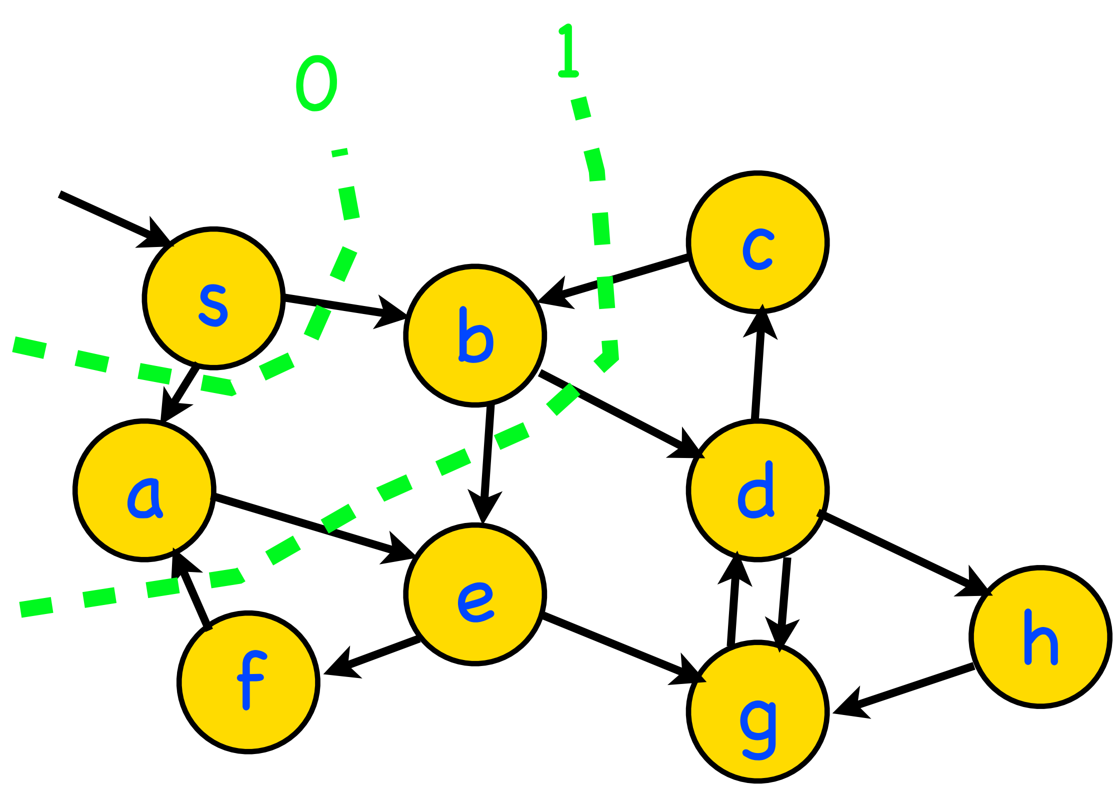

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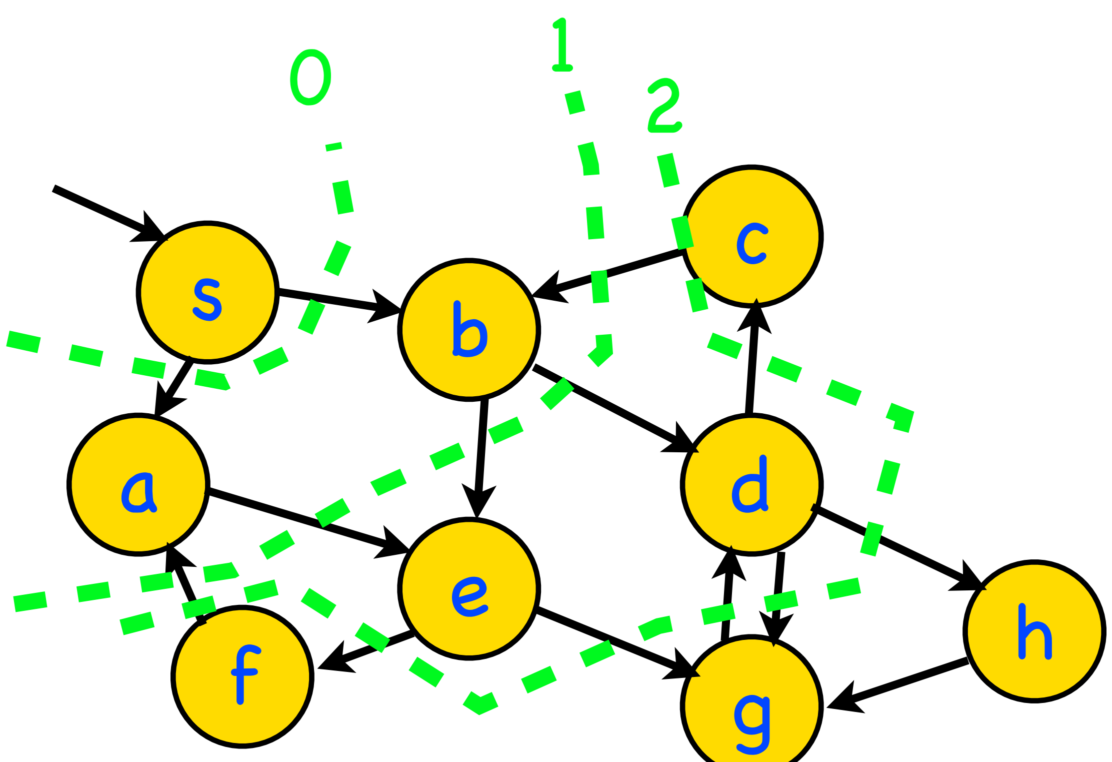

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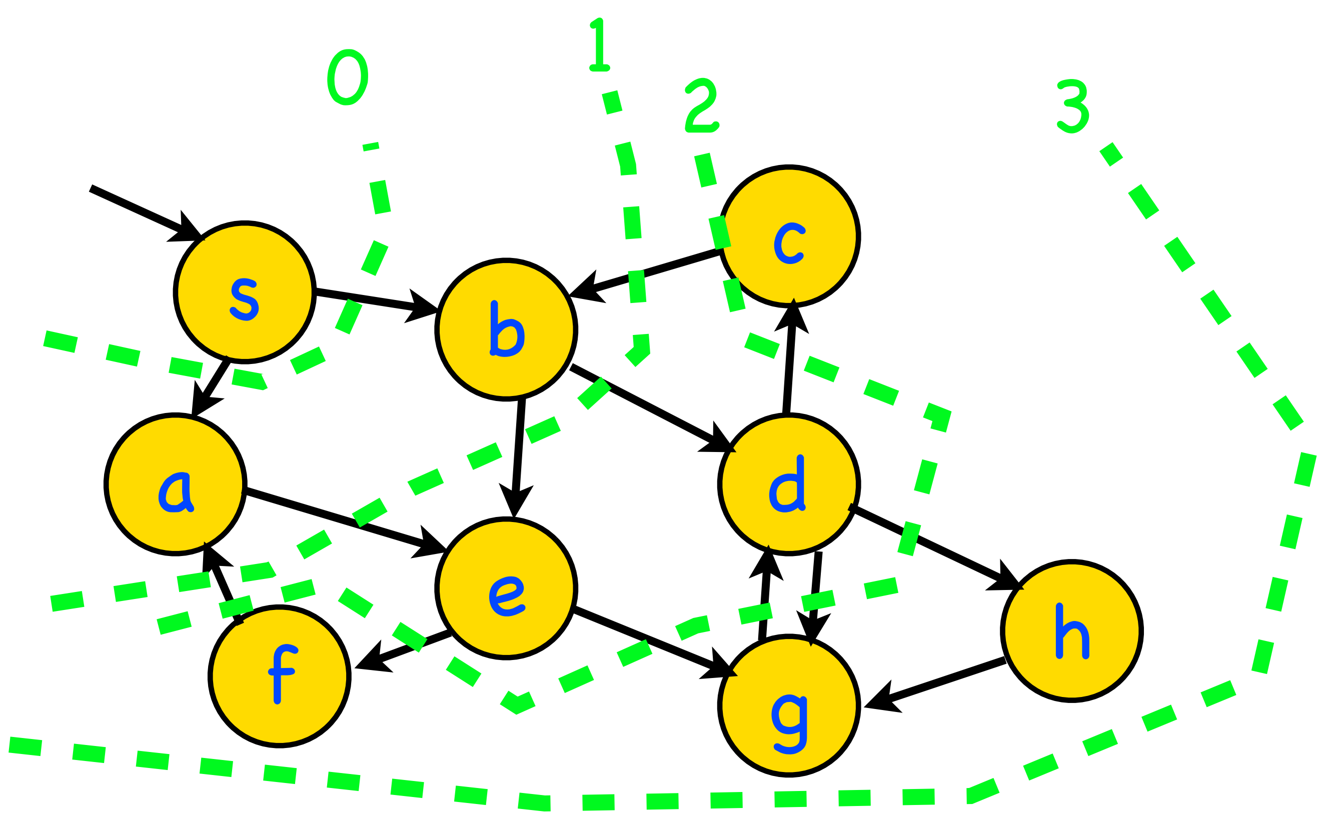


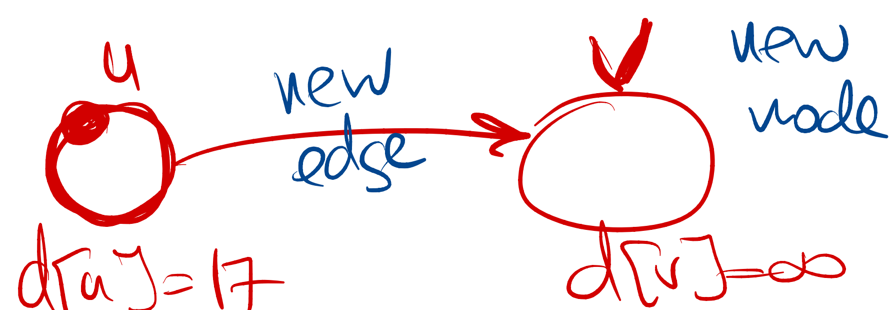
exercise: produce min_hoops(i,j) $\forall i,j$

Obvious: Run BFS(i) $\forall i$ $\Rightarrow O(V+E)\cdot V = O(VE)$

BFS

- use a queue to store processed vertices
  - for each vertex in the queue, follow adj matrix to get vertices of the next wave

```
▸ BFS(V,E,s)
▸ for each vertex v≠s, set d[v]=∞   = wave #
▸ init queue Q; enqueue(Q,s) //puts s in the queue
▸ while Q not empty
    ▸ u = dequeue(S) // takes the first elem available from the queue
    ▸ for each vertex v ∈ Adj[u]
        ▸ if (d[v]==∞) then
            ▸ d[v]=d[u]+1
            ▸ Enqueue(Q,v)
        ▸ end if
    ▸ end for
▸ end while
```



d[v]=18

usually O(E)

- Running time O(V+E) since each edge and vertex is considered once.

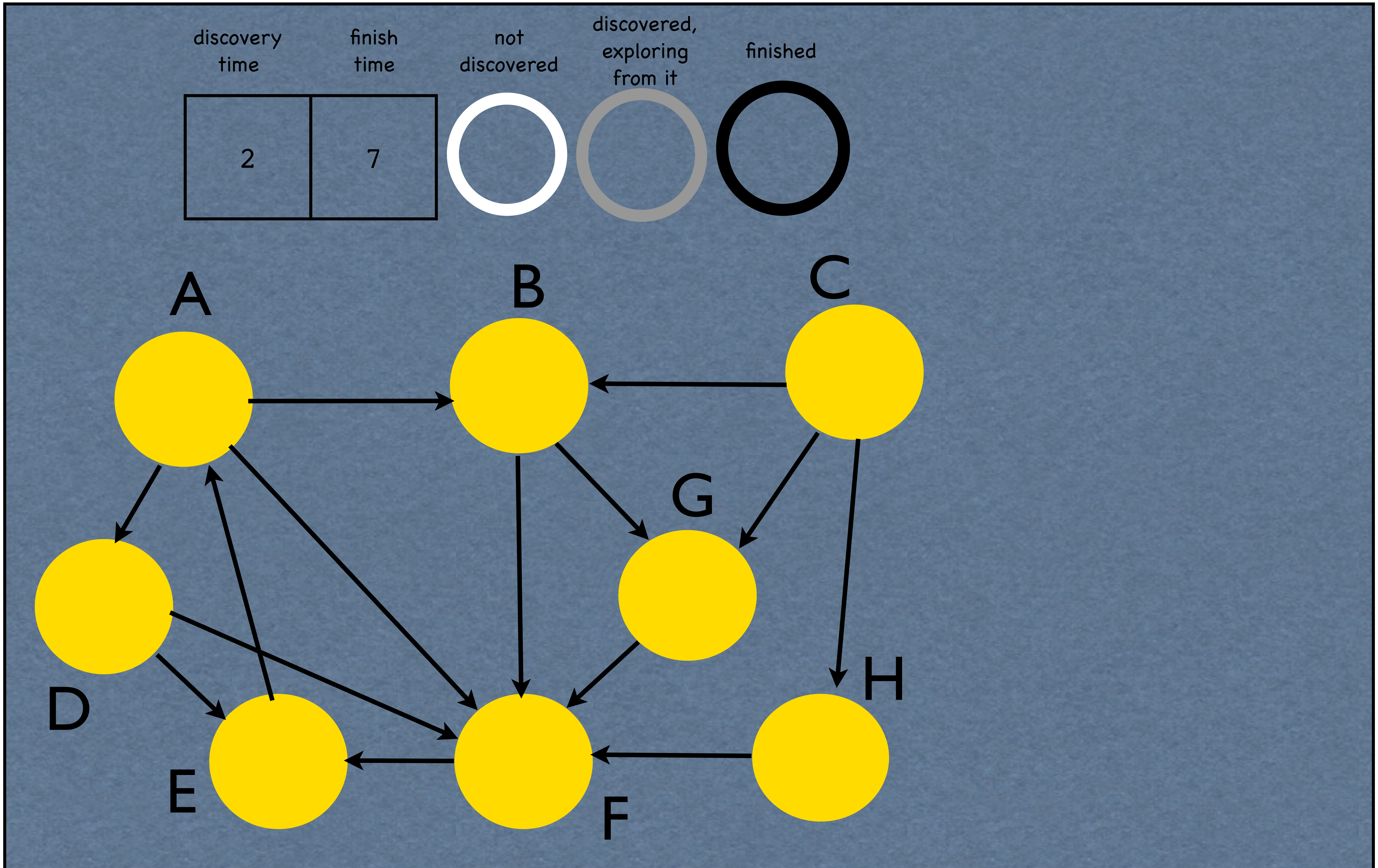
O(max(V,E))

Traverse/search graphs : DFS

- DFS = depth-first search
  - once a vertex is discovered, proceed to its adj vertices, or "children"(depth) rather than to its "brothers" (breadth)

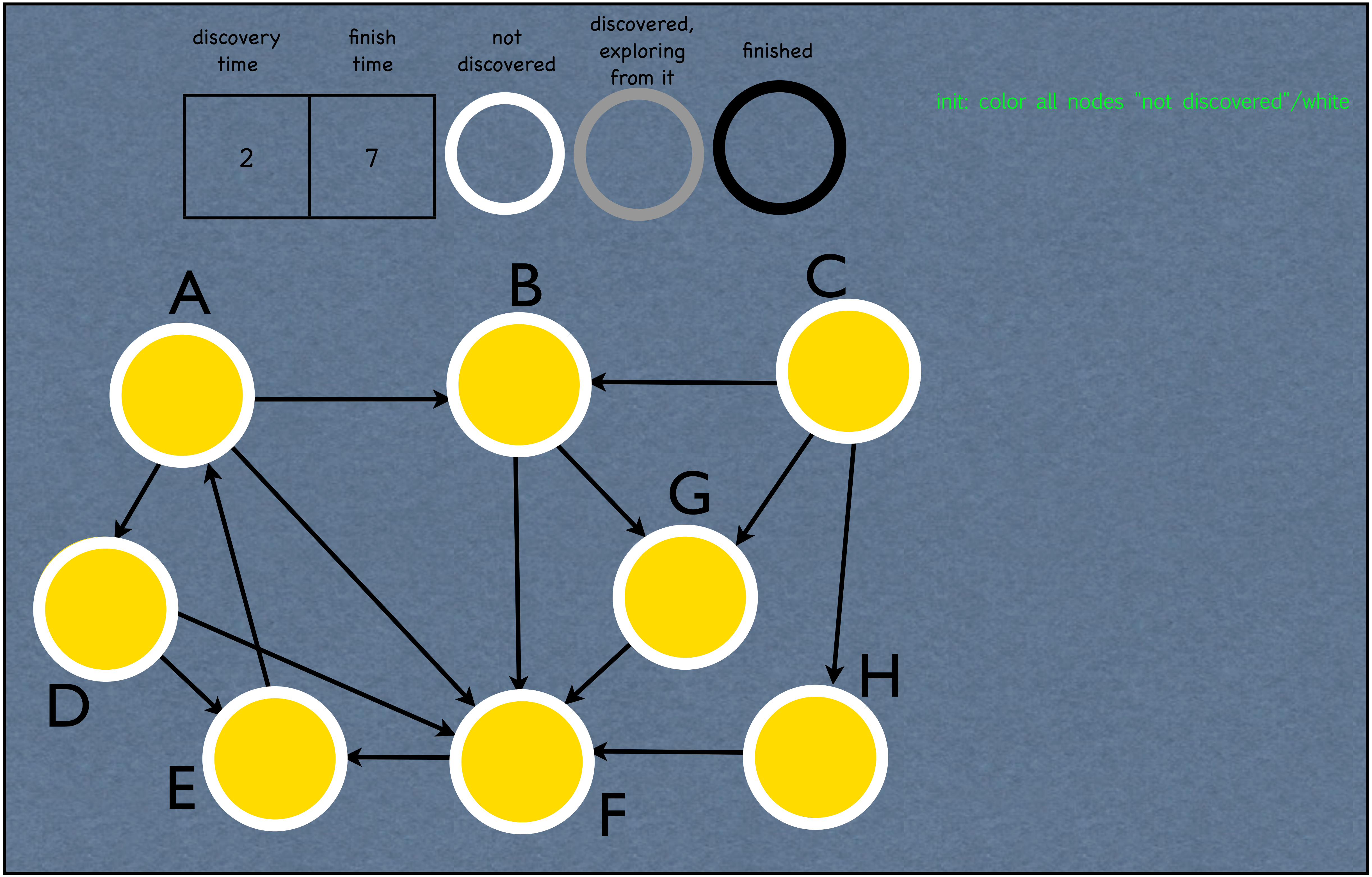
```
DFS-wrapper(V,E)
    foreach vertex u∈V {color[u] = white} end for //color all nodes white
    foreach vertex u∈V
        if (color[u]==white) then DFS-Visit(u)
    end for

DFS-Visit(u) //recursive function
    color[u] = gray; //gray means "exploring from this node"
    time++; discover_time[u] = time;//discover time
    for each v ∈ Adj[u]
        if (color[v]==white) then DFS-Visit(v)//explore from u
    end for
    color [u] = black; finish_time[u]=time; //finish time
```

DFS

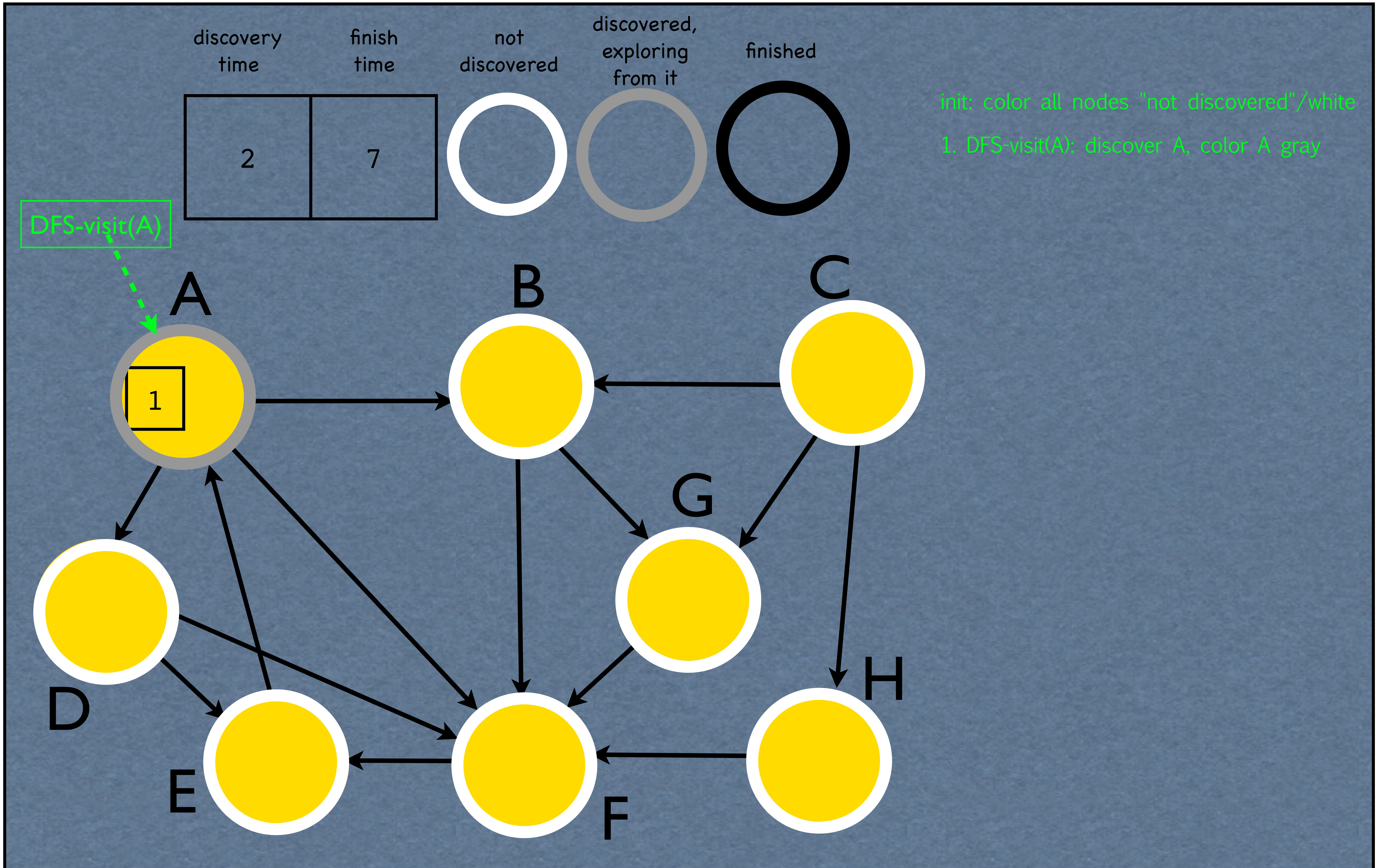


discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
A
B
C
D
E
F
G
H

DFS

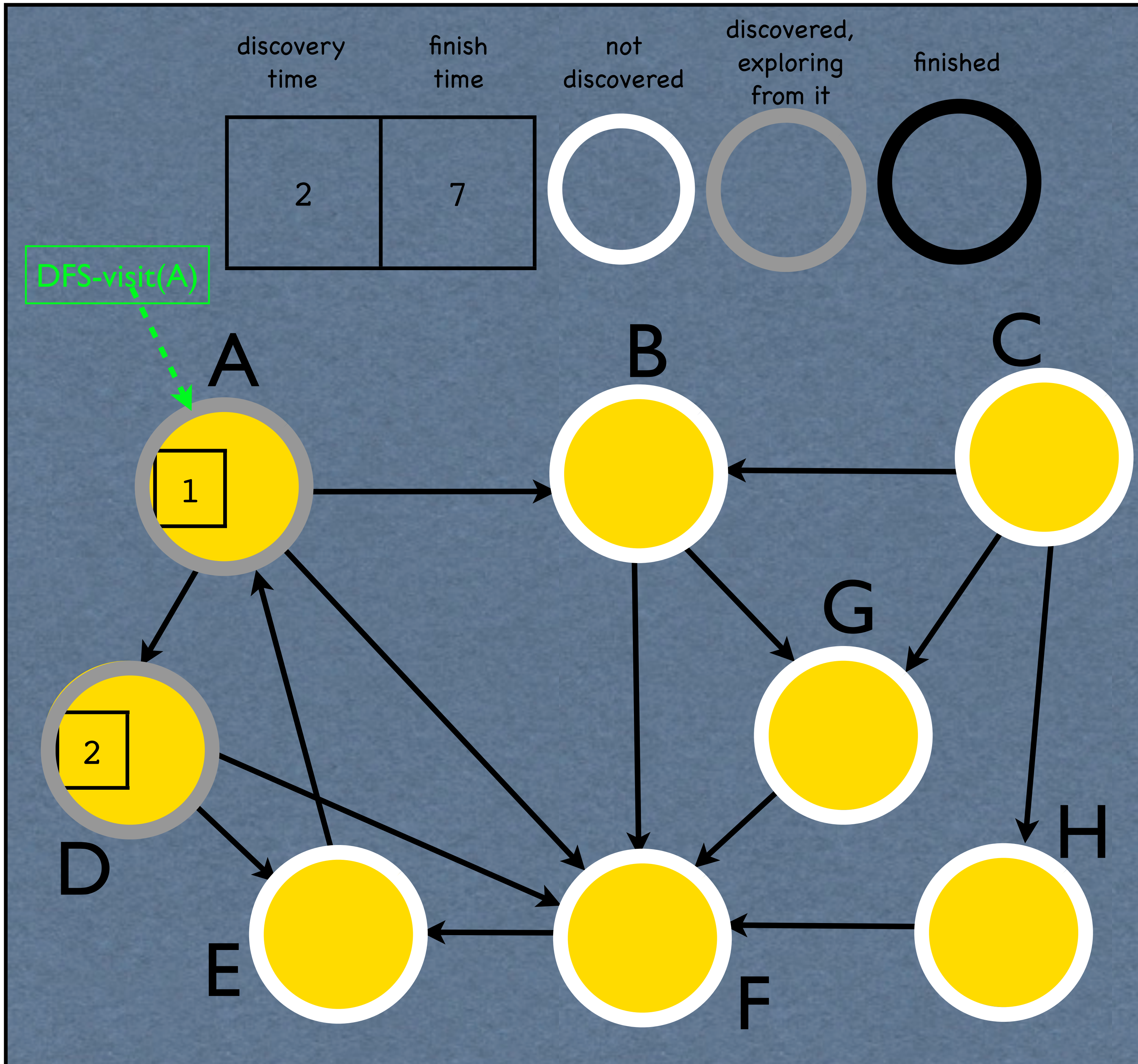

discovery time
finish time
2
7
not discovered
discovered, exploring from it
finished
init: color all nodes "not discovered"/white
A
B
C
D
E
F
G
H

DFS



discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
DFS-visit(A)
A
B
C
D
E
F
G
H
1
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray

DFS

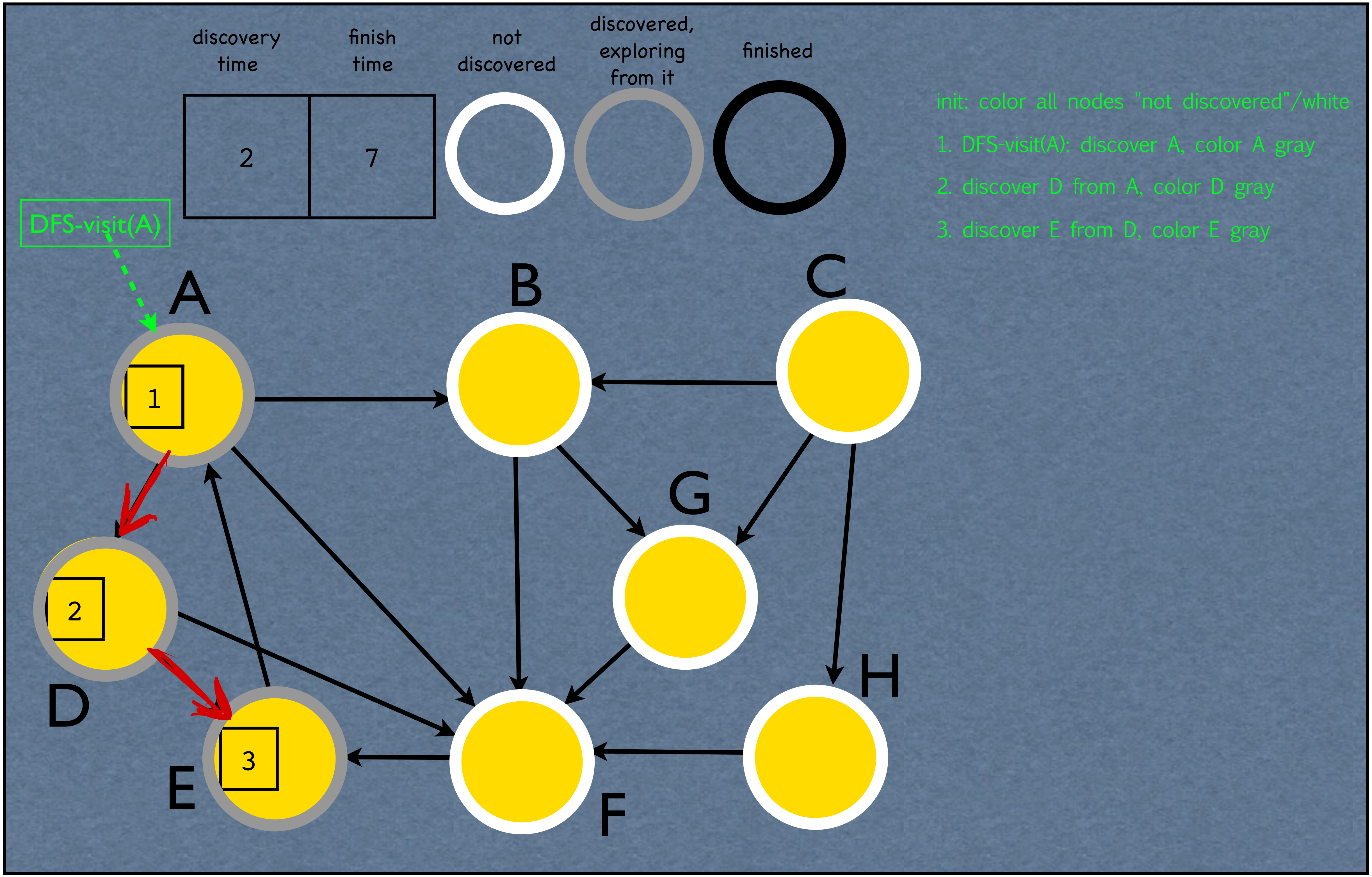


init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

DFS


discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
DFS-visit(A)
A
B
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D
E
F
H
1
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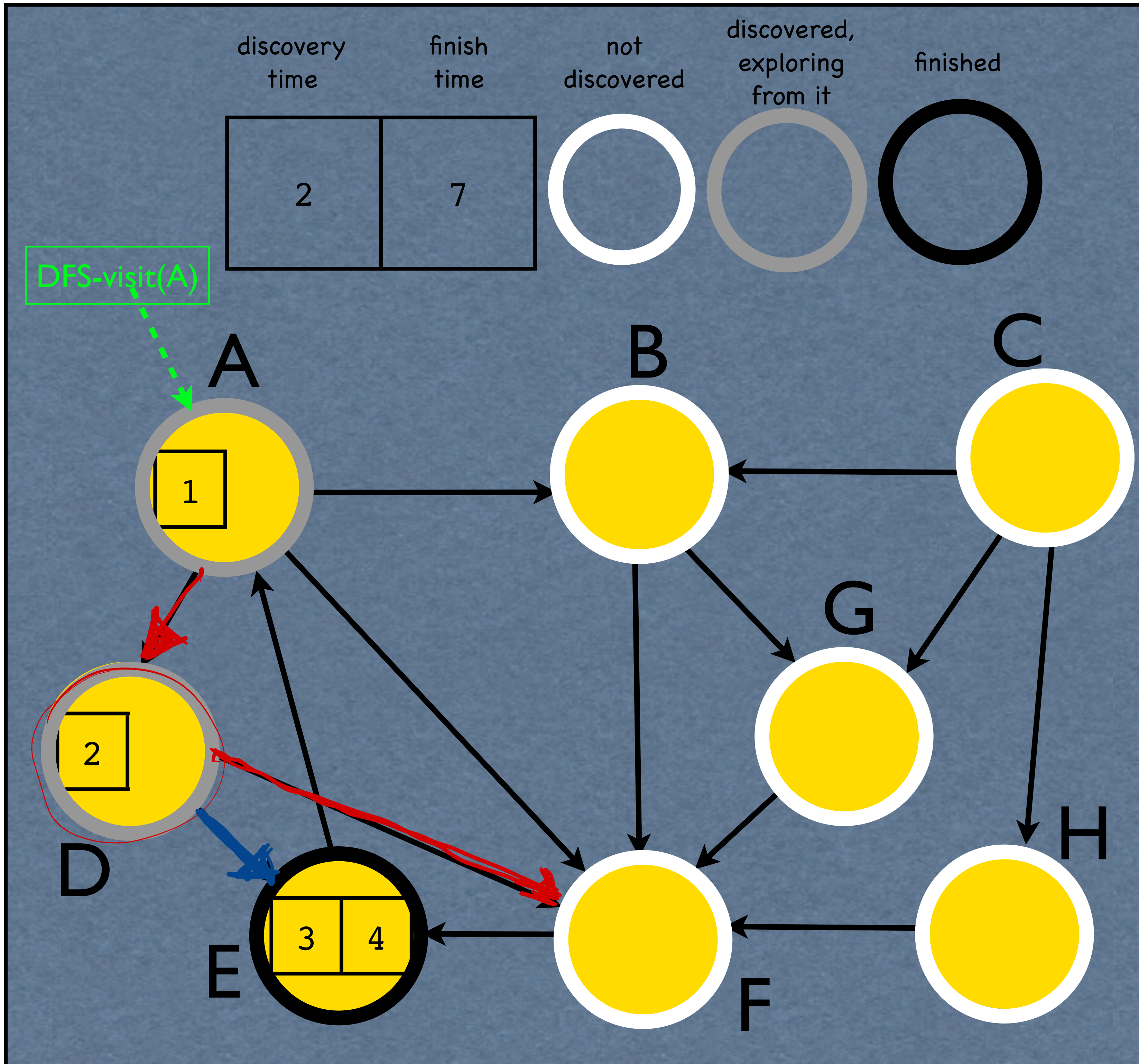
3

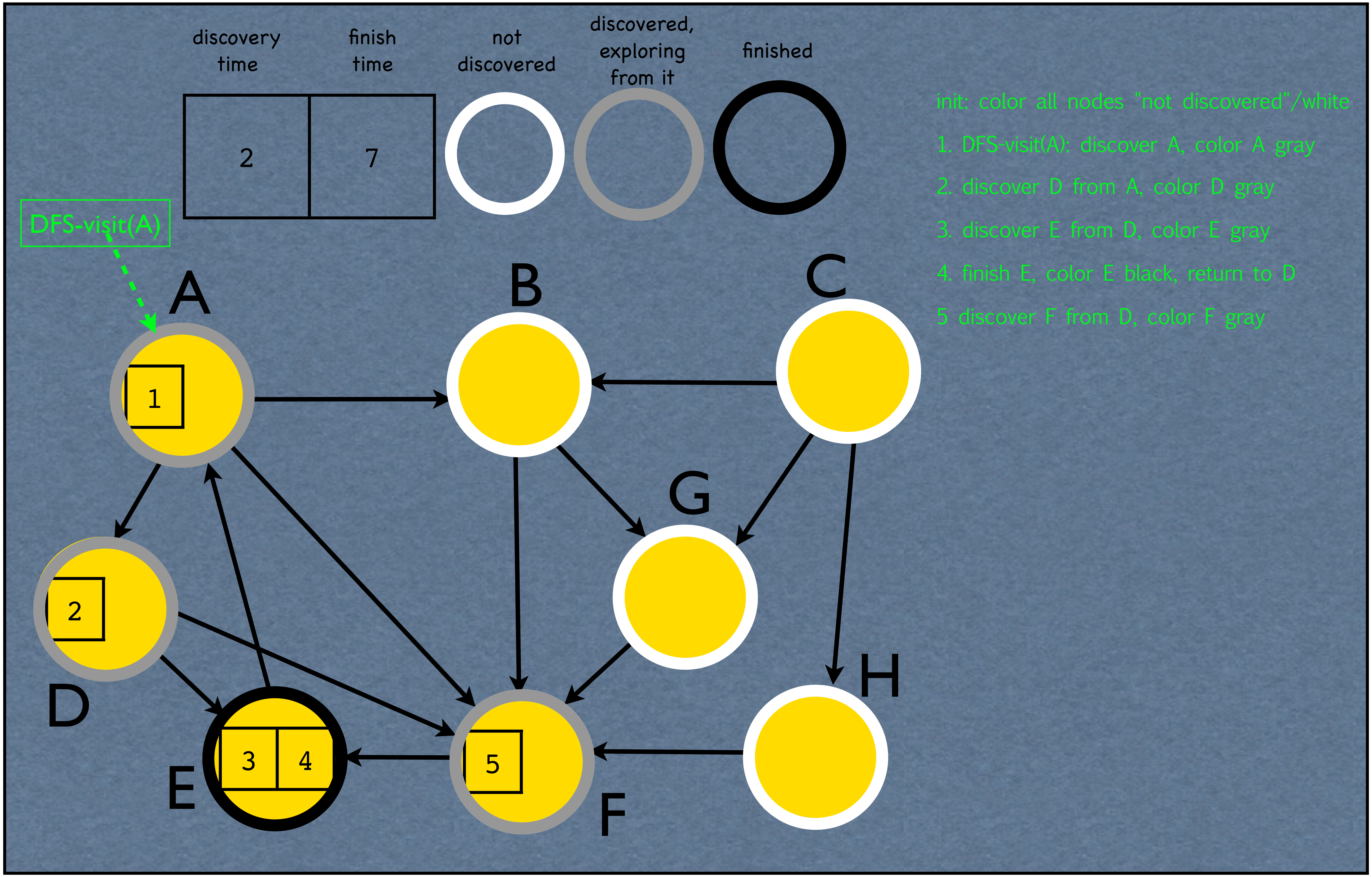
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D

DFS


discovery time
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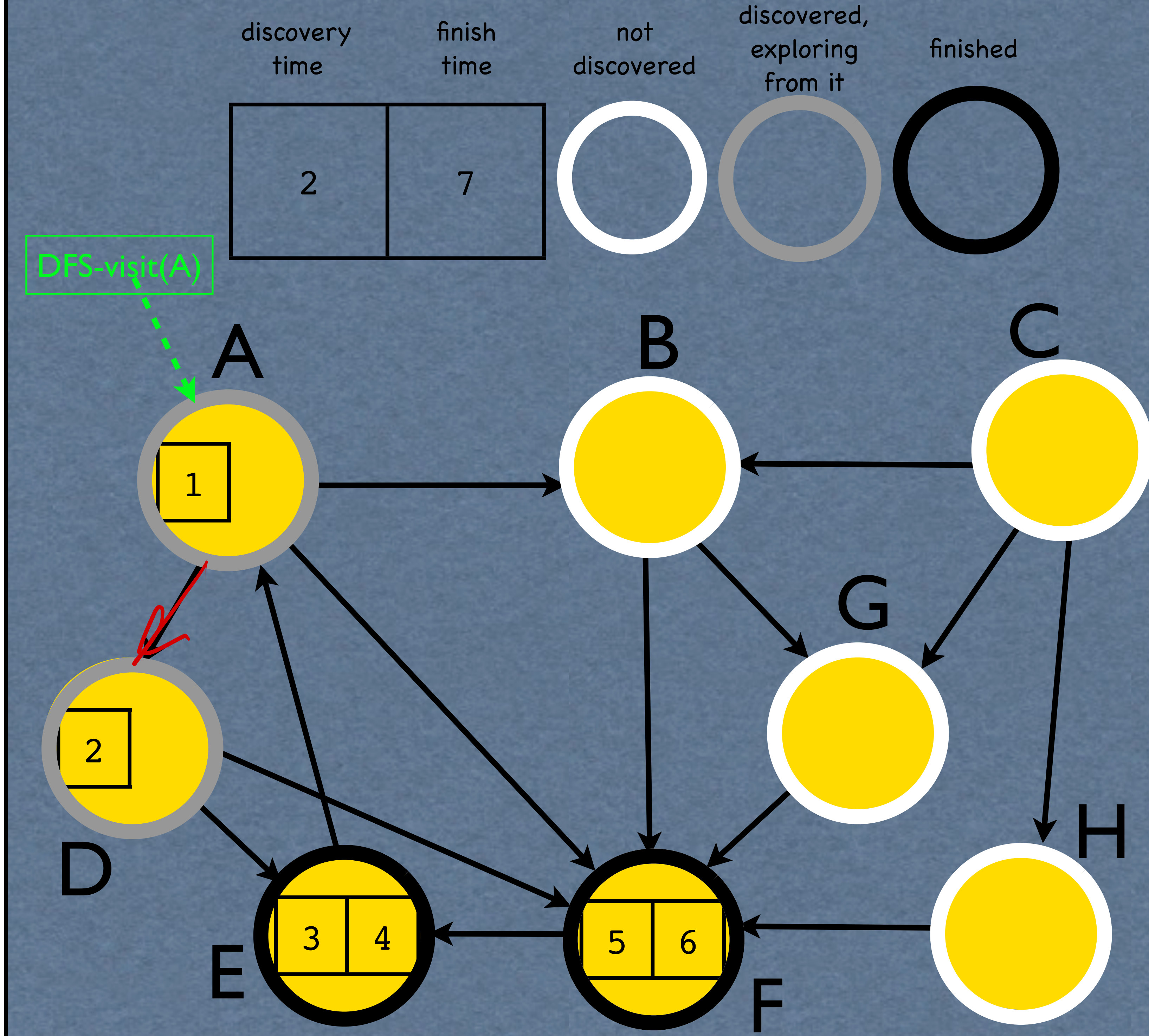
2
7
DFS-visit(A)
A
B
C
D
E
F
G
H
1
2
3
4
5
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray

DFS



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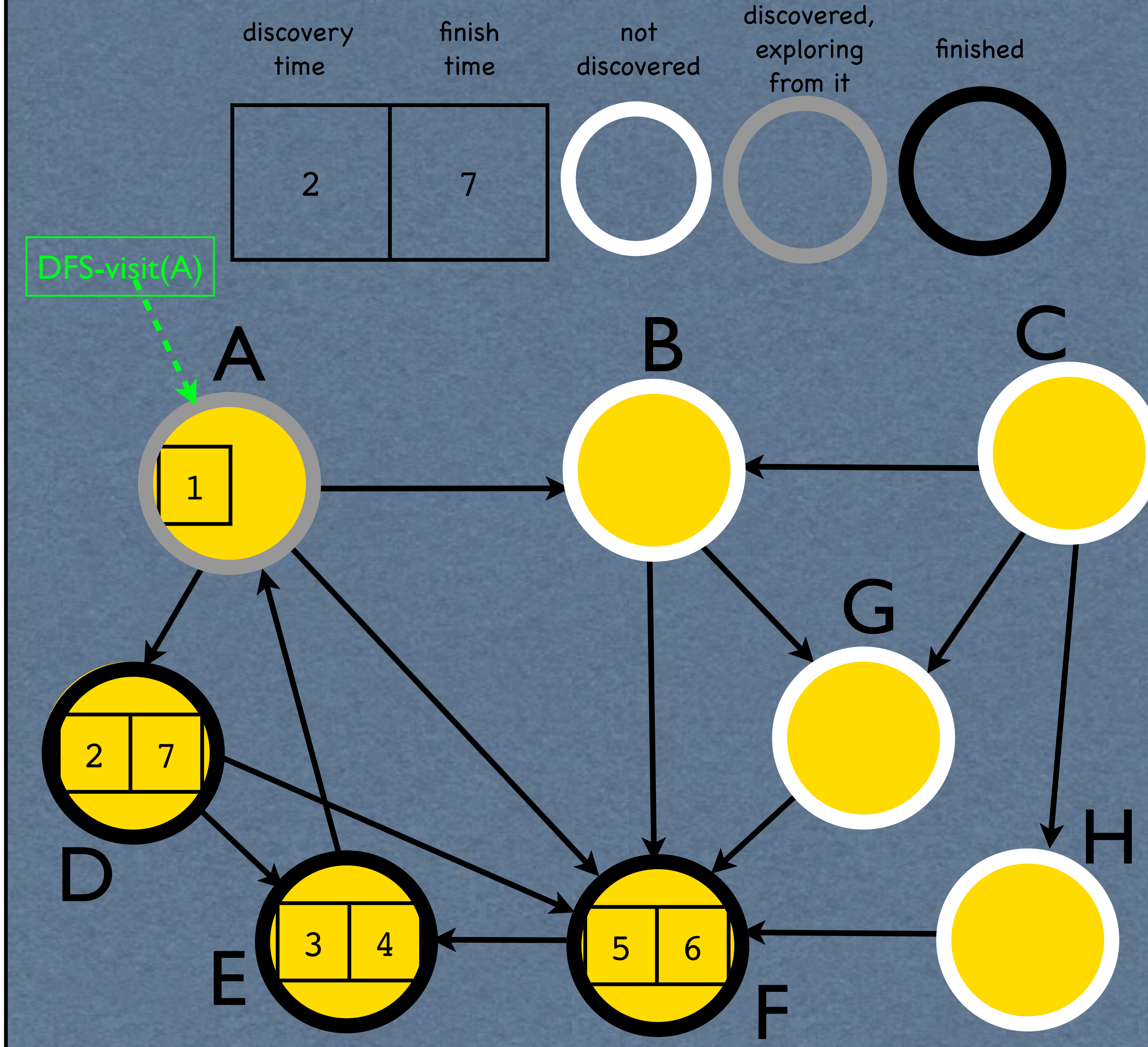
3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

DFS



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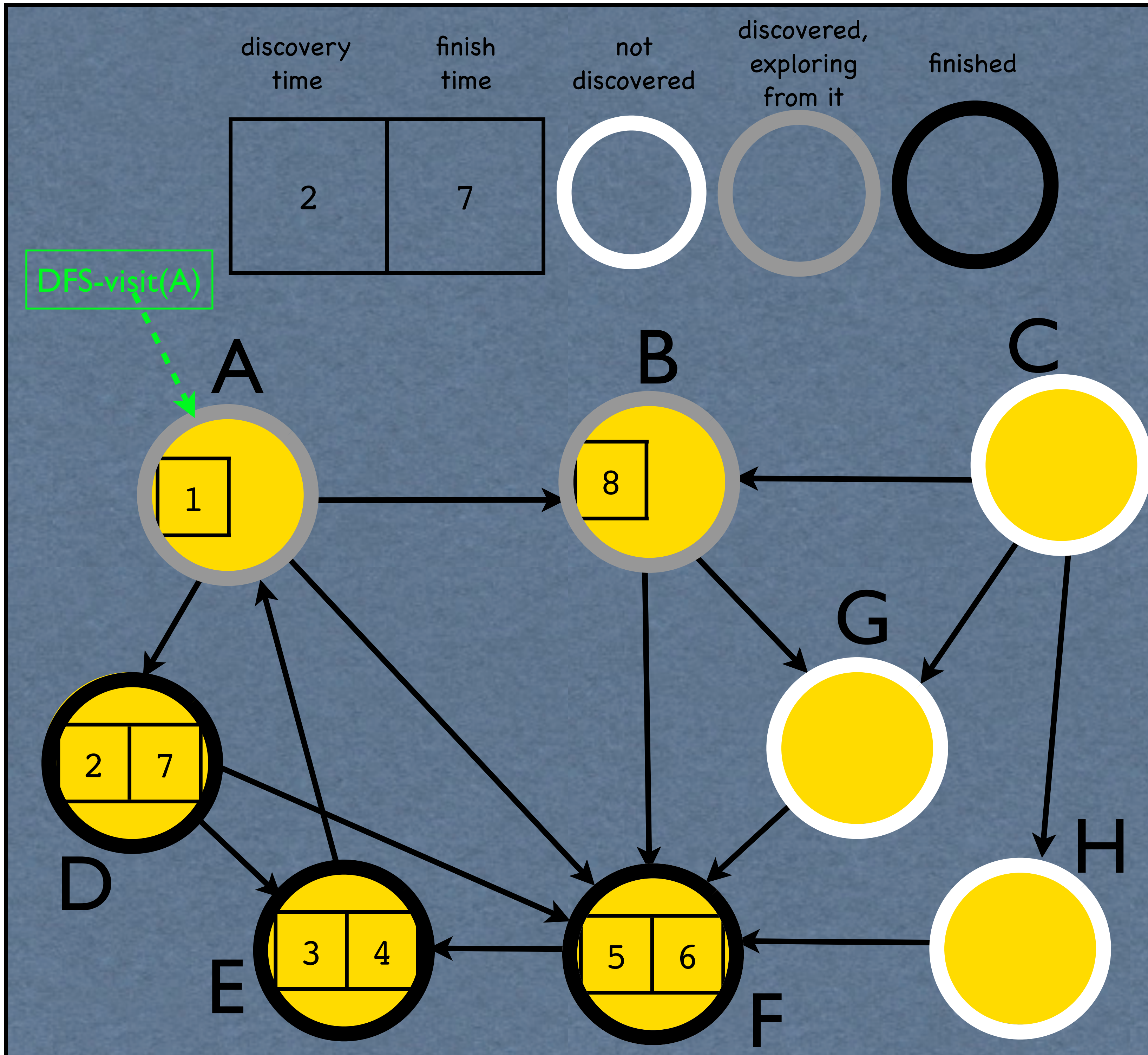
3. discover E from D, color E gray
4. finish E, color E black, return to D
5. discover F from D, color F gray
6. finish F, color F black, return to D
7. finish D, color D black, return to A

DFS



init: color all nodes "not discovered"/white

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2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

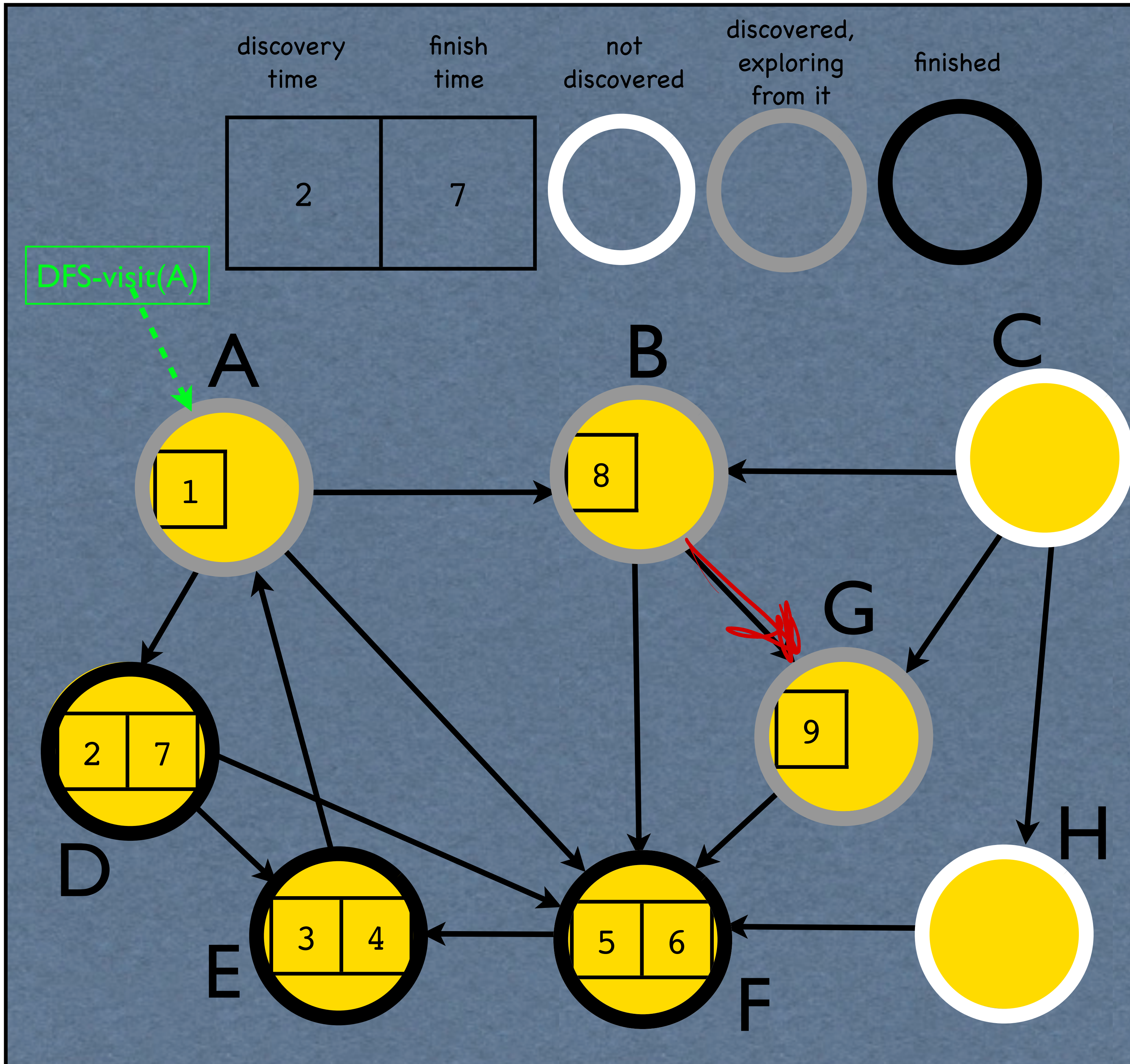
5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

8 discover B from A, color B gray

DFS



init: color all nodes "not discovered"/white

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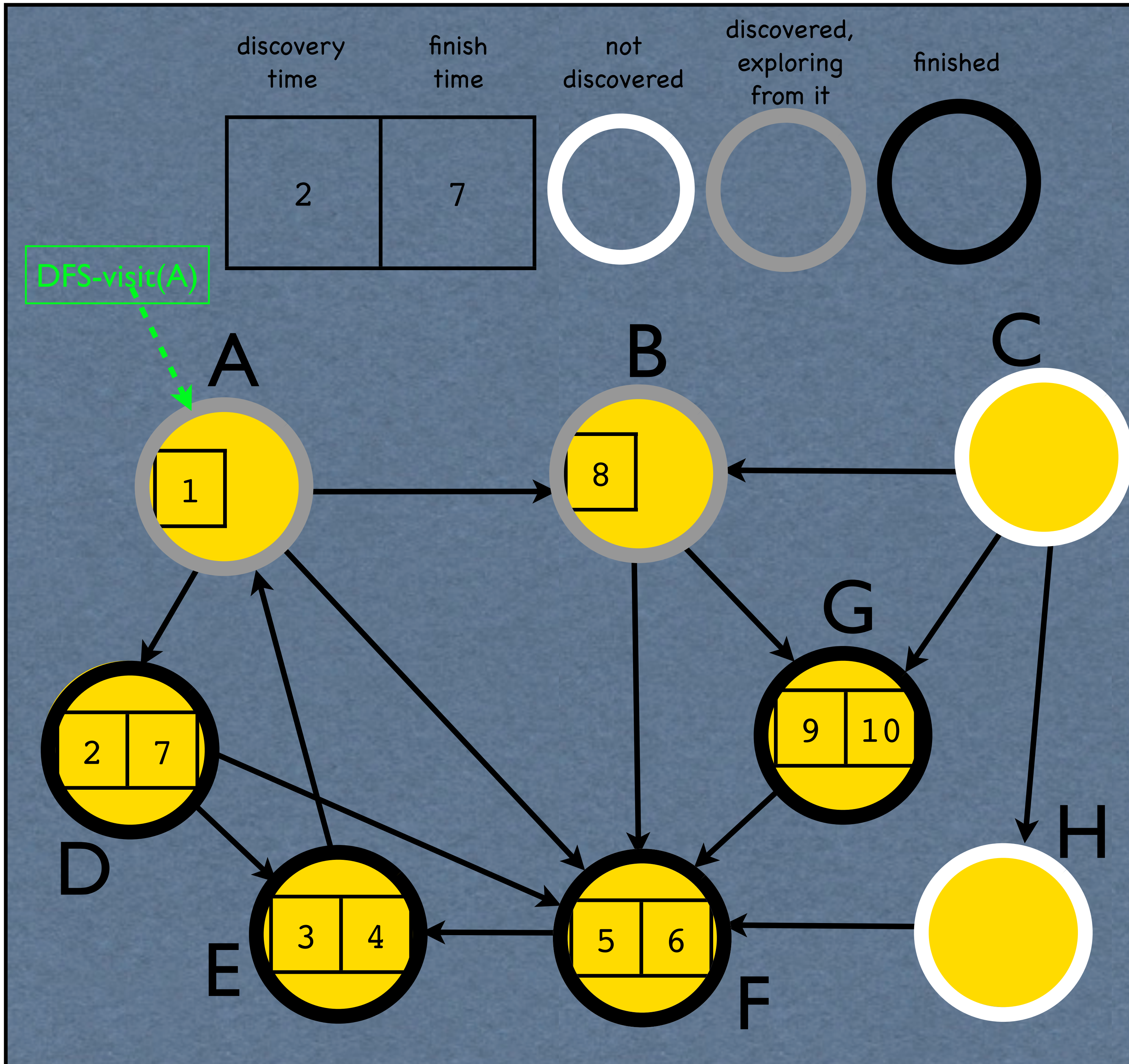
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5. discover F from D, color F gray
6. finish F, color F black, return to D
7. finish D, color D black, return to A
8. discover B from A, color B gray
9. discover G from B, color G gray

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

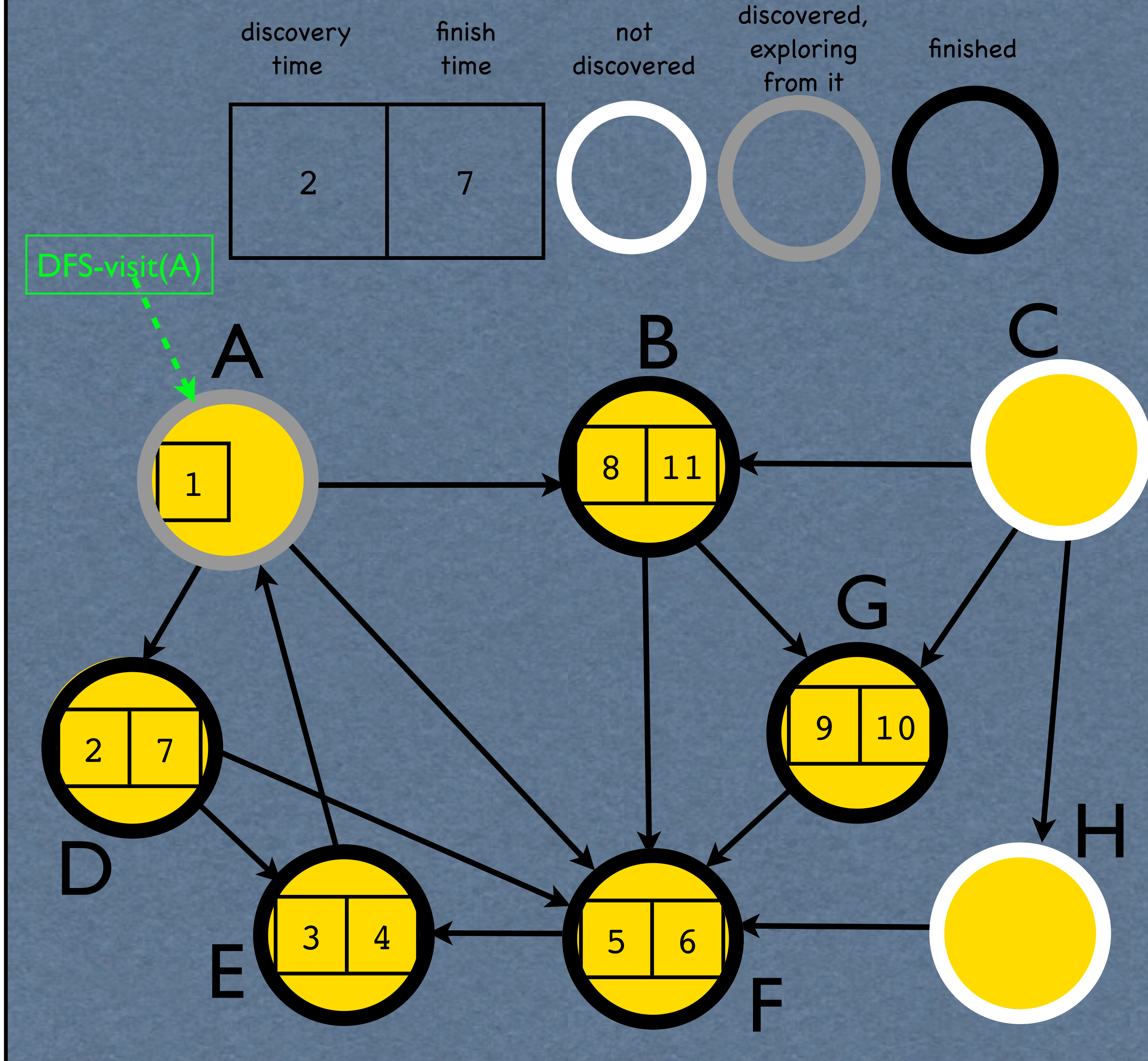
7 finish D, color D black, return to A

8 discover B from A, color B gray

9 discover G from B, color G gray

10 finish G, color G black, return to B

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

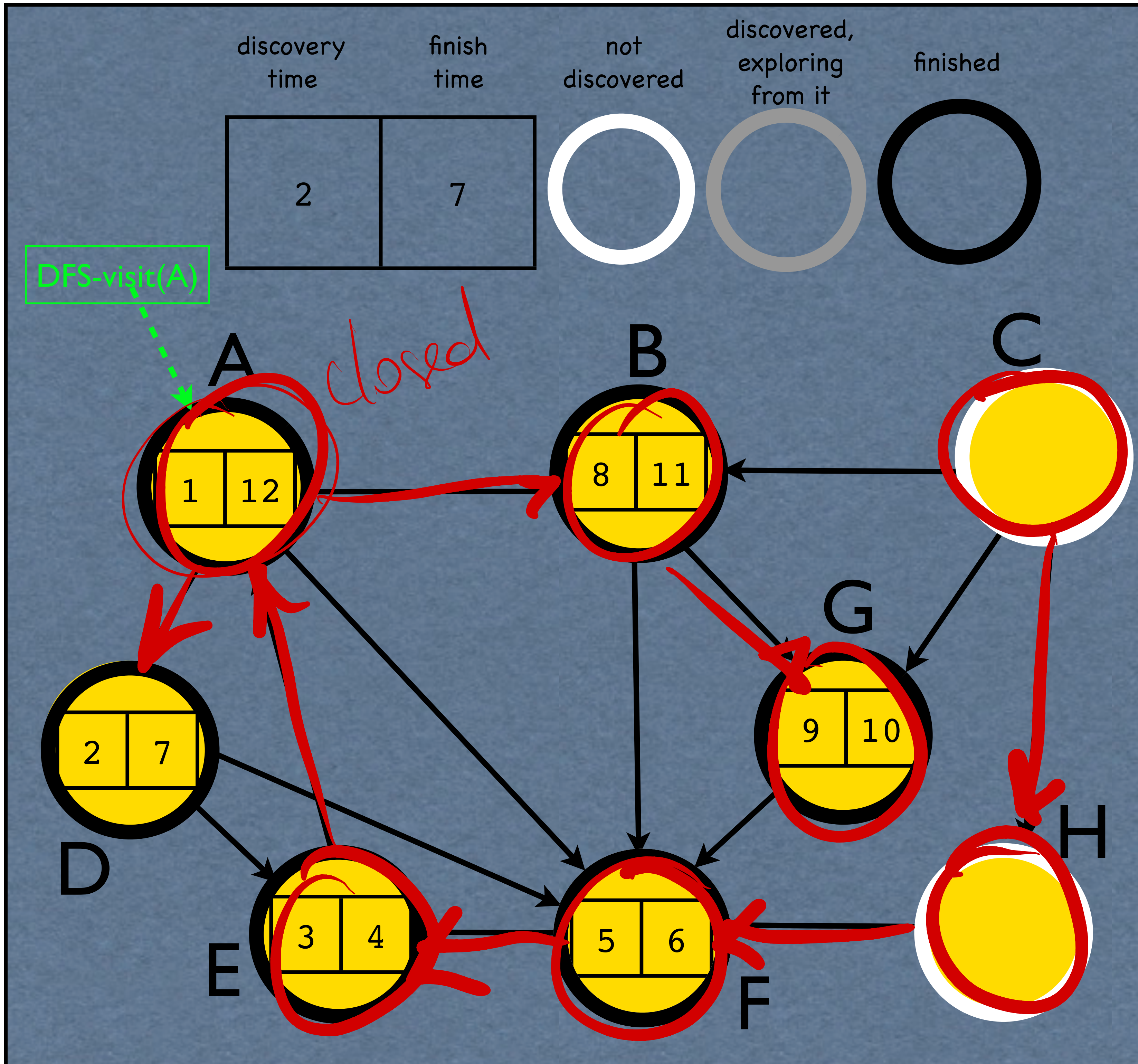
8 discover B from A, color B gray

9 discover G from B, color G gray

10 finish G, color G black, return to B

11 finish B, color B black, return to A

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray

2. discover D from A, color D gray

3. discover E from D, color E gray

4. finish E, color E black, return to D

5 discover F from D, color F gray

6 finish F, color F black, return to D

7 finish D, color D black, return to A

8 discover B from A, color B gray

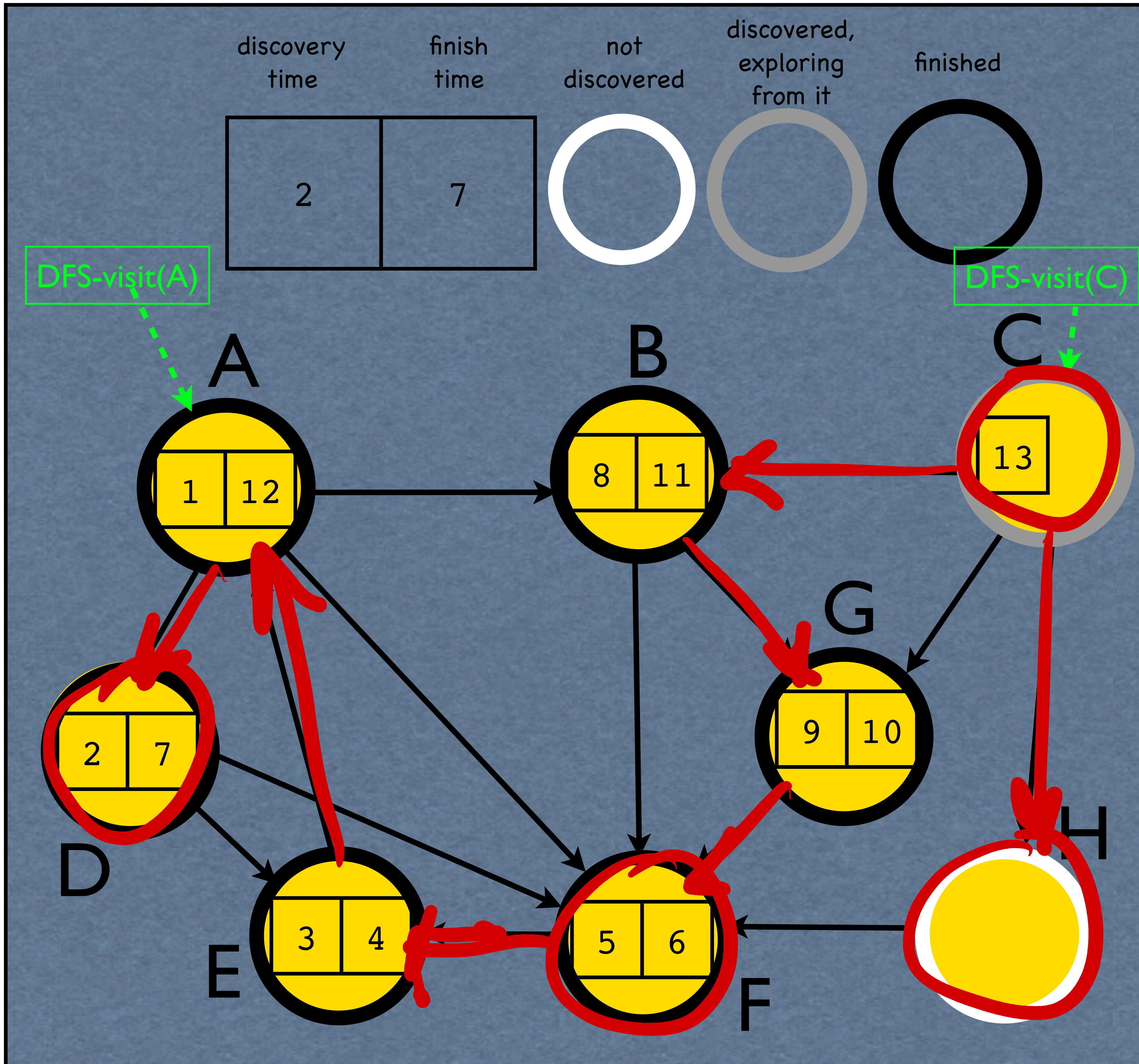
9 discover G from B, color G gray

10 finish G, color G black, return to B

11 finish B, color B black, return to A

12 finish A, color A black, done DFS-visit(A)

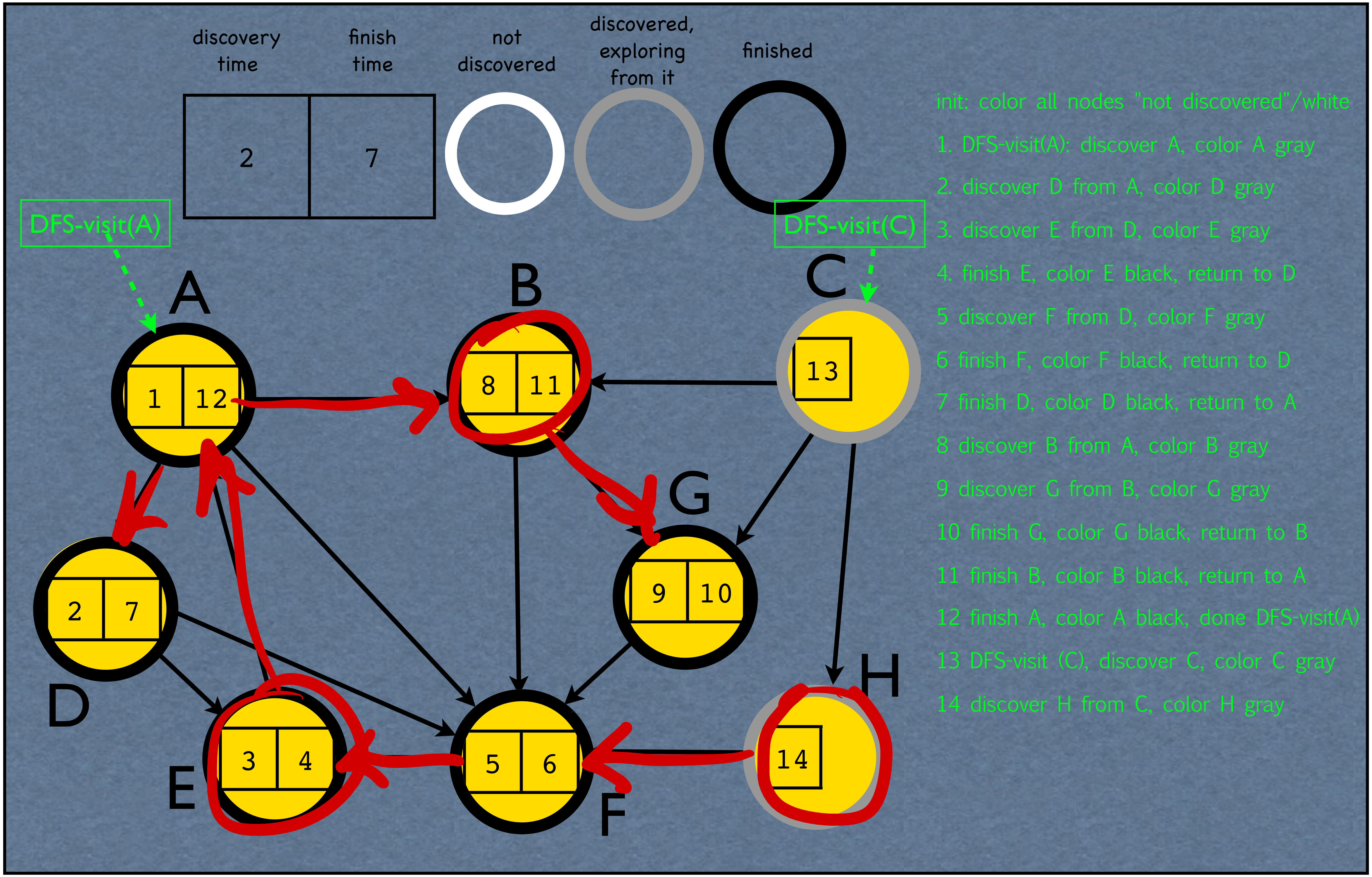
DFS



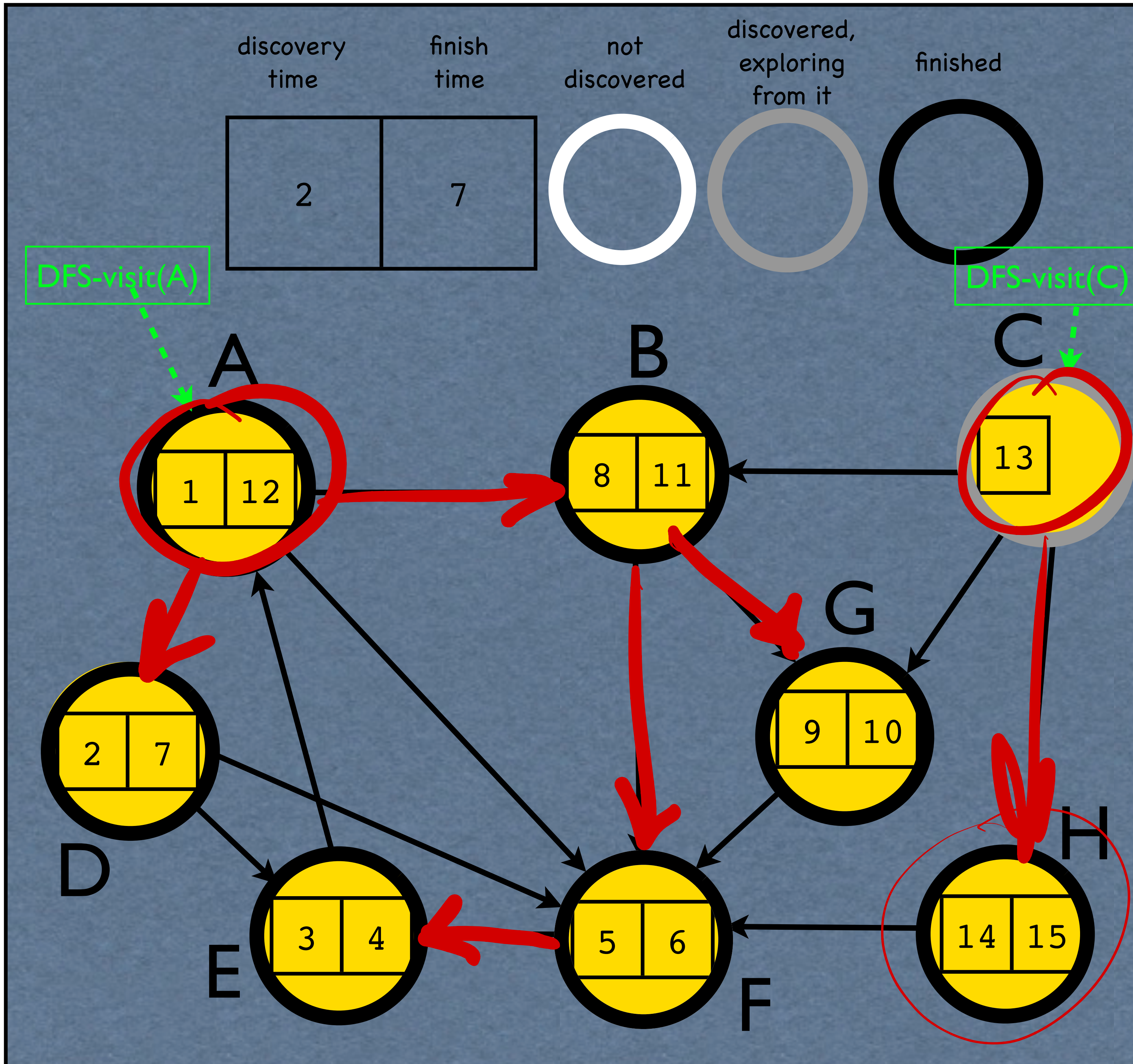
init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B
11 finish B, color B black, return to A
12 finish A, color A black, done DFS-visit(A)
13 DFS-visit (C), discover C, color C gray

DFS


discovery time
finish time
not discovered
discovered, exploring from it
finished
2
7
DFS-visit(A)
DFS-visit(C)
A
B
C
D
E
F
G
H
1
12
8
11
13
2
7
9
10
3
4
5
6
14
init: color all nodes "not discovered"/white
1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B
11 finish B, color B black, return to A
12 finish A, color A black, done DFS-visit(A)
13 DFS-visit (C), discover C, color C gray
14 discover H from C, color H gray

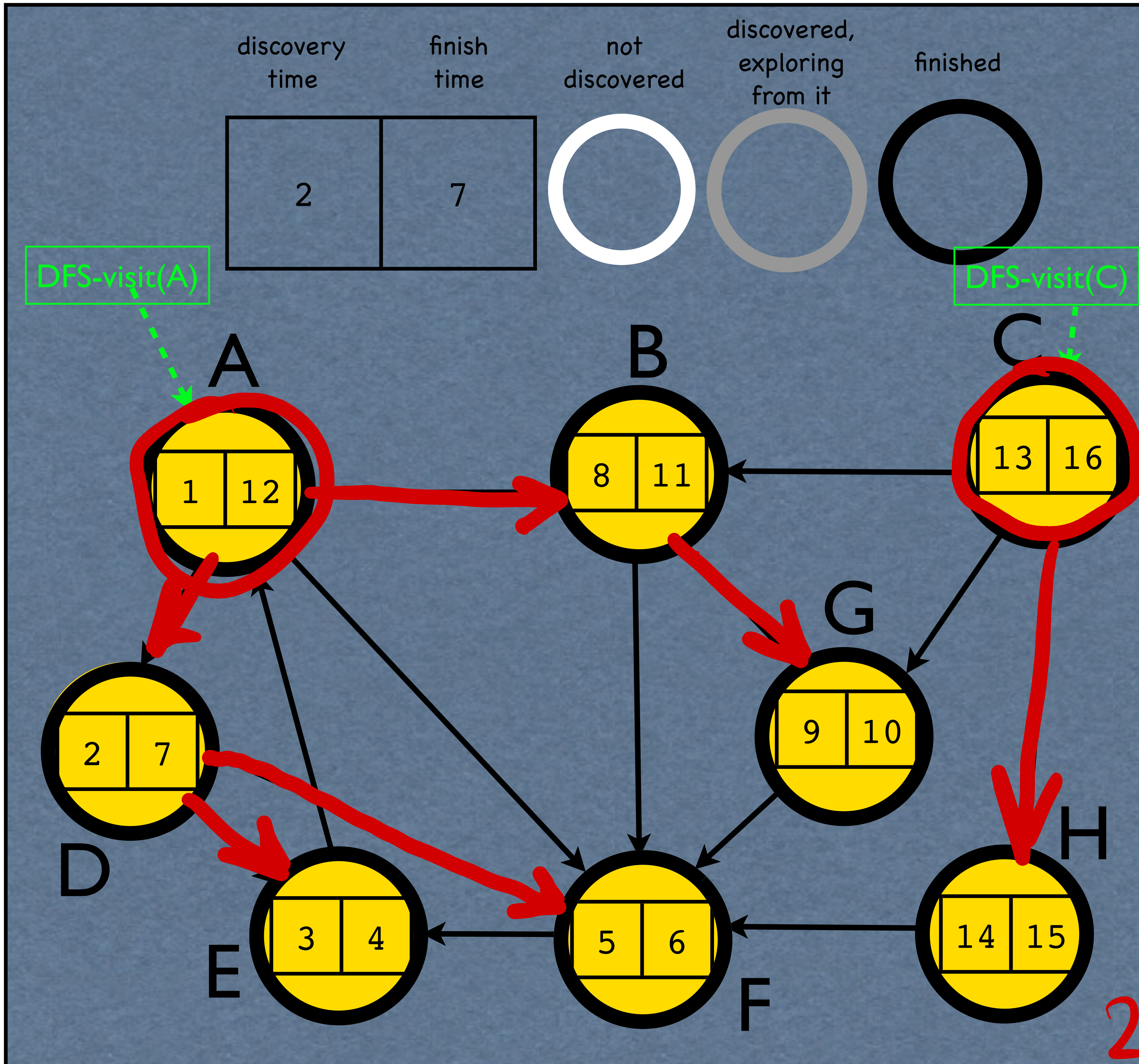
DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B
11 finish B, color B black, return to A
12 finish A, color A black, done DFS-visit(A)
13 DFS-visit (C), discover C, color C gray
14 discover H from C, color H gray
15 finish H, color H black, return to C

DFS



init: color all nodes "not discovered"/white

1. DFS-visit(A): discover A, color A gray
2. discover D from A, color D gray
3. discover E from D, color E gray
4. finish E, color E black, return to D
5 discover F from D, color F gray
6 finish F, color F black, return to D
7 finish D, color D black, return to A
8 discover B from A, color B gray
9 discover G from B, color G gray
10 finish G, color G black, return to B
11 finish B, color B black, return to A
12 finish A, color A black, done DFS-visit(A)
13 DFS-visit (C), discover C, color C gray
14 discover H from C, color H gray
15 finish H, color H black, return to C
16 finish C, color C black, finish DFS-visit(C)

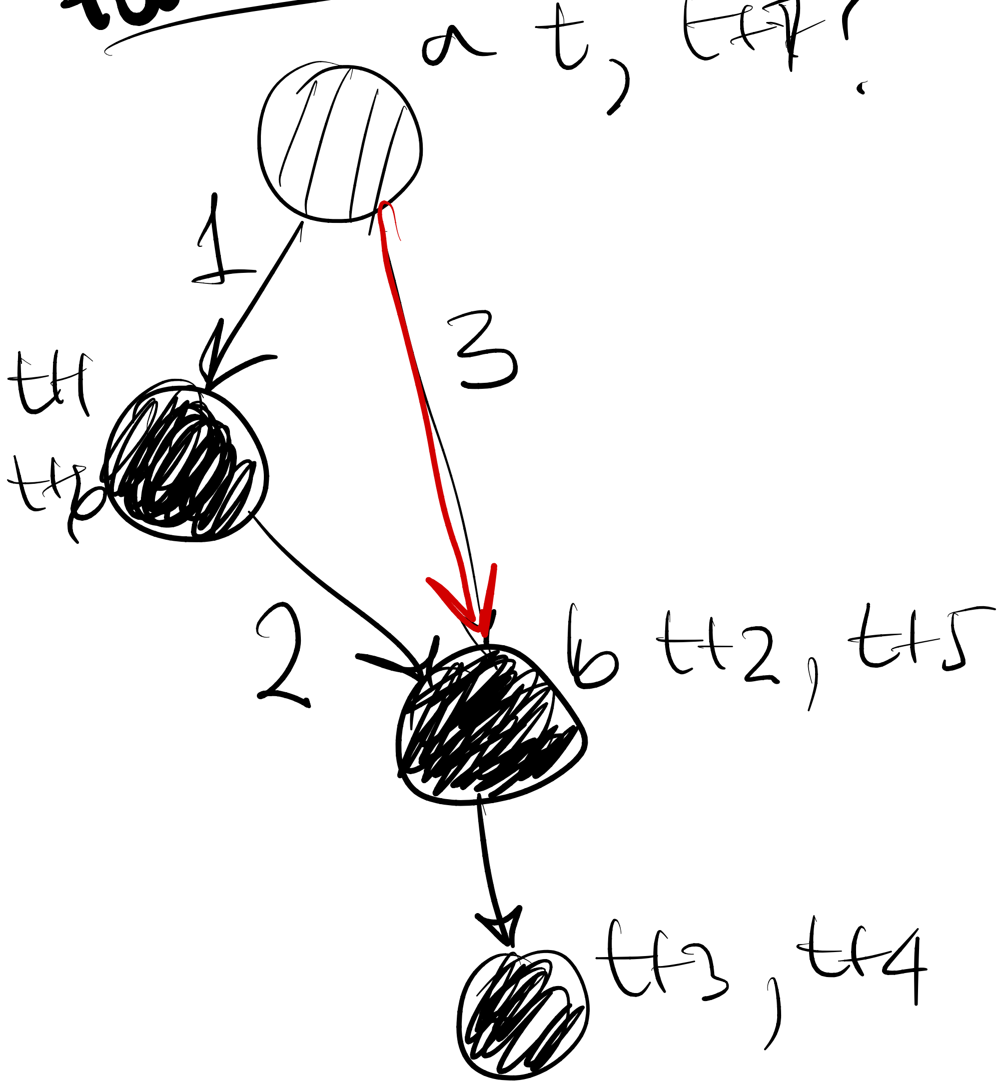
2 starts => 2 trees

DFS edge classification

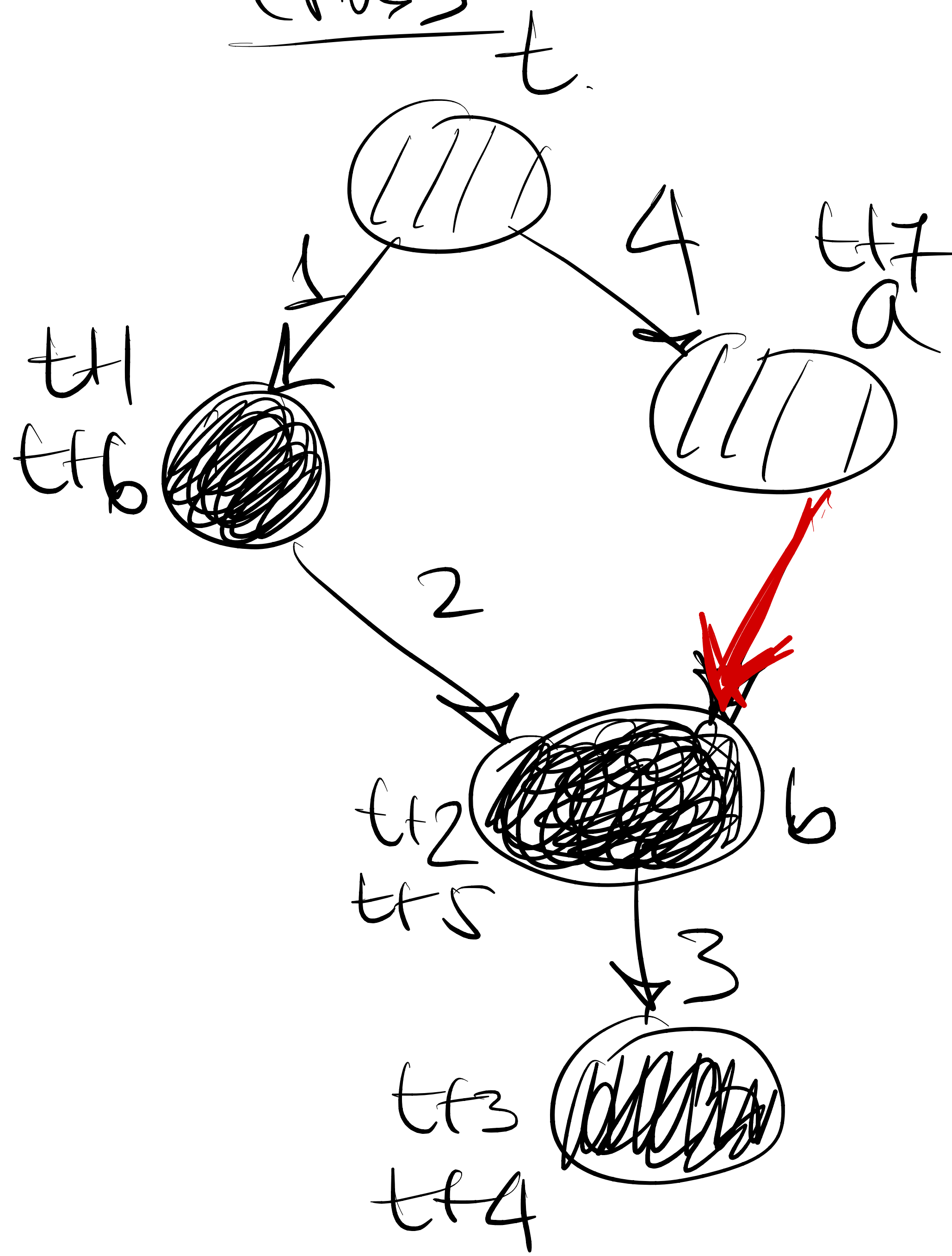
advancing ⟹ DFS tree

- "tree" edge : from vertices gray to white new node
  - a tree edge advances the graph exploration/traversal
- "back" edge : from vertices gray to gray
  - a back edge points to a cycle within the current exploration nodes
- "forward" edge : from vertices a(gray) to b(black), if a discovered first
  - discovery_time[a] < discovery_time[b]
  - points to a different part of the tree, already explored from a
- "cross" edge : from vertices a(gray) to b(black), if b discovered first
  - discovery_time[a] > discovery_time[b]
  - points to a different part of the tree, explored before discovering a

Forward

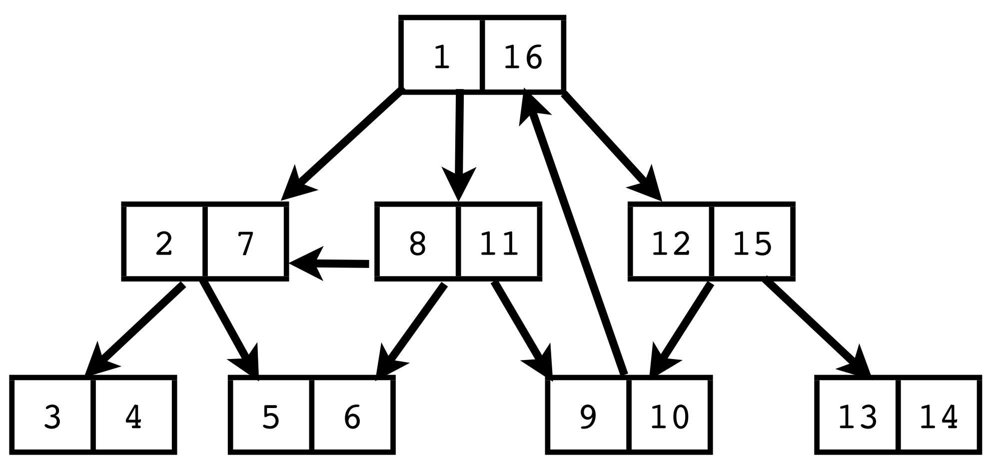


Cross



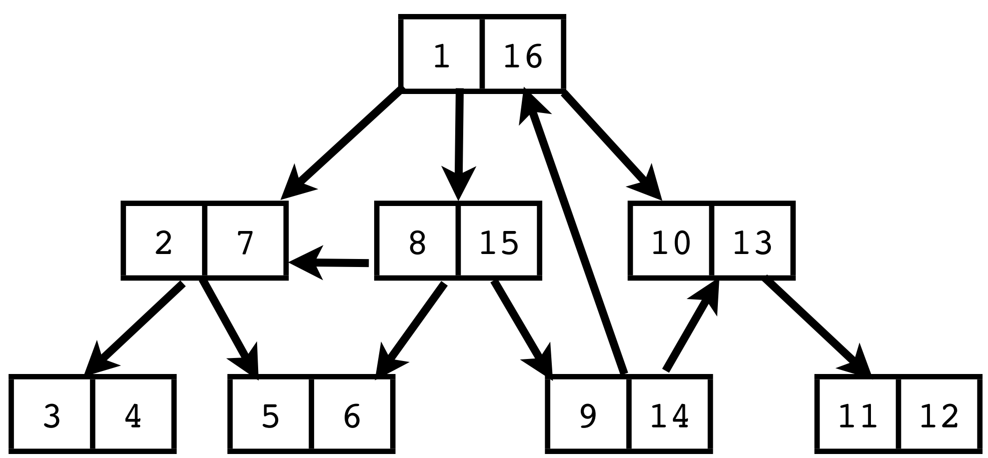
Checkpoint

- on the animated example, label each edge as "tree","back", "cross", or "forward"
- do the same on the following example (DFS discovery and finish times marked for each node)



Checkpoint

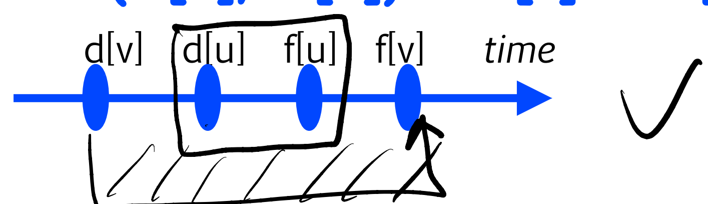
- almost same example, with a small modification: one edge was reversed



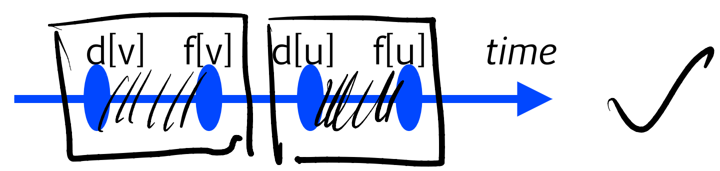
DFS observations

d[ ] f[ ]

- Running time O(V+E), same as BFS
- vertex v is gray between times discover[v] and finish[v]
- gray time intervals (discover[v], finish[v]) are inclusive of each other
  - (d[v], f[v]) can include (d[u], f[u]) : d[v] < d[u] < f[u] <f[v]



  - (d[v], f[v]) can separate from (d[u], f[u]) : d[v] < f[v] < d[u] <f[u]

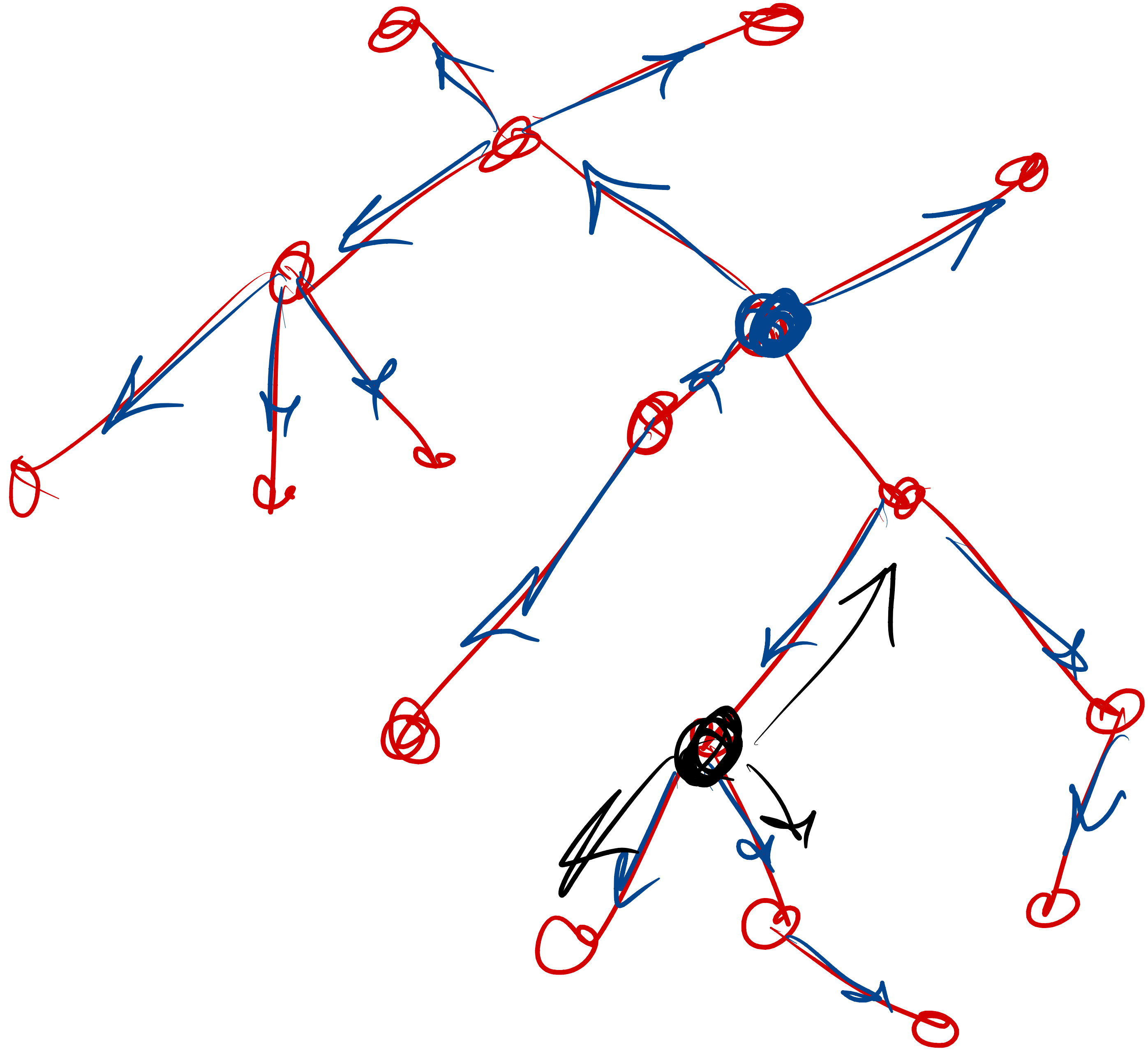


  - (d[v], f[v]) cannot intersect (d[u], f[u]) : d(v) < d(u) < f[v] <f[u]

- graph G=(V,E) is acyclic (does not have cycles) if DFS does not find any "back" edge

Undirected graphs cycles

- graph $G=(V,E)$ is acyclic (does not have cycles) if DFS does not find any "back" edge
- since G is undirected, no cycles implies $|E| \leq |V|-1$
- running DFS, if we find more than $|V|-1$ edges, there must be a cycle
- Undirected graphs: find-cycles algorithm takes $O(V)$



Directed graphs cycles

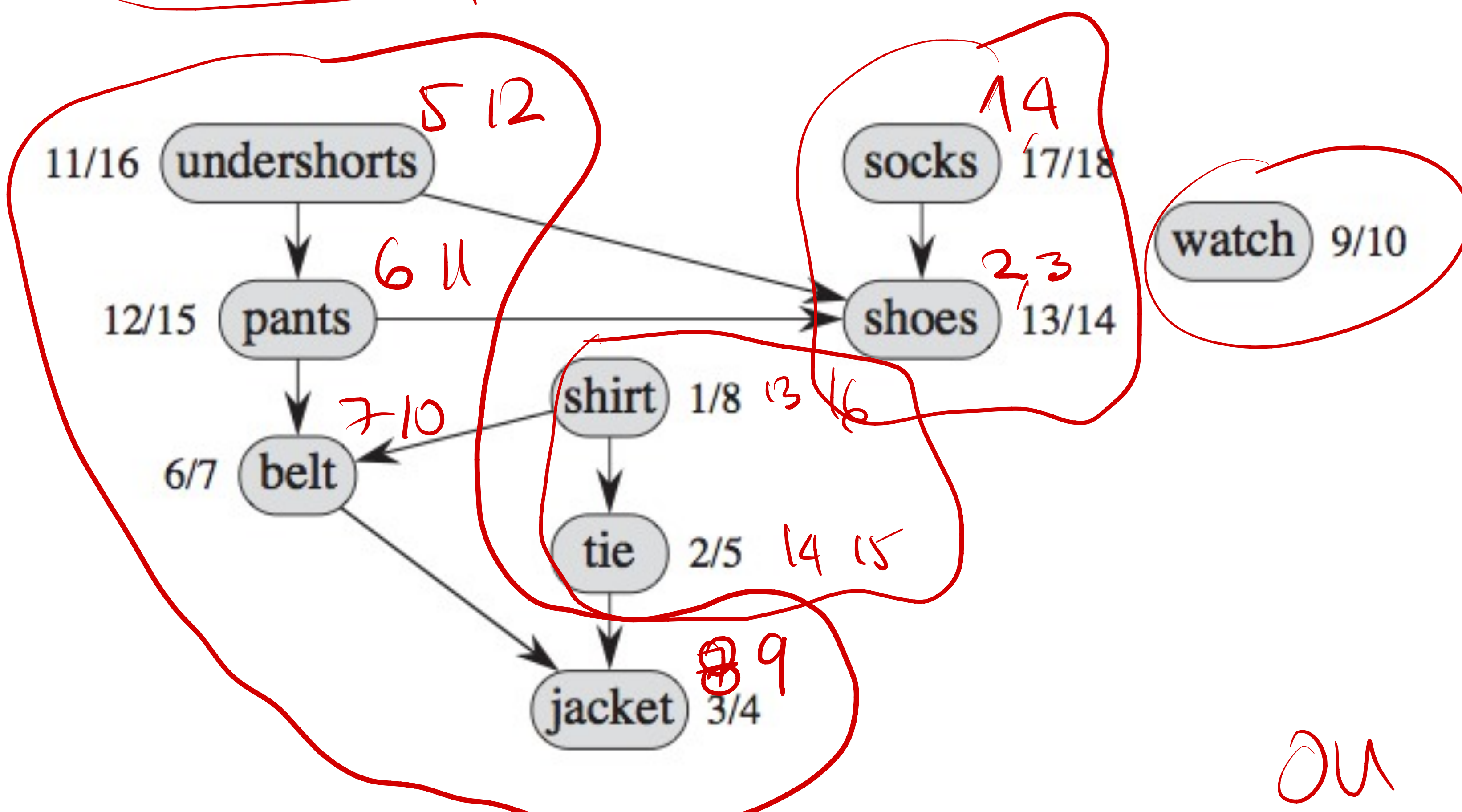
E = V-1 ⇔ no cycles connected ⇔ tree ??

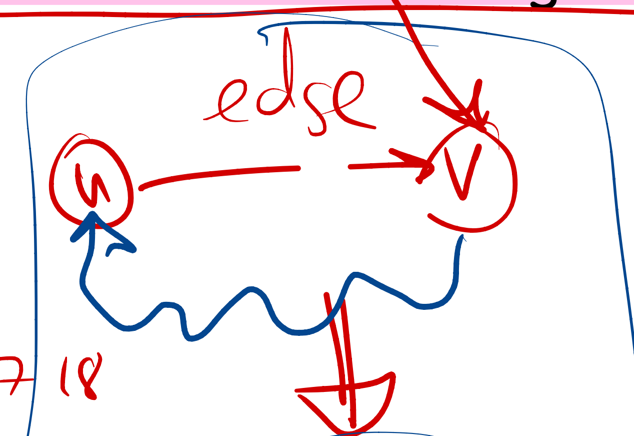
- graph G=(V,E) is acyclic (does not have cycles) if DFS does not find any "back" edge
- for directed graphs, even without cycles they can have more edges, |E| > |V|-1
- algorithm to determine cycles: run DFS, look for back edges - O(V+E) time
- DAG = directed acyclic graph ⇔ ~~TOPOLOGICAL~~ LINSORT (6)

no cycles ⇒ much easier

~~Topological~~ Line sort

- DAG admits topological sort: all vertices "sorted" on a line, such that all edges point from left to right-no cycles - 2 graphs below are the same-
- to do this: algorithm: run DFS, time O(V+E). Output vertices in reverse order given by finishing time



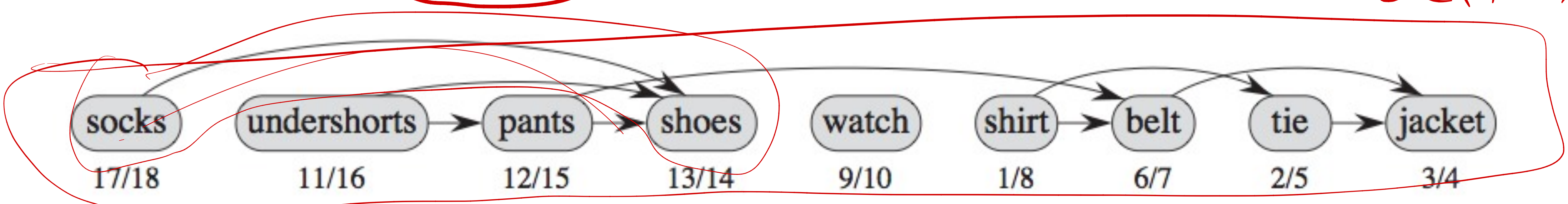


$f[u] > f[v]$

bigger = after

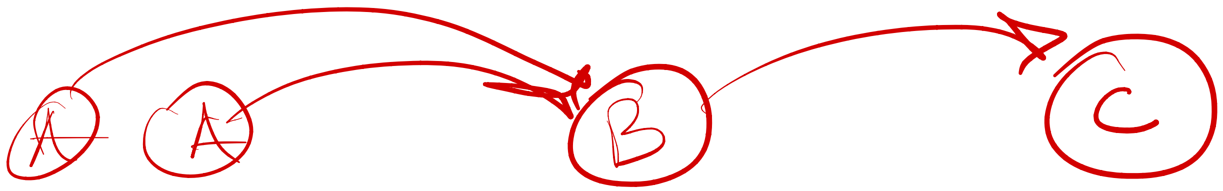
smaller = before

on line $\Leftrightarrow$ edges left $\rightarrow$ right



watch shirt tie undersh pants belt jacket socks shoes

Check Point

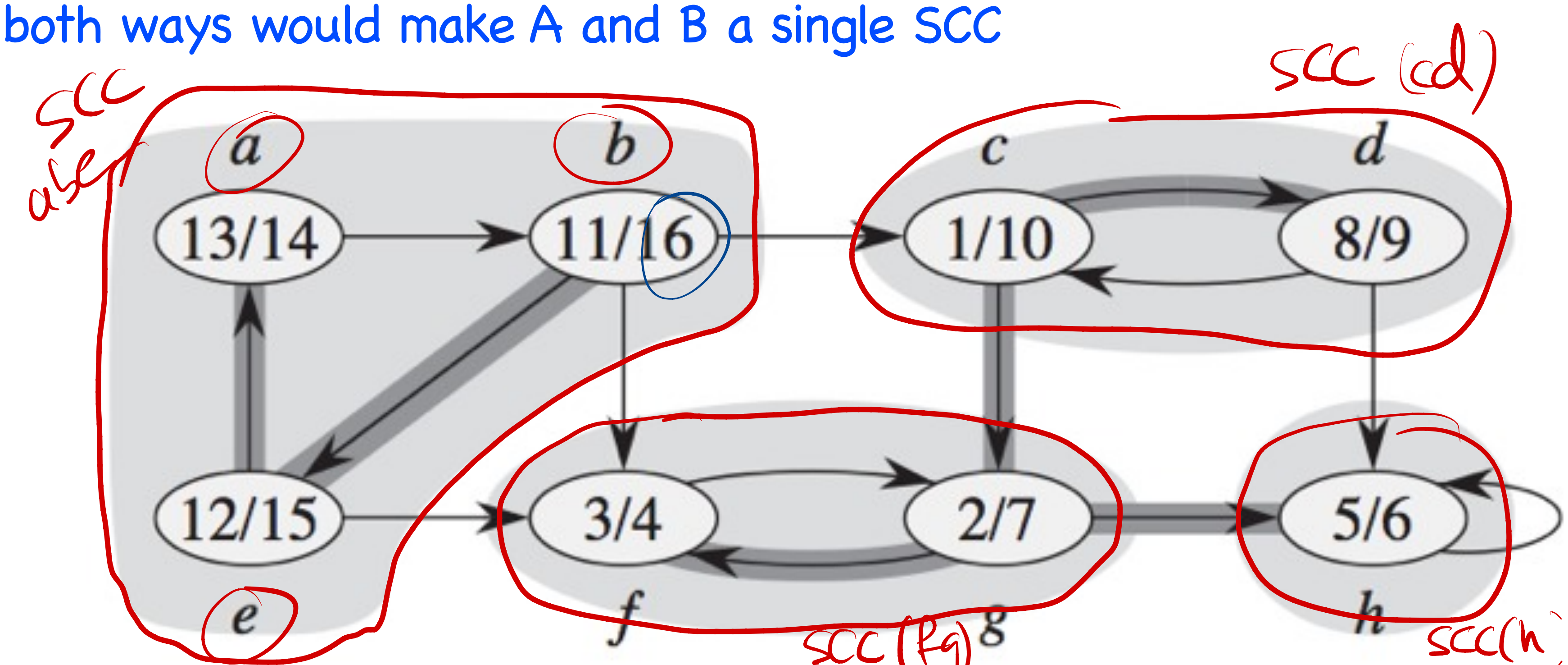


- * how can we use DFS to determine if there is ~~a path~~ all paths from u to v ?

- exercise
  prove that by sorting vertices in the reverse order of finishing times, we obtained a topological sort
  - assuming no cycles
  - in other words, all edges point in the same direction

Strongly connected components

- SCC = a set of vertices $S \subset V$, such that for any two $(u,v) \in S$, graph G contains a path $u \rightsquigarrow v$ and a path $v \rightsquigarrow u$
- trivial for undirected graphs
  - all connected vertices are in fact strongly connected
- tricky for directed graphs
- graph below has the DFS discover/finish times and marked 4 strongly connected components; "tree" edges highlighted
- between two SCC, A and B, there cannot exists paths both ways ($A \ni u \rightsquigarrow v \in B$ and $B \ni v' \rightsquigarrow u' \in A$)
  - paths both ways would make A and B a single SCC



Transitive Closure = matrix M

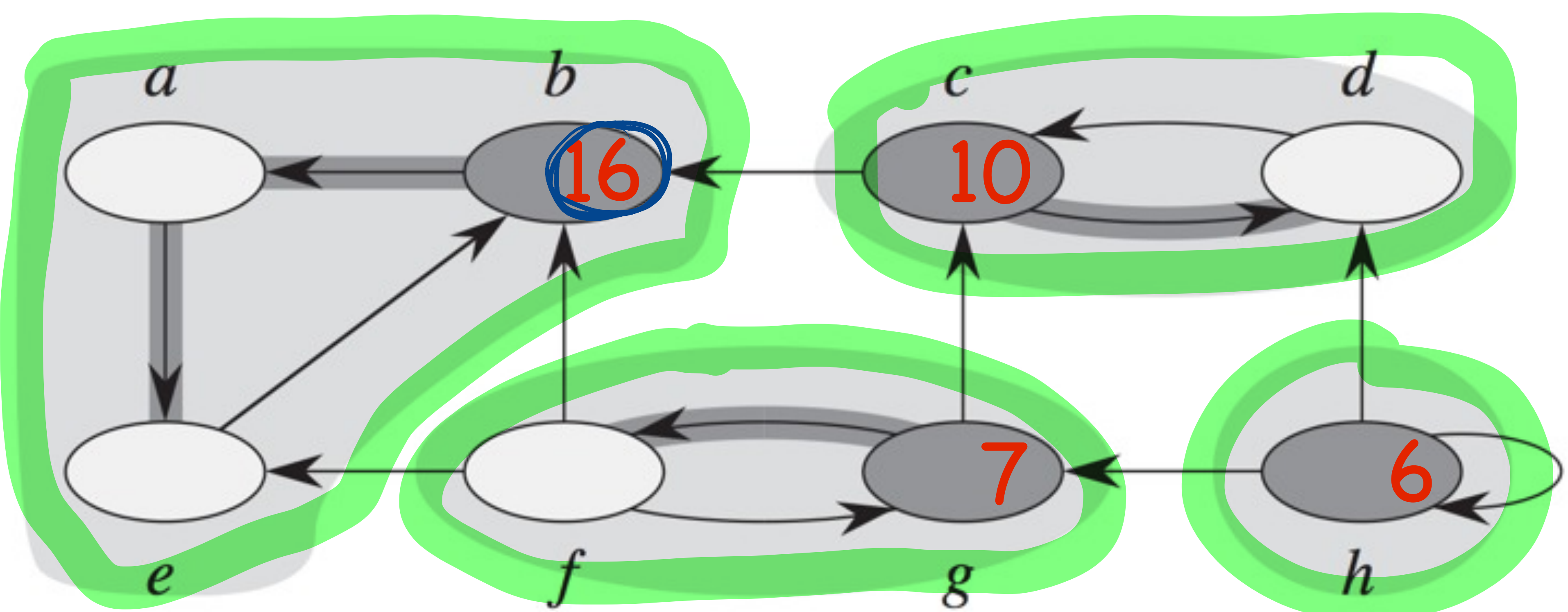
$$M_{ij} = \begin{cases} 1 & \text{if } i \rightsquigarrow j \text{ path} \\ 0 & \text{if no path} \end{cases}$$

(1) If M = trans closure given $\Longrightarrow$ SCC ?

(2) How to get the transitive closure?

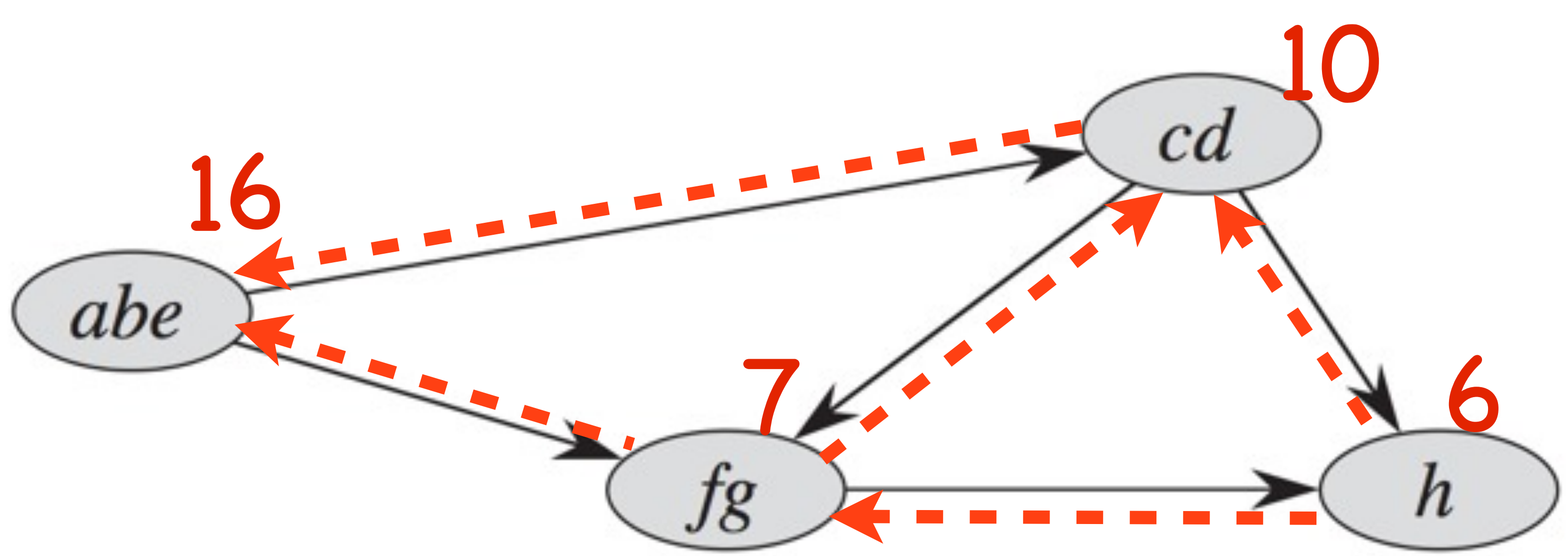
Strongly connected components

- run 1st DFS on G to get finishing times f[u]
- run 2nd DFS on G-reversed (all edges reversed -see picture), each DFS-visit in reverse order of f[u]
  - finishing times marked in red for the DFS-visit root vertices
- output each tree (vertices reached) obtained by 2nd DFS as an SCC

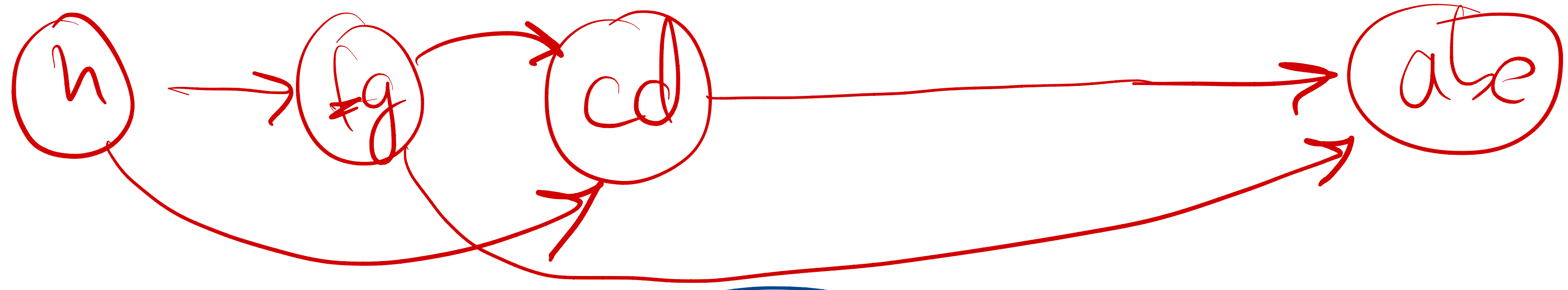


Strongly connected components

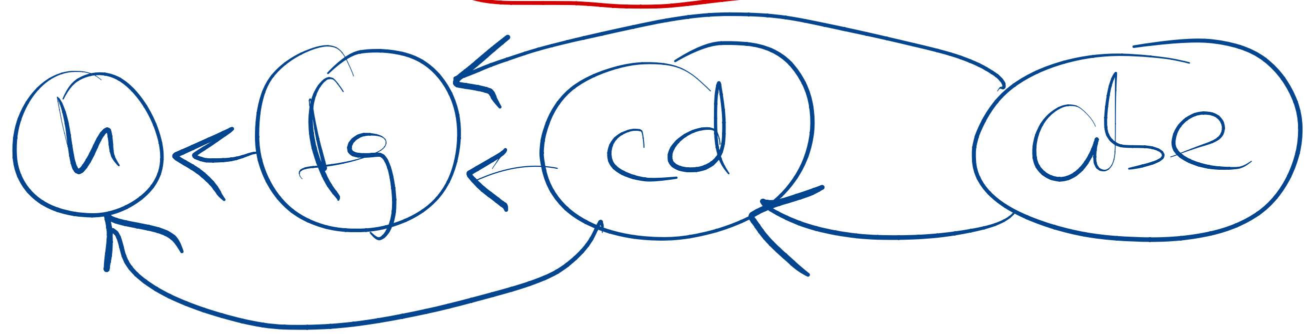
- why 2nd DFS produces precisely the SCC -s?
- SCC-graph of G: collapse all SCC into one SCC-vertex, keep edges between the SCC-vertices
- - SCC graph is a DAG;
  - contradiction argument: a cycle on the SCC-graph would immediately collapse the cycle's SCC-s into one SCC
- reversed edges (shown in red); reversed-SCC-graph also a DAG
- second DFS runs on reversed-edges (red); once it starts at a high-finish-time (like 16) it can only go through vertices in the same SCC (like abe)



Reversed



original



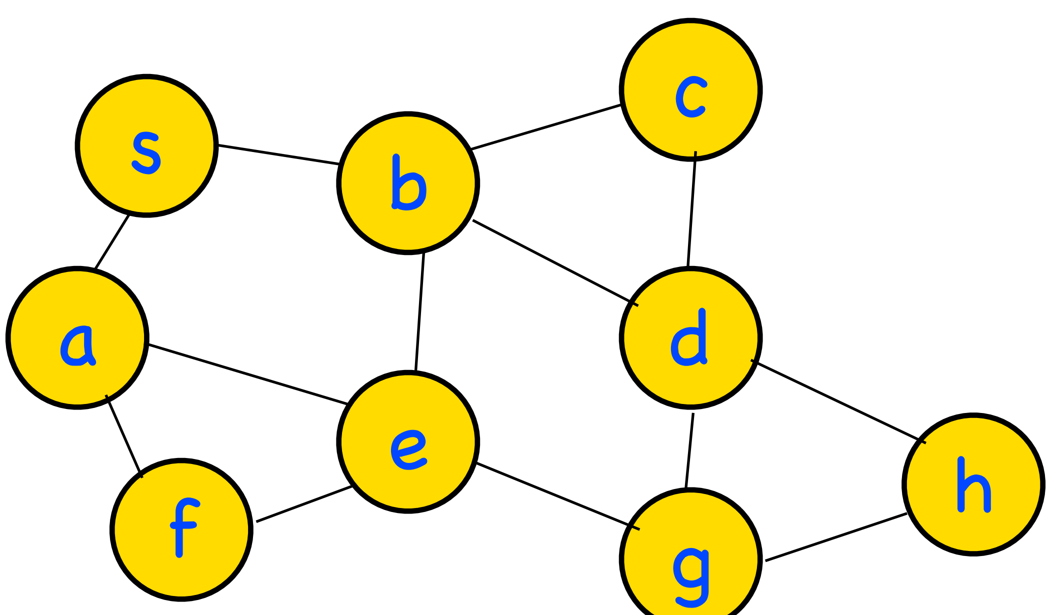
Th max finish time (1st DFS) ⇒ starts in the Right-most component DAG

Minimum Spanning Trees

Lesson 2

Spanning Trees

- context : undirected graphs
- a set of edges A that "span" or "touch" all vertices, and forms no cycles
  - necessary this set of edges A has size = |V|-1
- spanning tree: the tree formed by the set of spanning edges together with vertex set T = (V,F)



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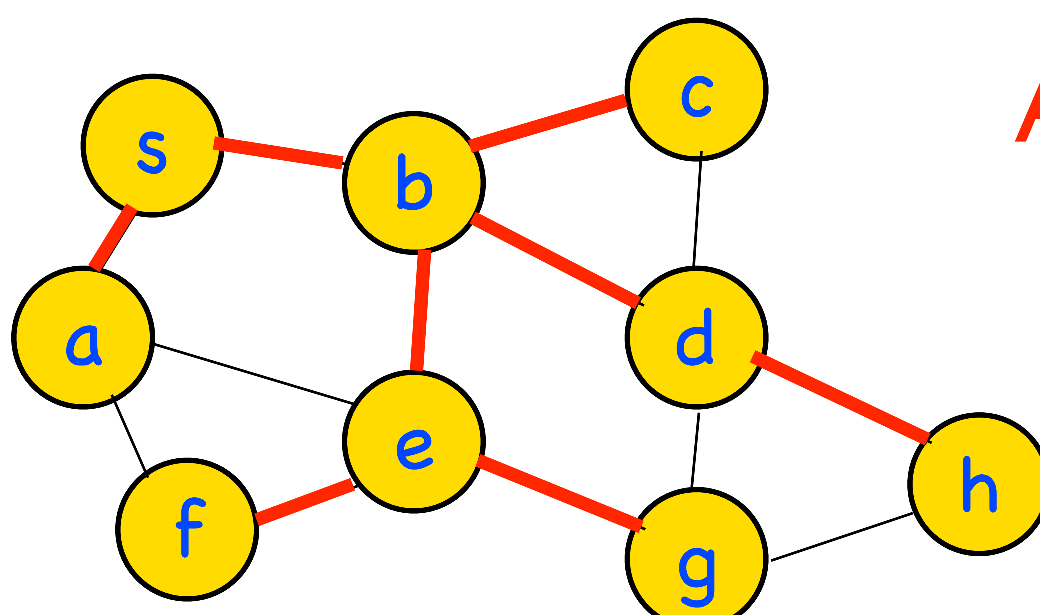
all nodes



A spanning tree

= a set of edges

|V|-1

Spanning Trees

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  - necessary this set of edges A has size = |V|-1
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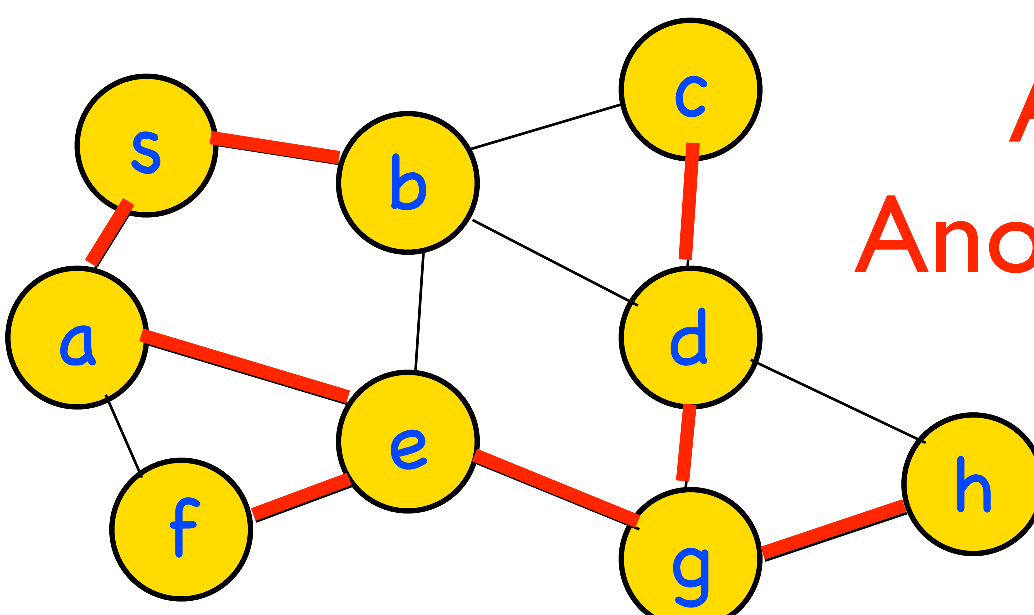
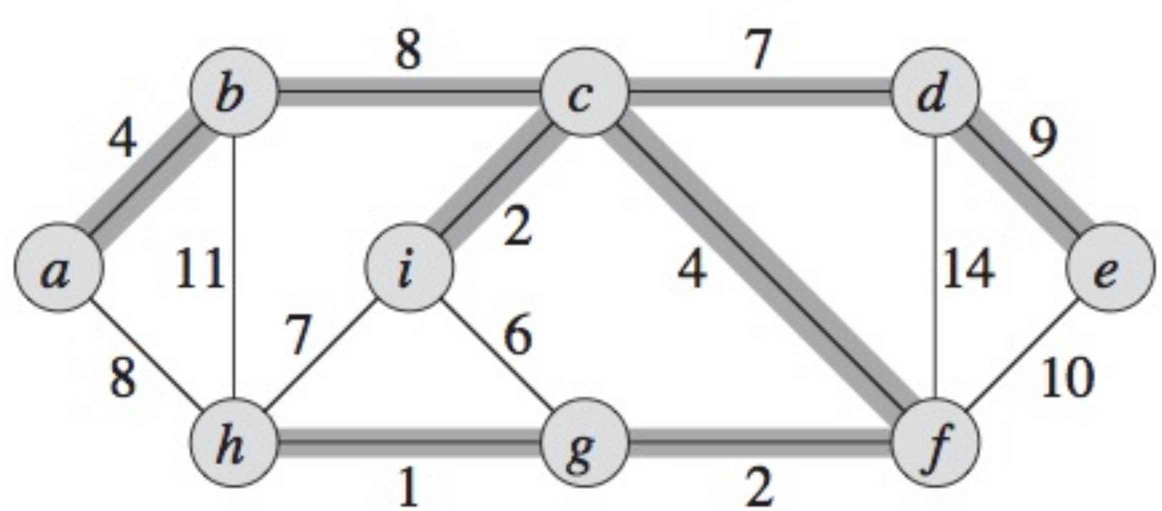


A spanning tree
Another spanning tree

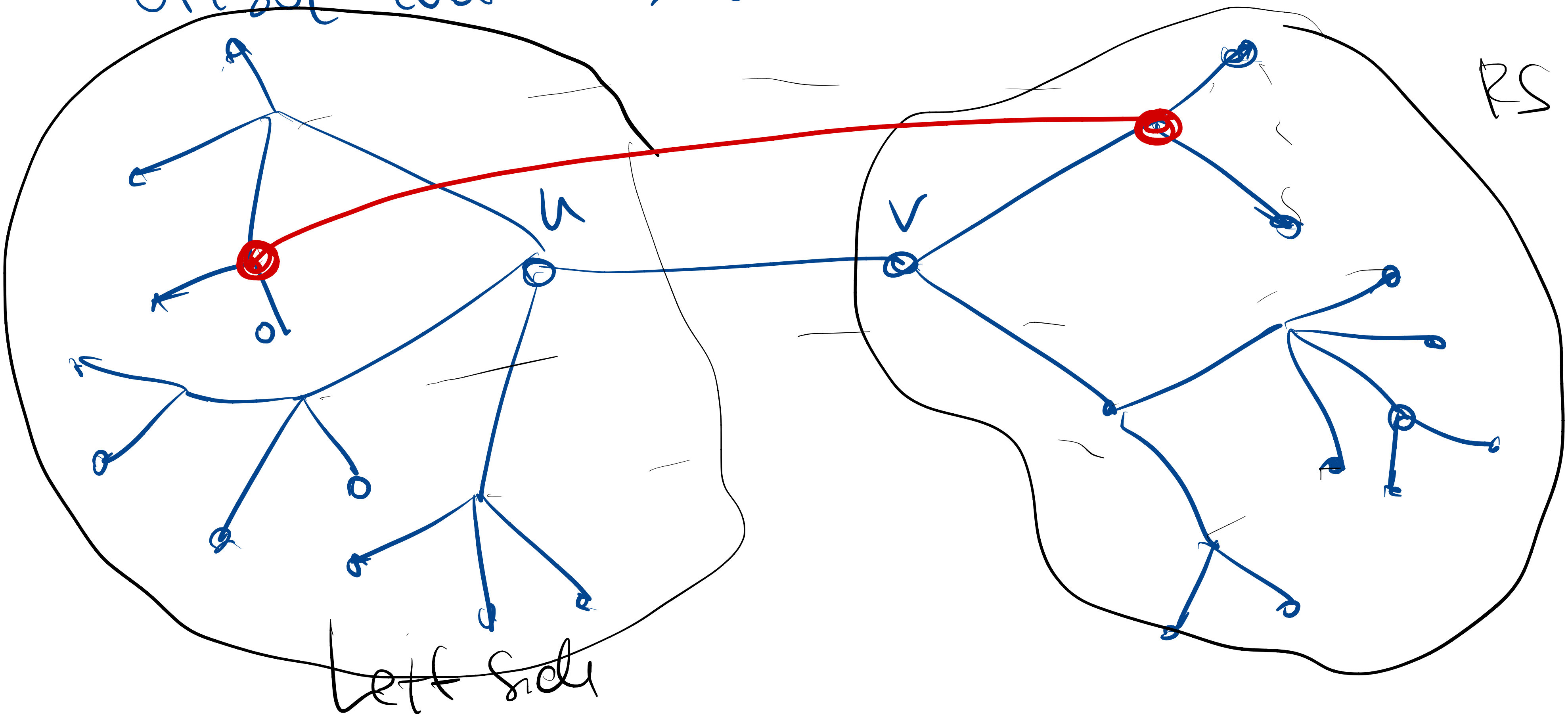
Minimum Spanning Tree (MST)

- context : undirected graph, edges have weights
  - edge $(u,v) \in E$ has weight $w(u,v)$
- MST is a spanning tree of minimum total weight (of its edges)
  - must span all vertices
  - exactly $|V|-1$ edges
  - sum of edges $\boxed{\text{weight be minimum}}$ among spanning trees

$$\text{weight(tree)} = \sum_{\substack{e = \text{edge} \\ \text{in tree}}} \text{weight}(e)$$



OPTSOL char. => set of edges = MST



① subtree(u) \ v => OPT MST for those nodes (LS)

subtree(v) \ u => OPT for R.S

② any edge not used across LS(u) — RS(v) ≥ edge(u,v)

③ in other used edge can cross LS-RS

Growing Minimum Spanning Trees

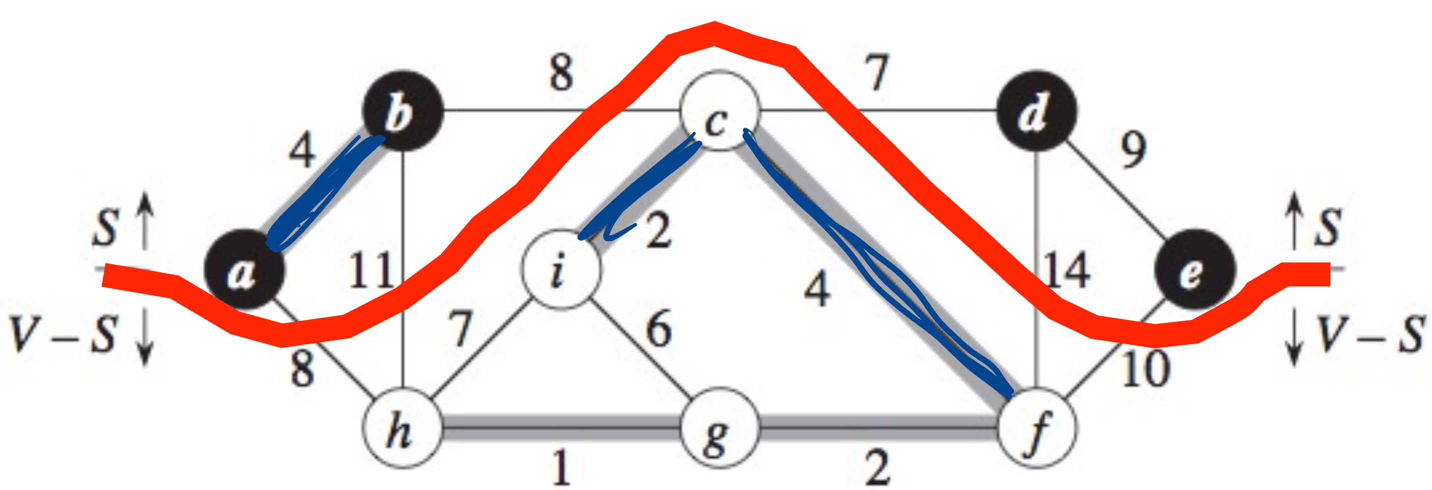
- "safe edge" (u,v) for a given set of edges A: there is a MST that uses A and (u,v)
  - that MST may not be unique

```
• GENERIC-MST (G)
• A = set of tree edges, initially empty
• while A does not form a spanning tree // meaning while |A| < |V|-1
  – find edge (u,v) that is safe for A
  – add (u,v) to A
• end while
```

- how to find a safe edge to a given set of edges A?
  - Prim algorithm
  - Kruskal algorithm

Cuts in the graph

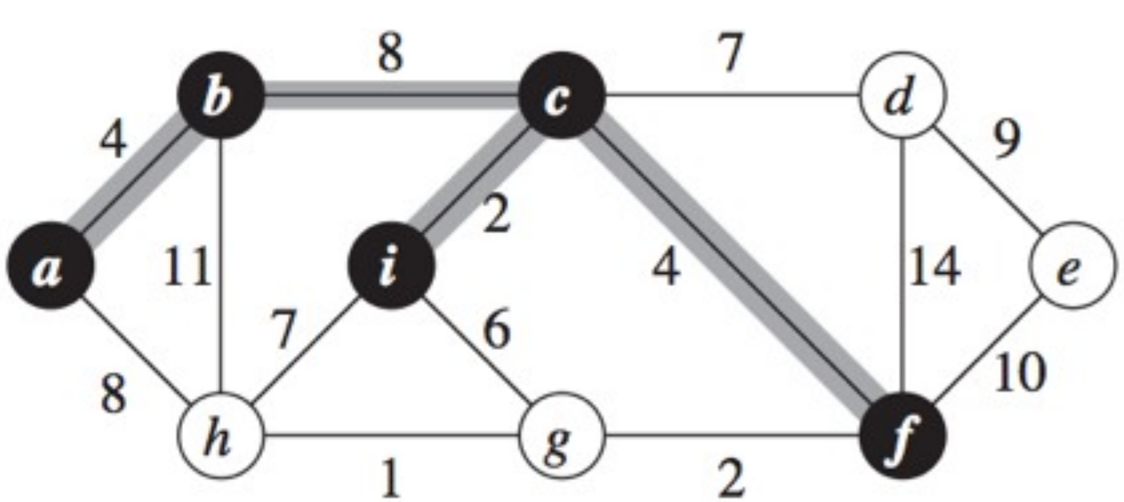
- "cut" is a partition of vertices in two sets : V=S ∪ V-S
- an edge (u,v) crosses the cut (S,V-S) if u and v are on different partitions (one in S the other in V-S)
- cut (S, V-S) respects set of edges A if A has no cross edge
- "min weight cross edge" is a cross edge for the cut, having minimum weight across all cross edges
- Cut Theorem : if A is a set of edges part of some MST, and (S,V-S) a cut respecting A , then a min-weight cross edge is "safe" for A (can be added to A towards an MST)



- A={ab, ic, cf, hg, fg}
- cut : S={a,b,d,e} V-S={h,i,c,g,f} respects A
- safe crossing edge : cd, weight(cd)=7

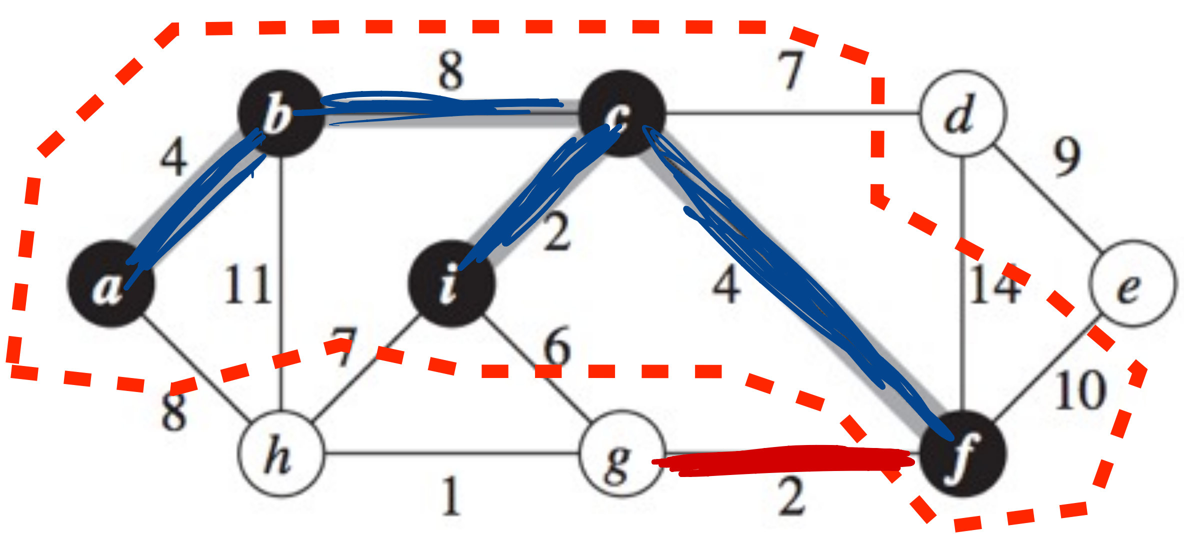
Prim algorithm

- grows a single tree A, S = set of vertices in the tree
  - as opposed to a forest of smaller disconnected trees
- add a safe edge at a time
  - connecting one more node to the current tree



Prim algorithm

- grows a single tree A, S = set of vertices in the tree
  - as opposed to a forest of smaller disconnected trees
- add a safe edge at a time
  - connecting one more node to the current tree
- define cut (S,V-S), which respects A. Using the cut theorem, the min-weight edge across the cut is the next edge added to A



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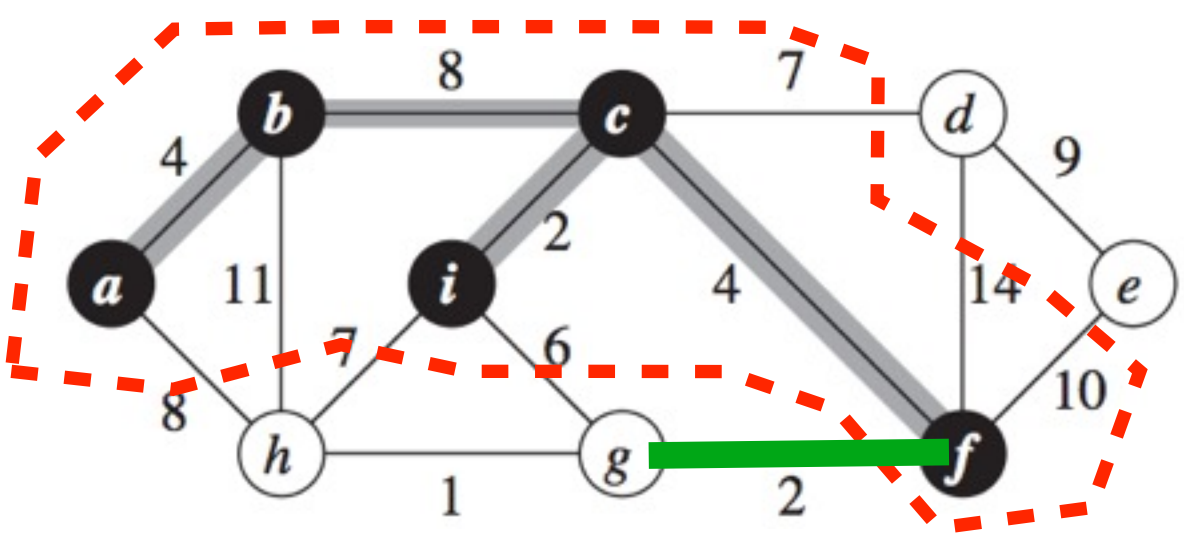
  - edge gf in the picture is added to A, vertex g added to the tree



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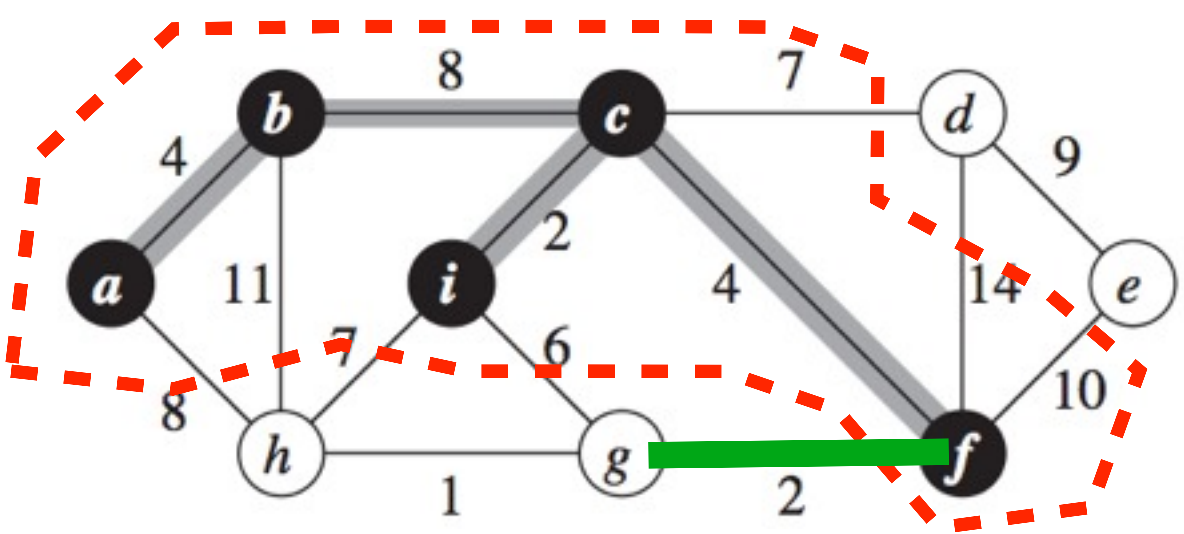
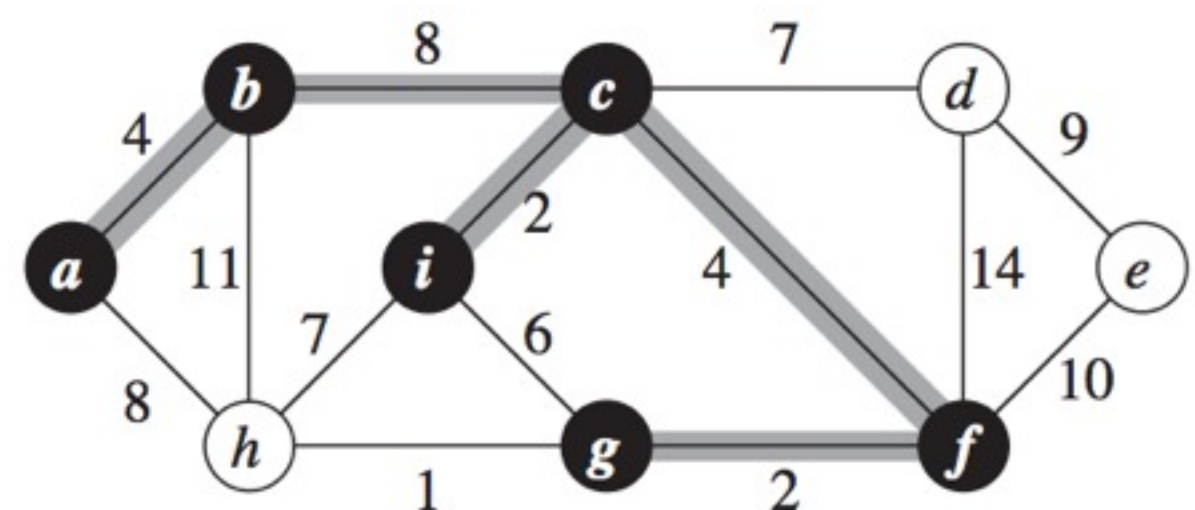
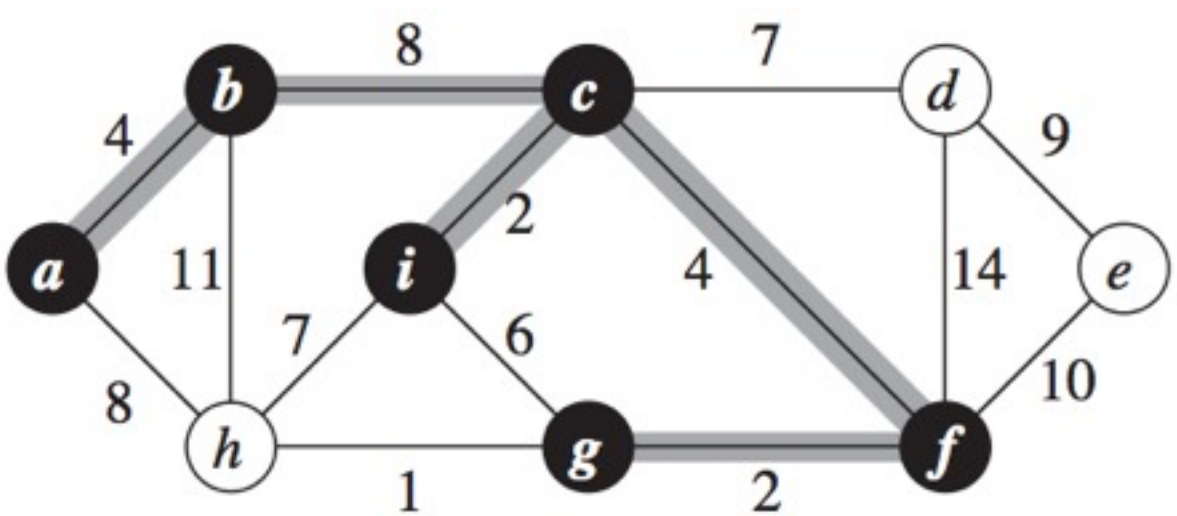




Prim algorithm

- add another(next) safe edge
  - connecting one more node to the current tree



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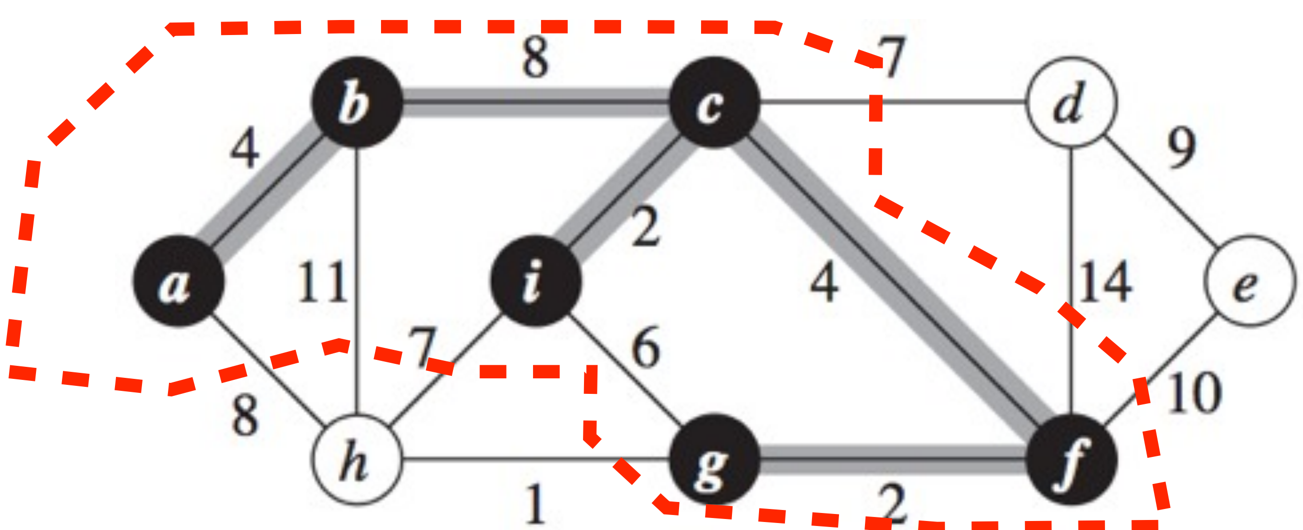

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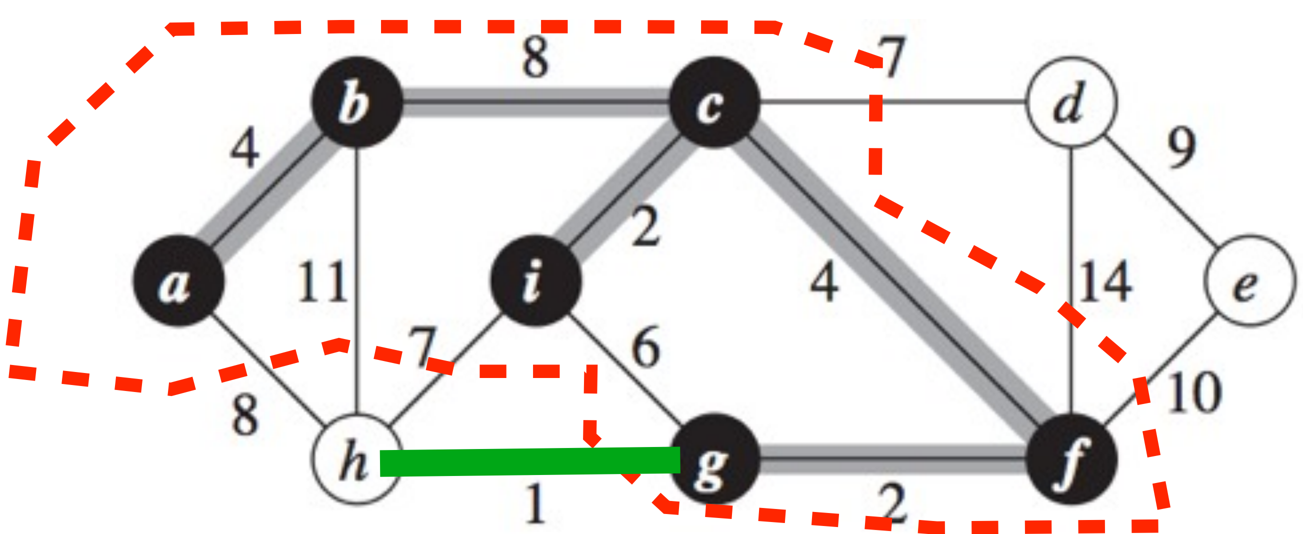

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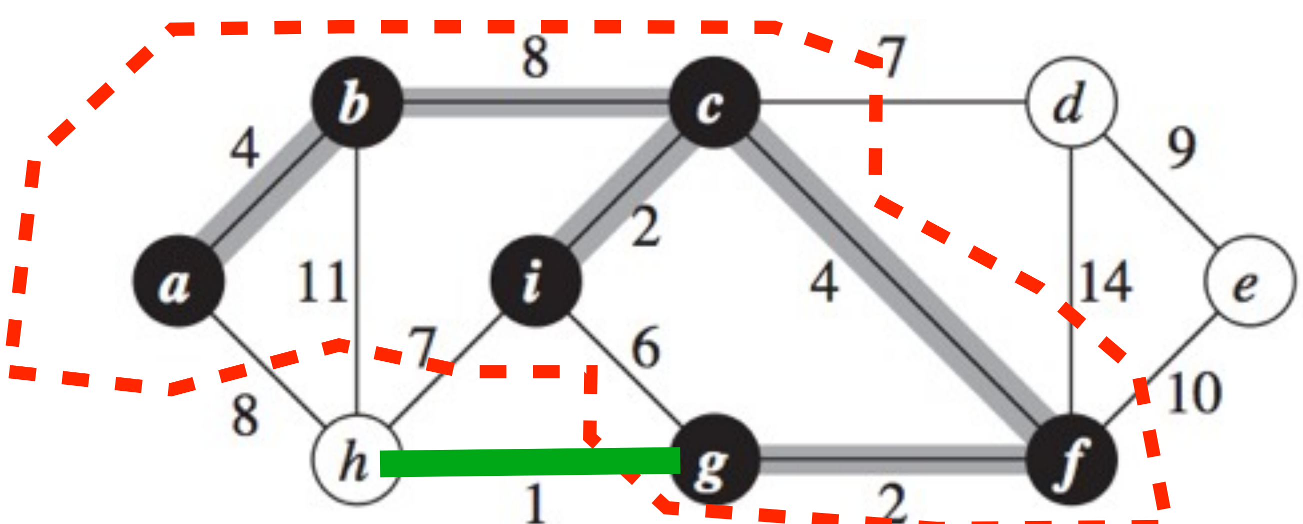
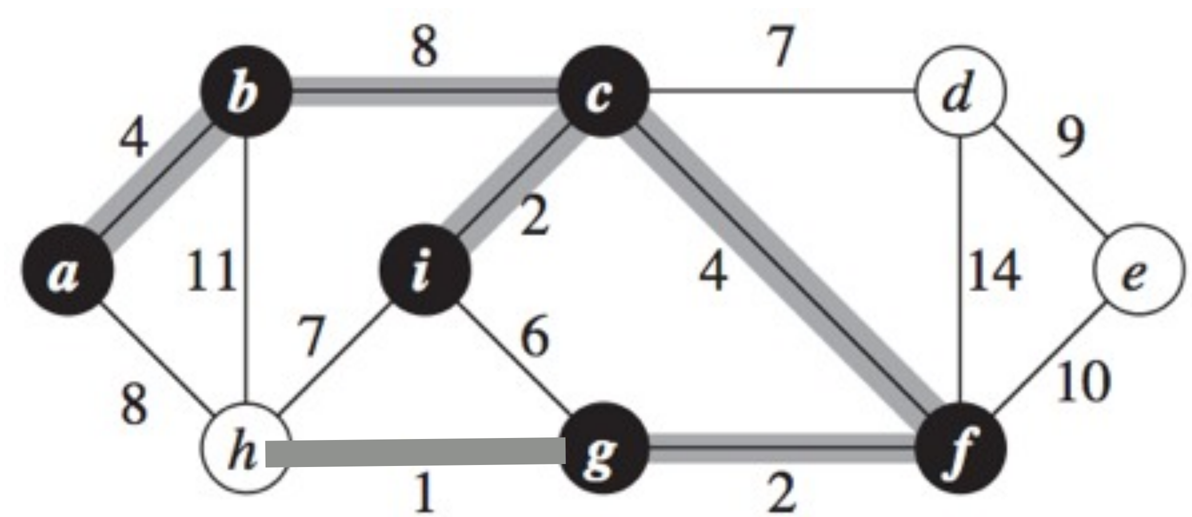




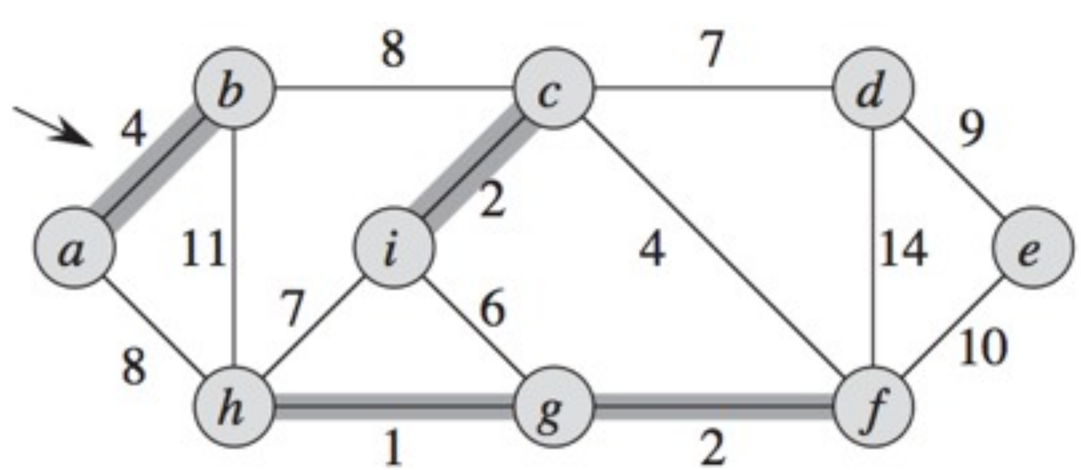
Prim MST algorithm

- Prim simple
  - but implementation a bit tricky
- Running Time depends on implementation of Extract-Min from the Queue
  - best theoretical implementation uses Fibonacci Heaps
  - also the most complicated
  - only makes a practical difference for very large graphs

```
MST-PRIM(G, w, r)
for each u ∈ G.V
    u.key = ∞
    u.π = NIL
r.key = 0
Q = G.V
while Q ≠ ∅
    u = EXTRACT-MIN(Q)
    for each v ∈ G.Adj[u]
        if v ∈ Q and w(u, v) < v.key
            v.π = u
            v.key = w(u, v)
```

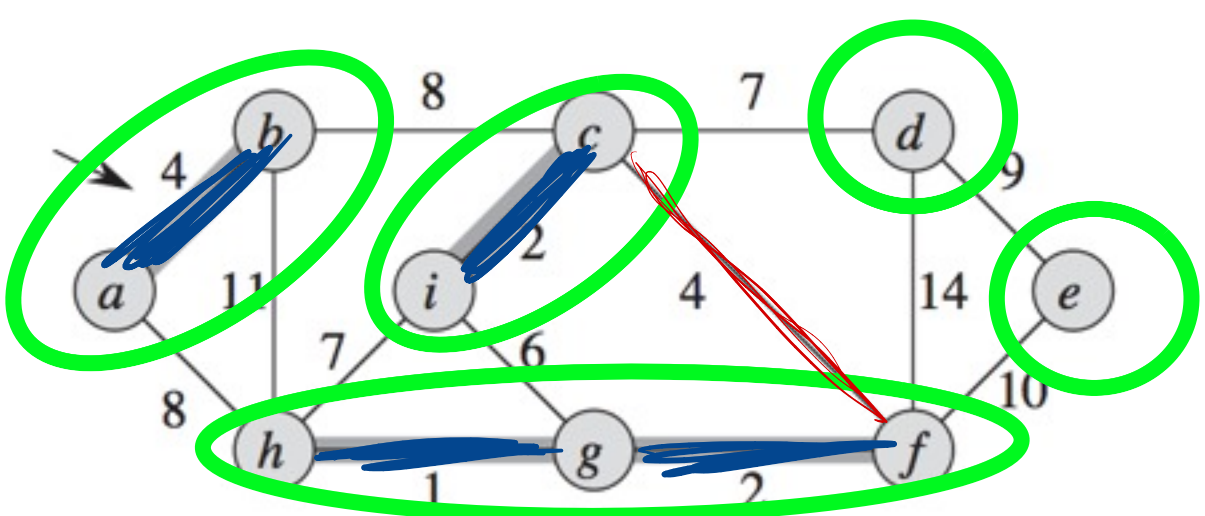
Kruskal MST algorithm

- Grows a forest of trees Forrest = (V,A)
  - eventually all connected into a MST
  - initially each vertex is a tree with no edges, and A is empty



Kruskal MST algorithm

- Grows a forest of trees Forrest = (V,A)
  - eventually all connected into a MST
  - initially each vertex is a tree with no edges, and A is empty
- each edge added connects two trees (or components)



Kruskal MST algorithm

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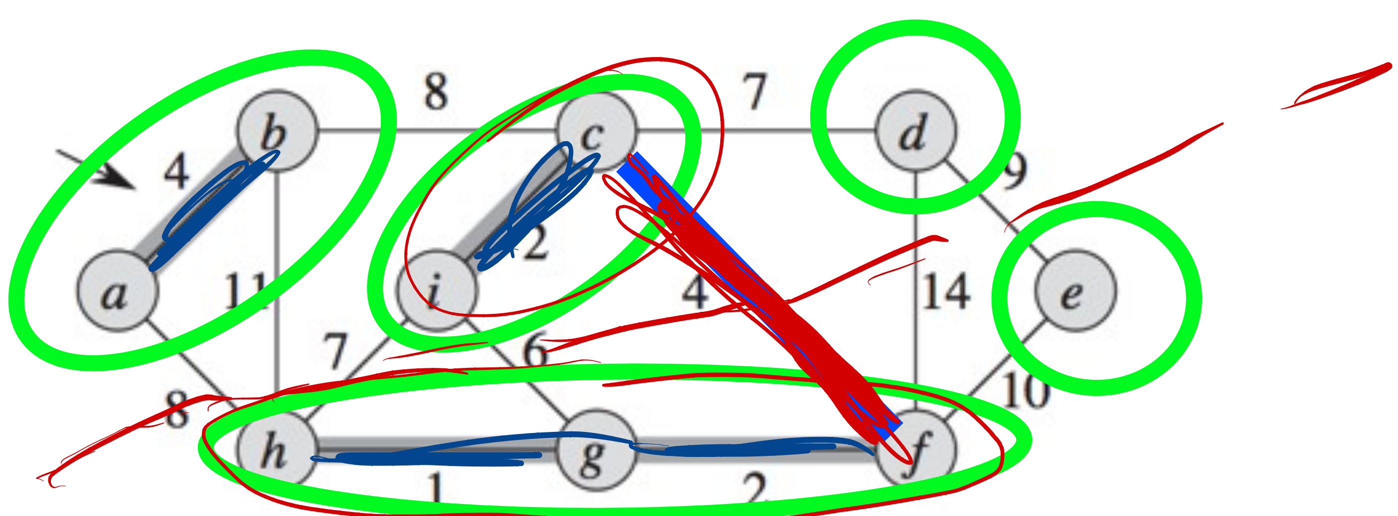
  - initially each vertex is a tree with no edges, and A is empty
- each edge added connects two trees (or components)
  - find the minimum weight edge (u,v) across two components, say connecting trees T1∋v and T2∋u (edges between nodes of the same trees are no good because they form cycles) (blue in the picture)



Kruskal MST algorithm

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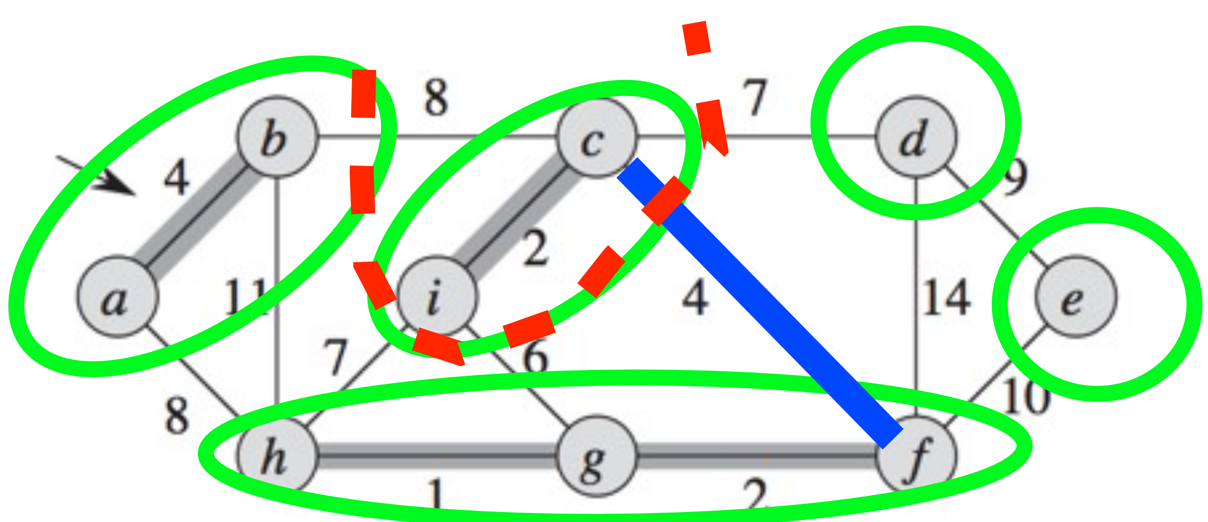

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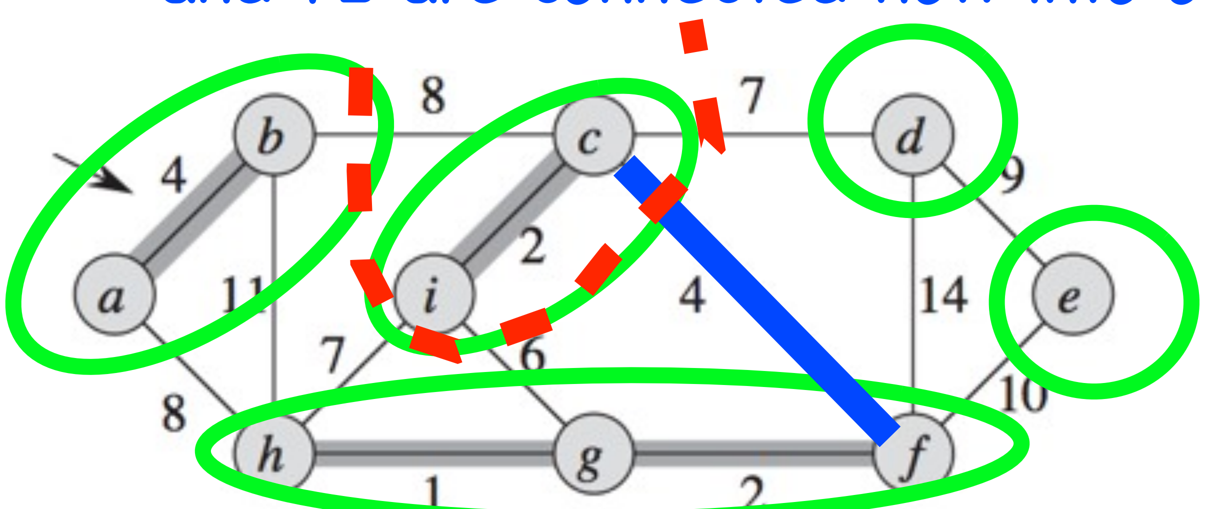
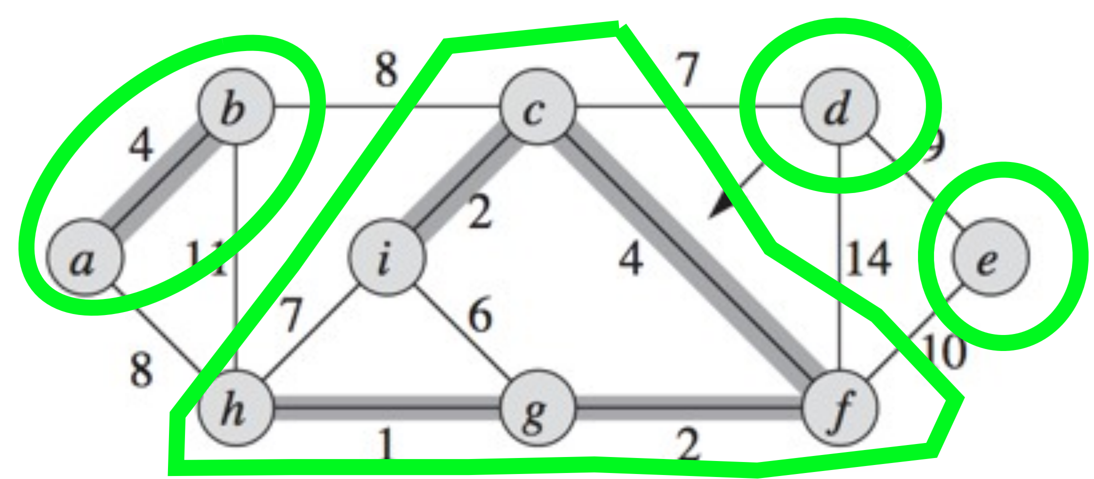
  - define cut (S,V-S); S = vertices of T1 (in red). This cut respects set A
  - edge (u,v) is the minimum cross edge, thus a safe edge to add to A. T1 and T2 are connected now into one tree





Kruskal algorithm

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)    edge valid
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

- Kruskal is simple
- implementation and running time depend on FIND-SET and UNION operations on the disjoint-set forest.
  - chapter 21 in the book, optional material for this course
- running time O(E logV)

MST algorithm comparison

- if you know graph density (edges to vertices)

	Kruskal	Prim with array implement.	Prim w/ binomial heap	Prim w/ Fibonacci heap	in practice
sparse graph $E=O(V)$	$O(V\log V)$	$O(V^2)$	$O(V\log V)$	$O(V\log V)$	Kruskal, or Prim+binom heap
dense graph $E=\Theta(V^2)$	$O(V^2\log V)$	$O(V^2)$	$O(V^2\log V)$	$O(V^2)$	Prim with array
avg density $E=\Theta(V\log V)$	$O(V\log^2 V)$	$O(V^2)$	$O(V\log^2 V)$	$O(V\log V)$	Prim with Fib heap, if graph is large