Amortized Analysis
Move to Front

Self-organizing lists

- List L of n elements
- The operation ACCESS(x) costs

$$rank_L(x) = \text{distance of } x \text{ from the head of } L.$$

- L can be reordered by swapping adjacent elements at a cost of 1
- Goal: access to a sequence of n items with minimal cost

List access algorithms

- Off-line Algorithm: if the sequence of access S is known in advance, one can design an optimal algorithm to rearrange the list based on how often items are accessed
- On-line Algorithm: if the sequence is not known in advance, one can design an algorithm based on some heuristics.

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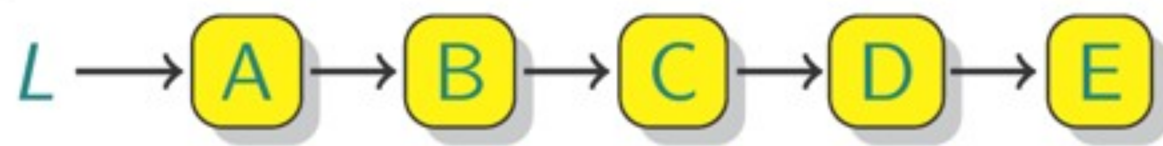

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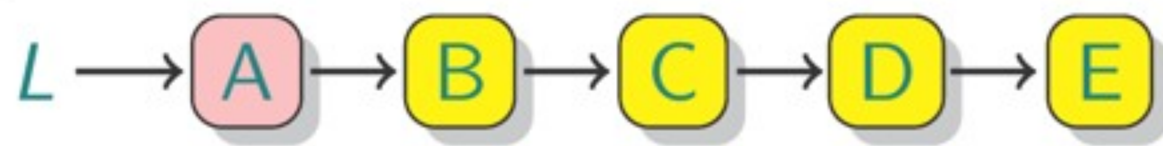

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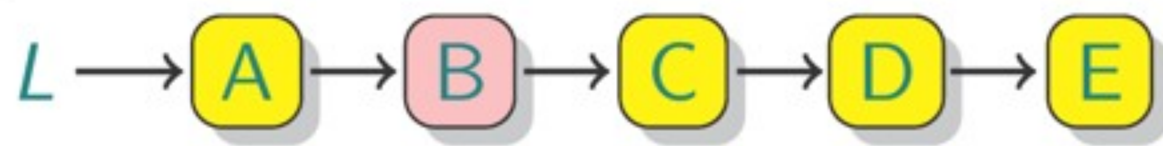

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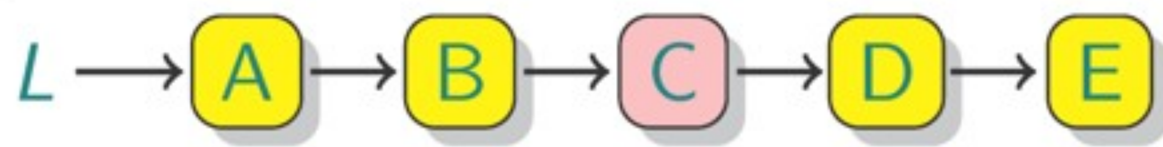

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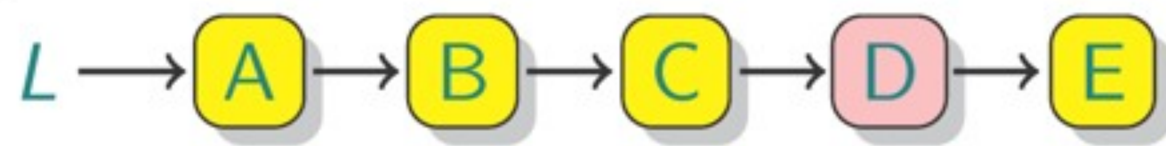

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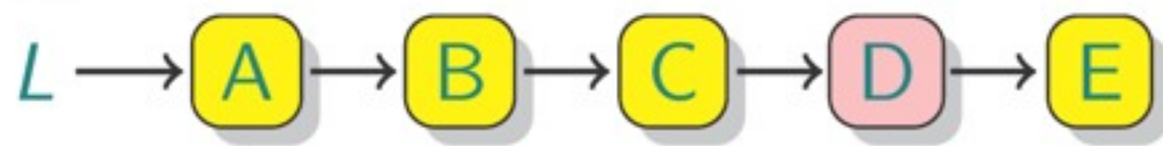

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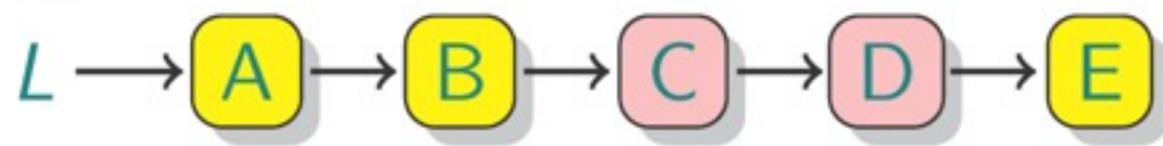

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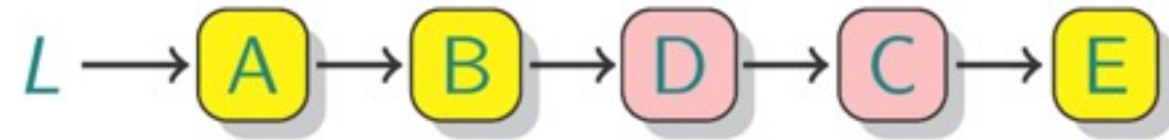

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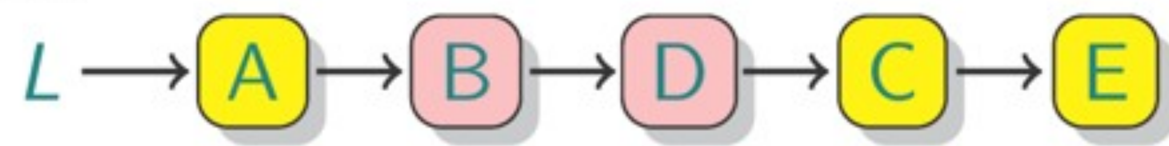

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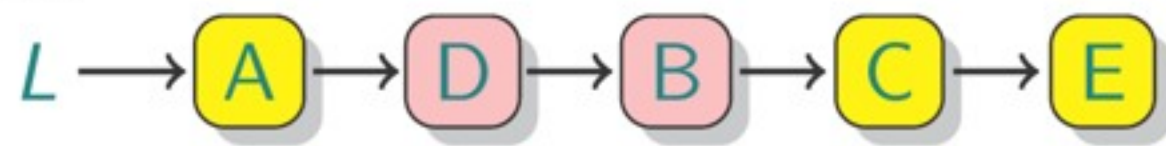

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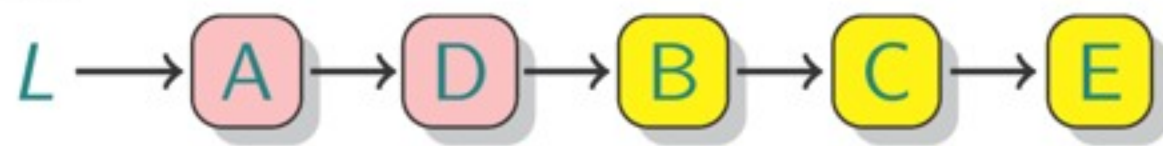

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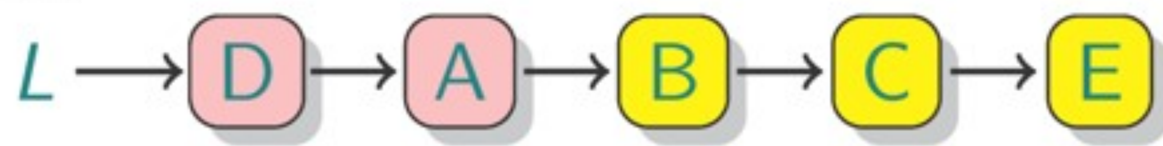

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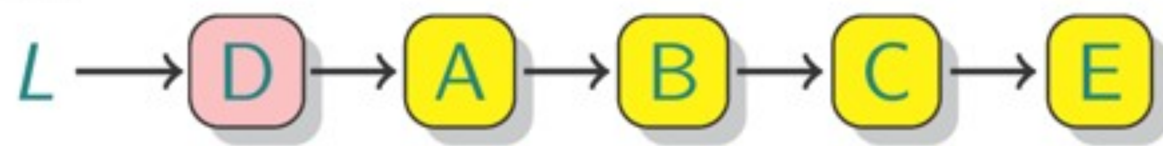

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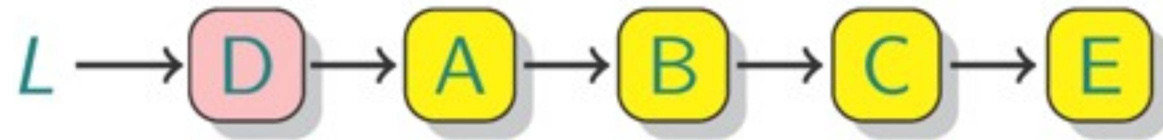


- Heuristic: if x is accessed at time t, it is likely to be accessed again soon after time t.
- Cost: MTF always performs within a factor of 4 of the optimal algorithm.

Amortized analysis of MTF

Theorem: $C_{MTF}(S) \leq 4C_{OPT}(S)$
Proof: Let L_i be MTF's list after the ith access, and let L_i^* be OPT's list after the ith access. Let

$$\begin{aligned} c_i &= \text{ MTF's cost for the } i\text{th operation} \\ &= 2 \cdot rank_{L_{i-1}}(x) \text{ if it accesses } x; \\ c_i^* &= \text{ OPT's cost for the } i\text{th operation} \\ &= rank_{L_{i-1}^*}(x) + t_i, \end{aligned}$$

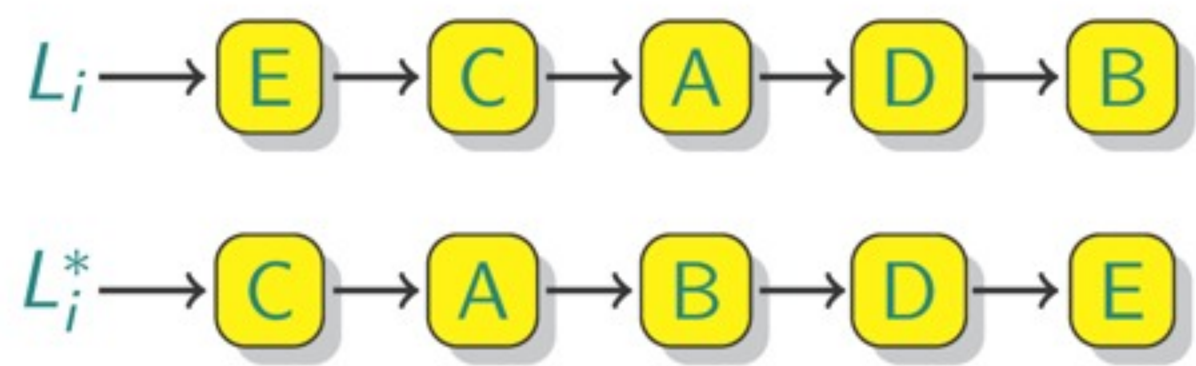
where t_i is the number of swaps that OPT performs.

Potential function

Define the potential function $\Phi : L_i \to \mathcal{R}$ by

$$\begin{aligned}\Phi(L_i) =& 2 \cdot |\{(x, y) : x \prec_{L_i} y \text{ and } y \prec_{L_i^*} x\}| \\ =& 2 \cdot \#\text{ inversions}\end{aligned}$$

Example:



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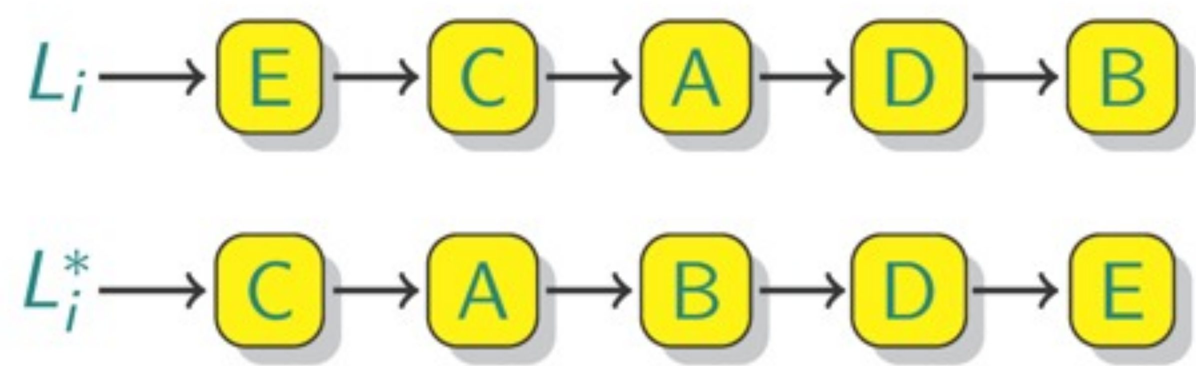

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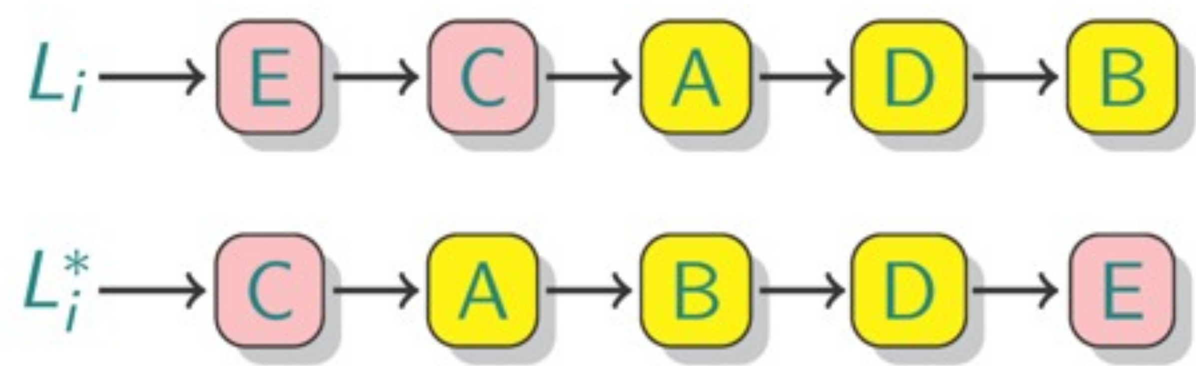

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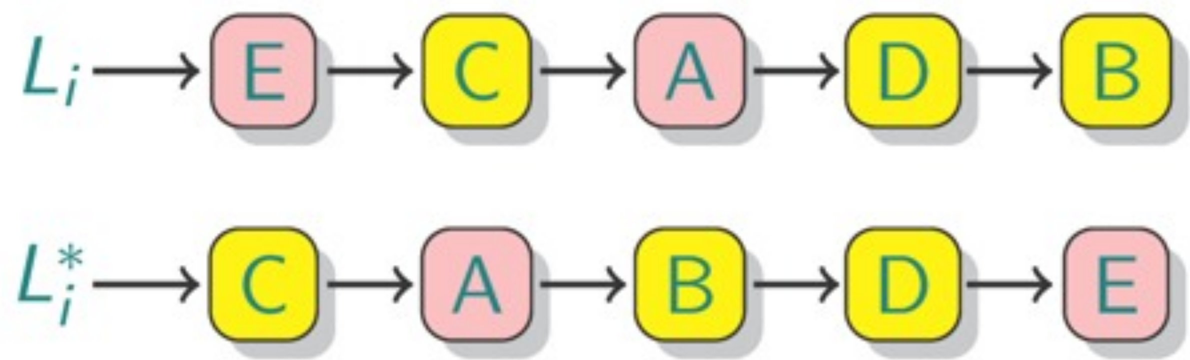

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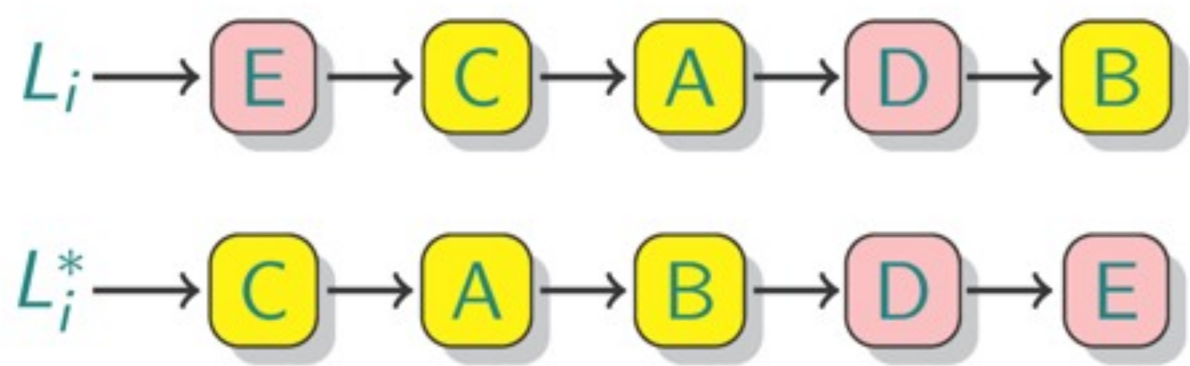

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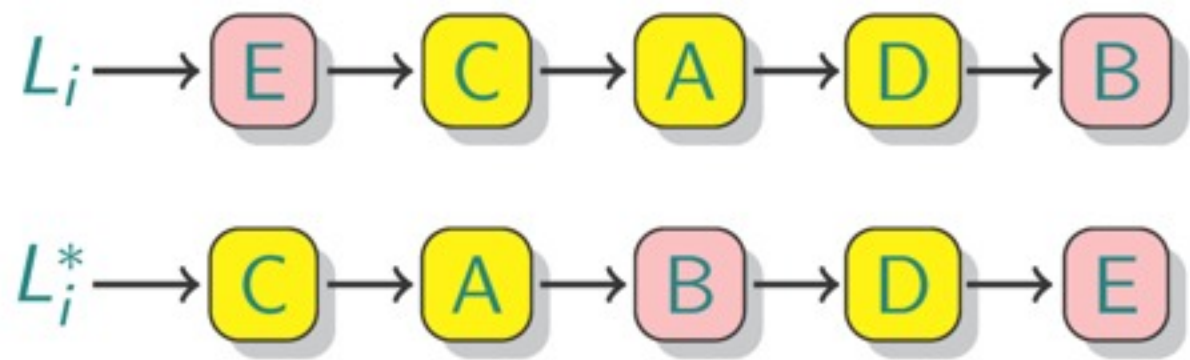

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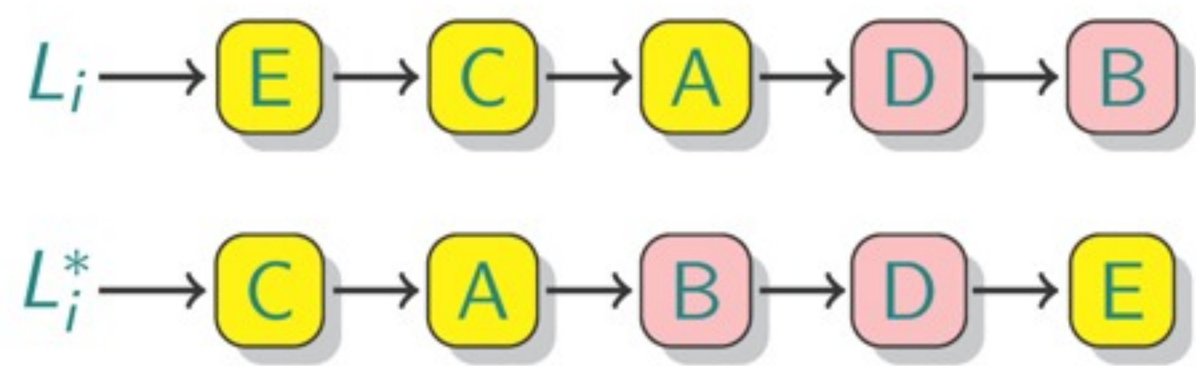

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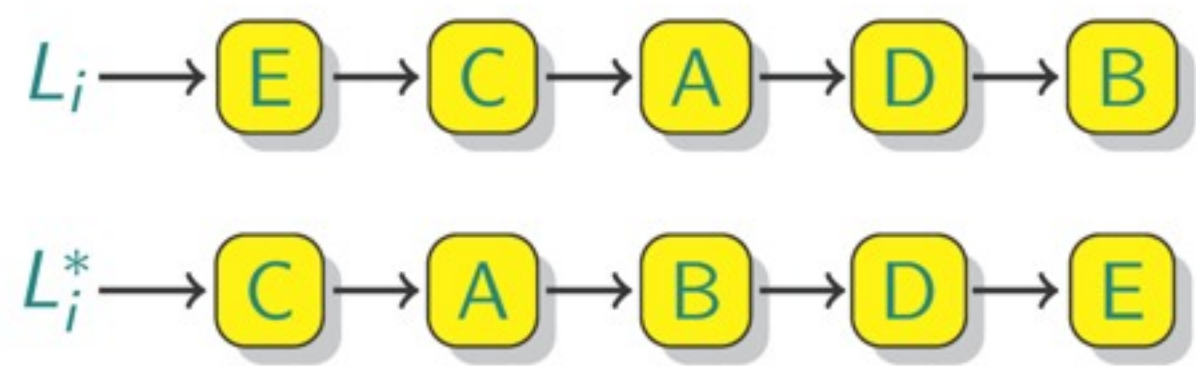

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Note that:

- $\Phi(L_i) \geq 0$ for $i = 0, 1, ...$
- $\Phi(L_0) = 0$ if MTF and OPT start with the same list.

How much does Φ change from one swap?

- a swap creates/destroys 1 inversion
- $\Delta\Phi = \pm 2$

What happens on access?

Suppose that operation i access item x, and define

$$A = \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \prec_{L^*_{i-1}} x\},$$
$$B = \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \succ_{L^*_{i-1}} x\},$$
$$C = \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \prec_{L^*_{i-1}} x\},$$
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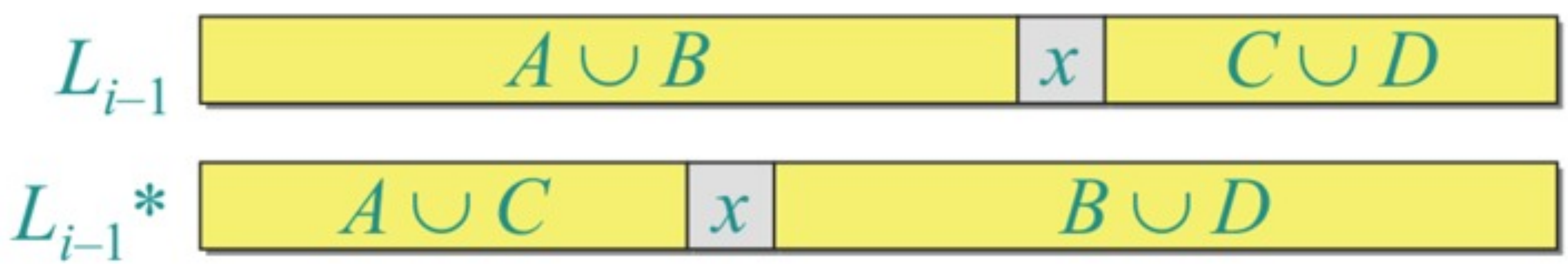

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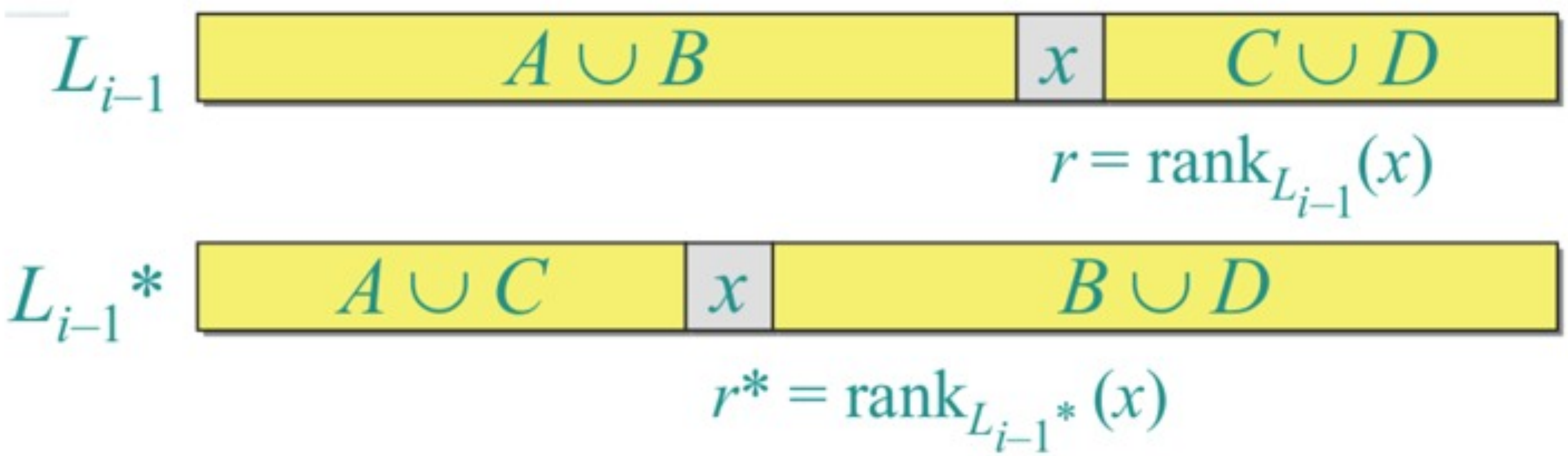

We have $r = |A| + |B| + 1$ and $r^* = |A| + |C| + 1$.
When MTF moves x to the front, it creates $|A|$ inversions and destroys $|B|$ inversions. Each swap by OPT creates ≤ 1 inversion. Thus, we have

$$\Phi(L_i) - \Phi_{L_{i-1}} \leq 2(|A| - |B| + t_i).$$

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\end{aligned}$$

(since $r = |A| + |B| + 1$)

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Thus, we have

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C_{MTF}(S) &= \sum_{i=1}^{|S|} c_i \\
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&\leq 4C_{OPT}(s)
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(since $\Phi(L_0) = 0$ and $\Phi(L_{|S|}) \geq 0$)