Dynamic Programming
part 2

Week 7 Objectives

- More dynamic programming examples
  - Matrix Multiplication Parenthesis
  - Longest Common Subsequence
- Subproblem Optimal structure
- Defining the dynamic recurrence
- Bottom up computation
- Tracing the solution

Subproblem Optimal Structure

- Divide and conquer - optimal subproblems
- divide PROBLEM into SUBPROBLEMS, solve SUBPROBLEMS
- combine results (conquer)
- critical/optimal structure: solution to the PROBLEM must include solutions to subproblems (or subproblem solutions must be combinable into the overall solution)
- PROBLEM = {DECISION/MERGING + SUBPROBLEMS}

Optimal Structure - NON GREEDY

- Cannot make a choice decision/CHOICE without solving subproblems first
- Might have to solve many subproblems before deciding which results to merge.

Matrix Multiplication (Parenthesis)

- Task: multiply matrices $A_1*A_2*...*A_n$
- Ai matrix has p_{i-1} rows and p_i columns (size p_{i-1} X p_i)
  - #rows of matrix A_{i+1} has to be the same as #columns of A_i
- Minimize the number of scalar multiplications
- Note that matrices can be multiplied in any order:
  - $A_1*(A_2*A_3)*A_4$; $(A_1*A_2)*(A_3*A_4)$; $A_1*(A_2*A_3*A_4)$
  - A_1(size p_0xp_1) * A_2(size p_1xp_2) takes $p_0*p_1*p_2$ scalar multiplications
  - order matters, example: A_1(10x100), A_2(100x5); A_3(5x50) (p_0= 10; p_1=100; p_2=5; p_3=50)
    - then $A_1*(A_2*A_3)$ takes 75000 scalar multiplications
    - while $(A_1*A_2)*A_3$ takes 7500 scalar multip., 10 times less.

Matrix Multiplication (Parenthesis)

- NAIVE SOLUTION: try all ways to put parenthesis to see which one is best/minimum
  - $A_1*((A_2*A_3)*A4)$; $(A_1*A_2)*(A_3*A_4)$; $A_1*(A_2*(A_3*A4))$
  - $((A_1*A_2)*A_3)*A_4$; $(A_1*(A_2*A_3))*A_4$
- P(n) = number of ways to parenthesize n matrices
- recursion on n

$$P(n) = \begin{cases} 1 & \text{if } n = 1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2 \end{cases}$$

- why? proof this recursion
- show that this P(n) is exponential in n

Matrix Multiplication (Parenthesis)

- 1) characterize optimal solution structure
- optimal solution SOL parenthesis has a "main split", or "last product" - that is the last matrix multiplication
  - say it is between matrices A_k and A_{k+1}

$$\overbrace{((A_i A_{i+1} \ldots A_k)}^{\text{prefix subchain}}\overbrace{(A_{k+1} A_{k+2} \ldots A_j))}^{\text{suffix subchain}}$$

- then SOL parenthesis on the left side (A_i*...*A_k) must be optimal
- same for right side: parenthesis on (A_{k+1}*...*A_j) must be optimal
  - why? use an exchange argument

A_1	A_2	A_3
5×2	2×10	10×3

5×10

$(A_1 \times A_2) \times A_3$ cmp $5 \times 10 \times 2 + 5 \times 10 \times 3 = 250$

$A_1 \times (A_2 \times A_3)$

5×2 $\quad$ 2×3

$2 \times 10 \times 3 + 5 \times 2 \times 3 = 90$

Matrix Multiplication (Parenthesis)

- 2) dynamic programming recursion
- C[i,j] = min scalar multip. to multiply $A_i*A_{i+1}*...*A_j$
  - C[i,i]=0; C[i,i+1] = $p_{i-1}*p_i*p_{i+1}$
- $A_i*A_{i+1}*...*A_j$ can be computed by first deciding the main split at some k, 1<k<j
  - for that split C[i,j] = C[i,k] + C[k+1,j] + pi-1*pk*pj

C[i,k] $p_{i-1}*p_k*p_j$ C[k+1,j]

$$((A_i A_{i+1} \ldots A_k)(A_{k+1} A_{k+2} \ldots A_j))$$

$p_{i-1} \times p_k$ $p_k \times p_j$

$i \leq j$

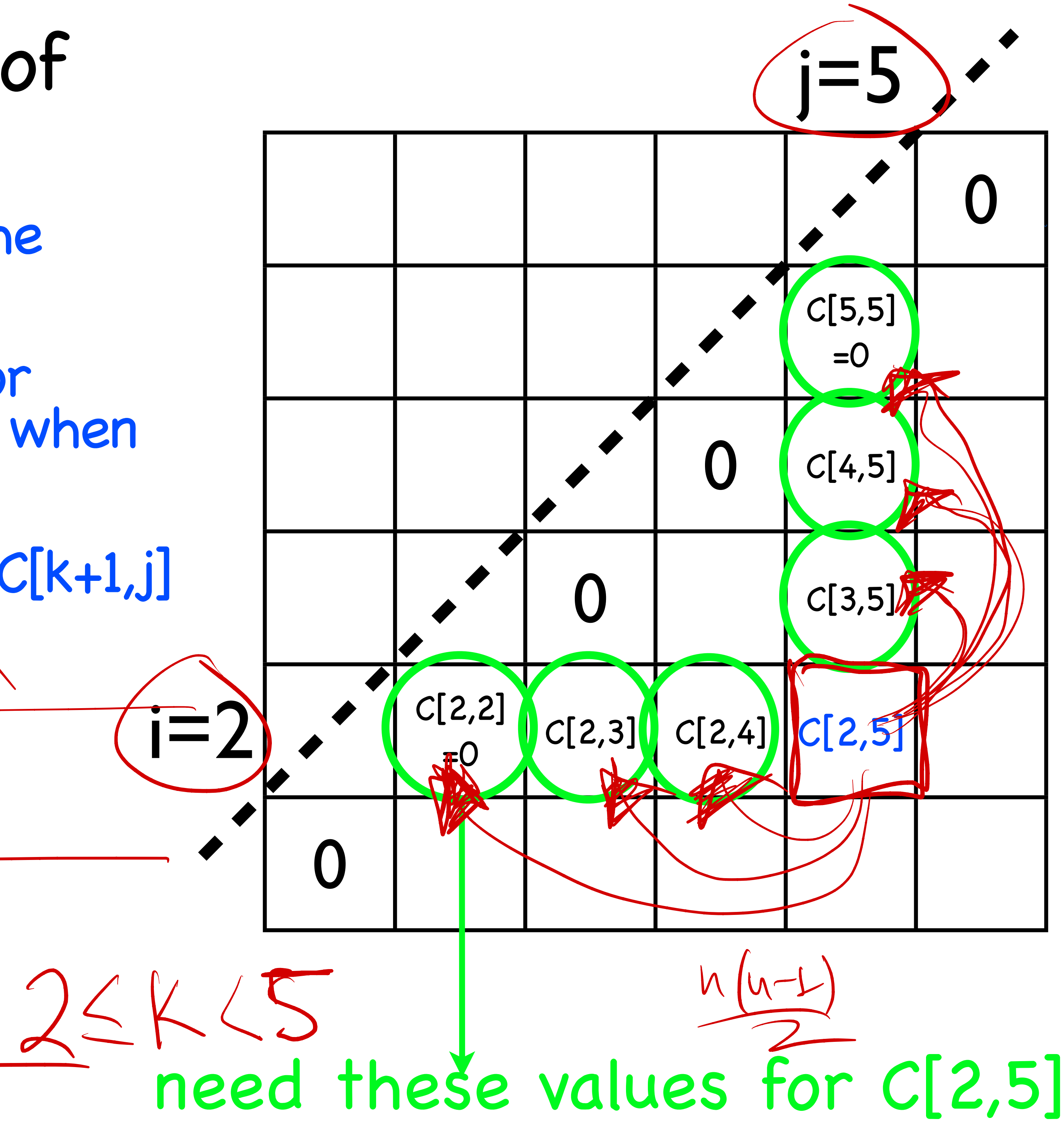
  - but we dont know what k is best, so we have to try all of them

$$C[i,j] = \begin{cases} 0 & \text{if } i = j, \\ \min_{i \leq k < j} \{C[i,k] + C[k+1,j] + p_{i-1} p_k p_j\} & \text{if } i < j. \end{cases}$$

Matrix Multiplication (Parenthesis)

- 3) bottom up computation of table C[]
  - what is the right order to fill the table?
  - guarantee that values needed for recursion are already computed when we compute C[i,j]
  - might need any value C[i,k] and C[k+1,j]

	Bottom Up	Memoization
recompute a pb	X	X
max # of pb (whole table)	$\frac{n(n-1)}{2}$	$\frac{n(n-1)}{2}$
compute ALL pbs	YES	No



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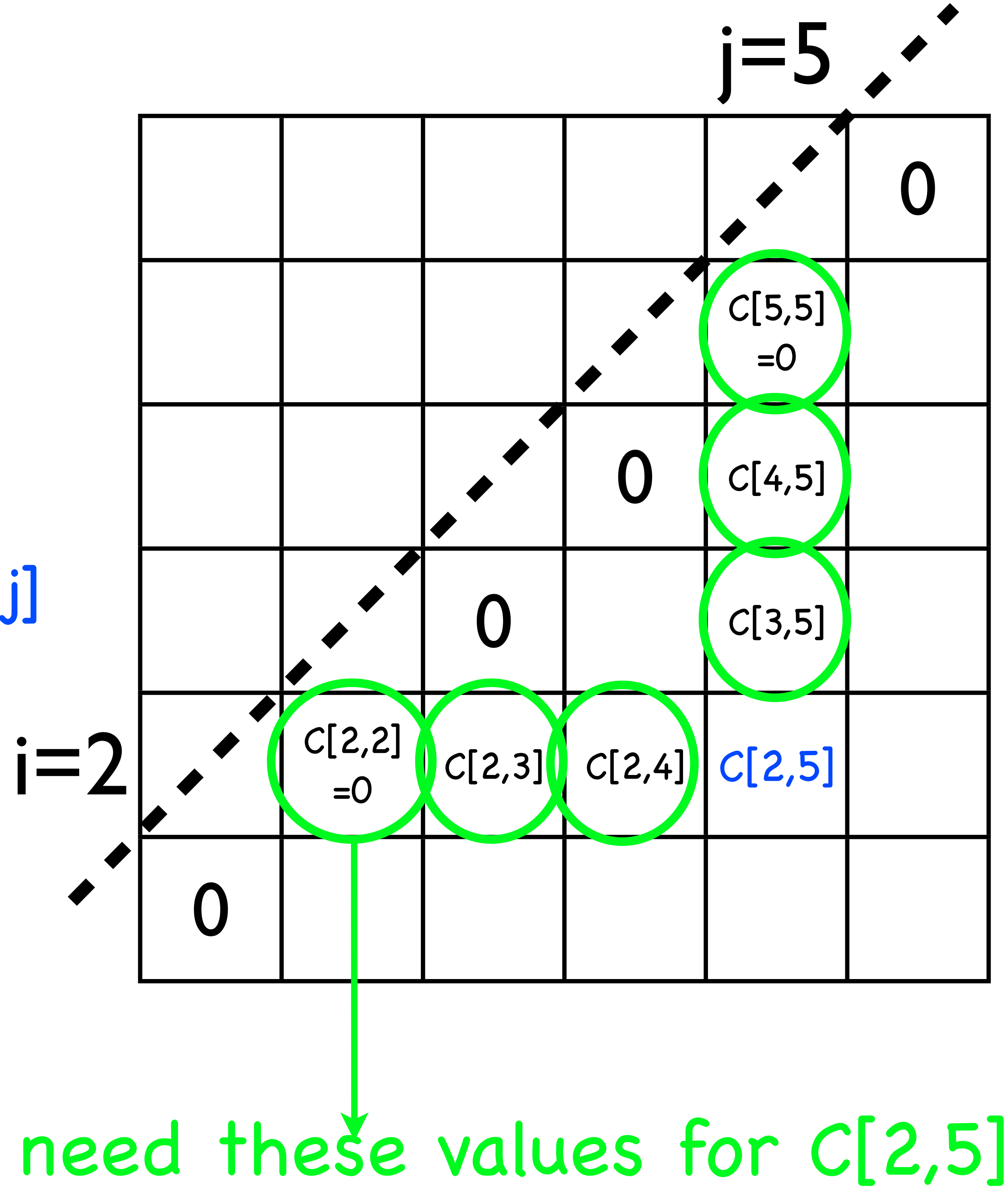

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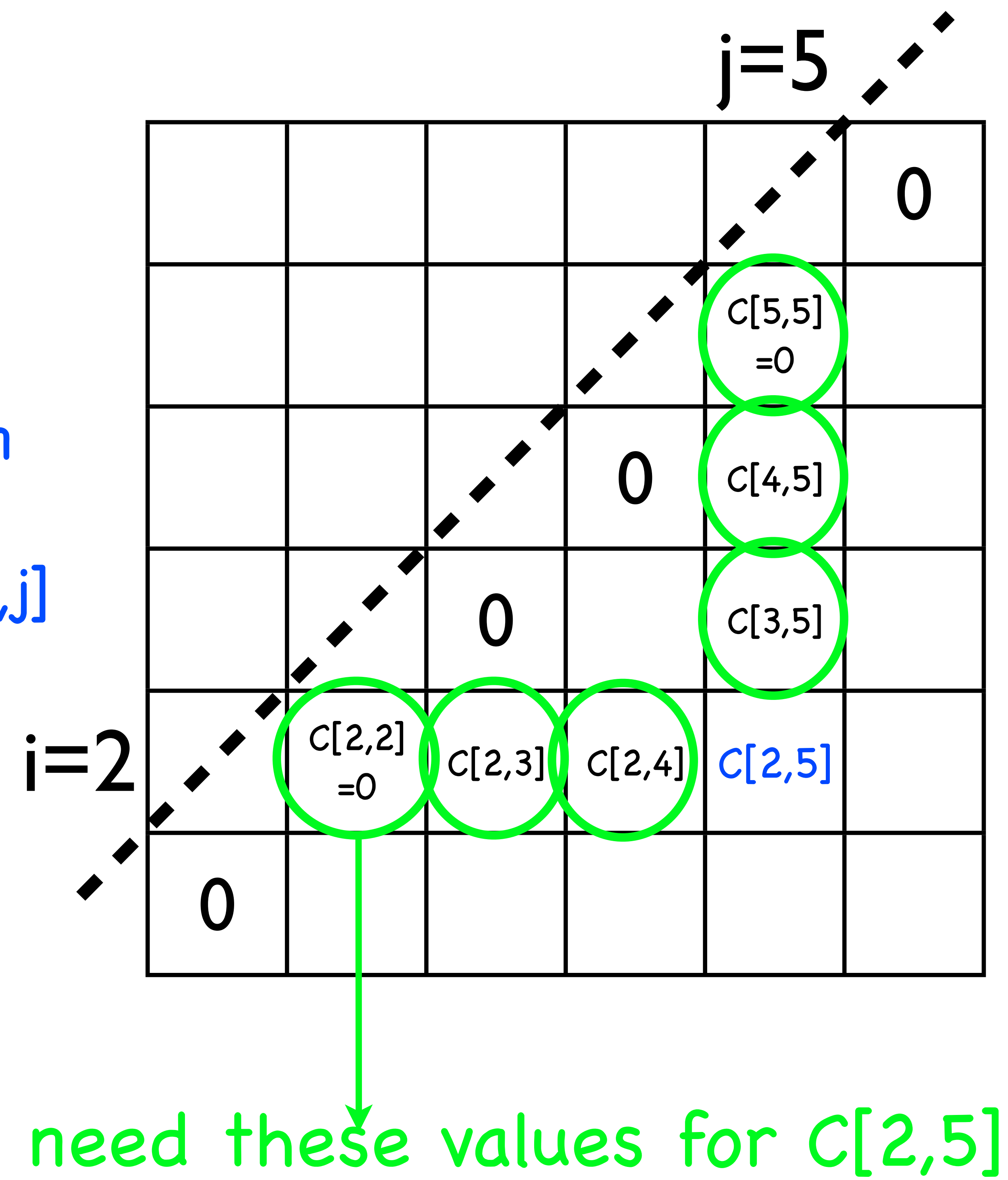
- fill table C[] by length
  - from cells with small length (main diagonal) to cells of high lengths (corners)



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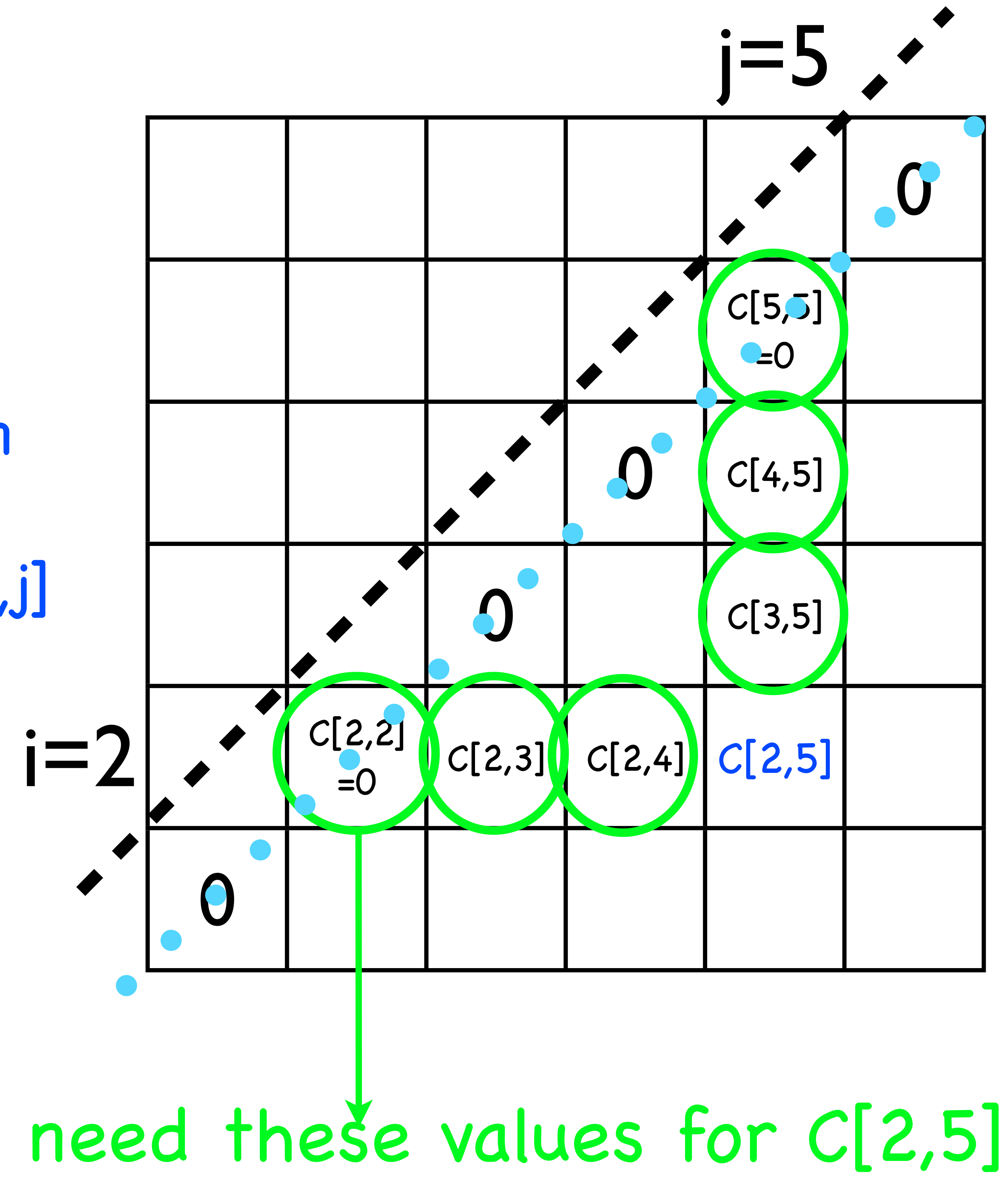

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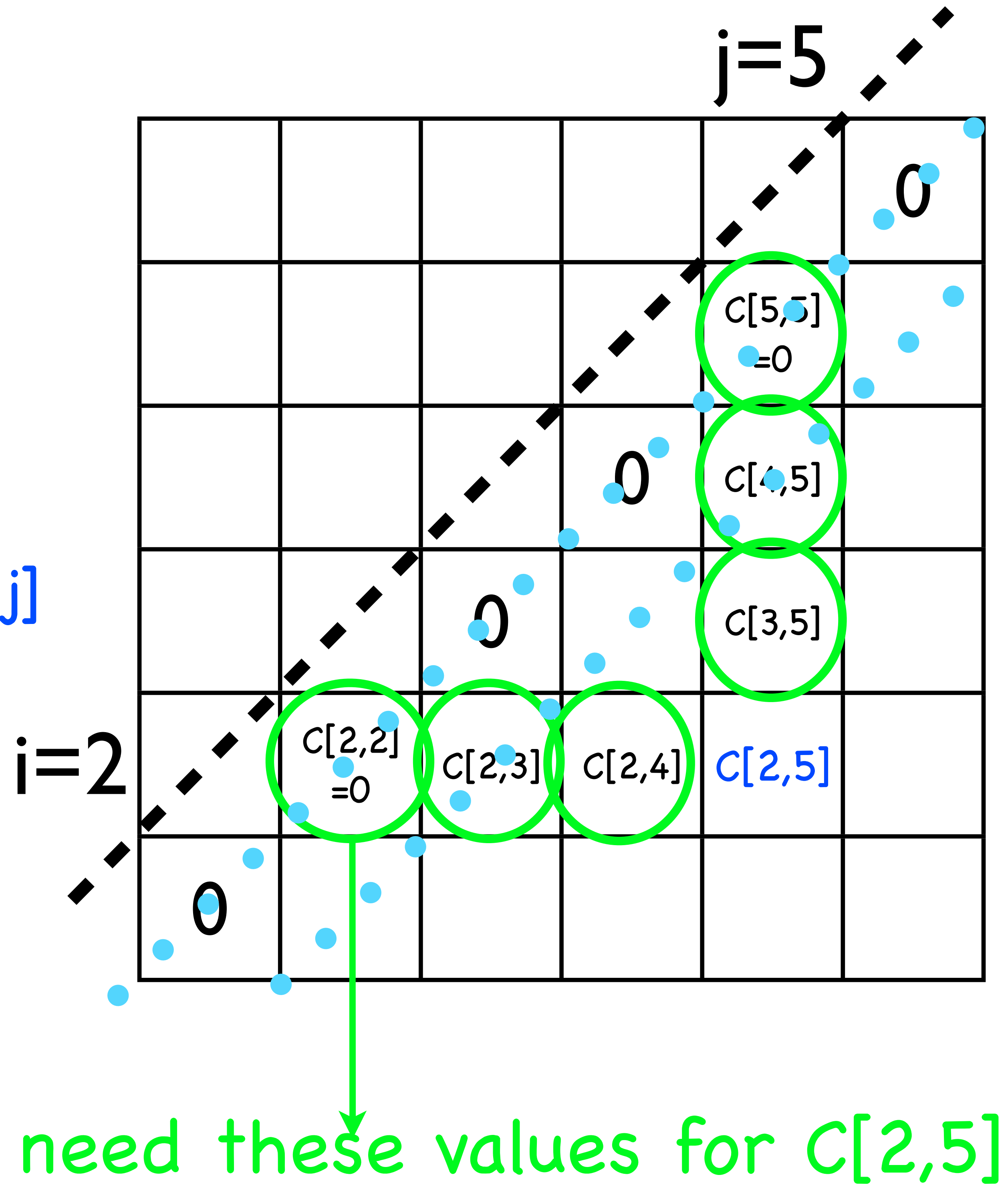

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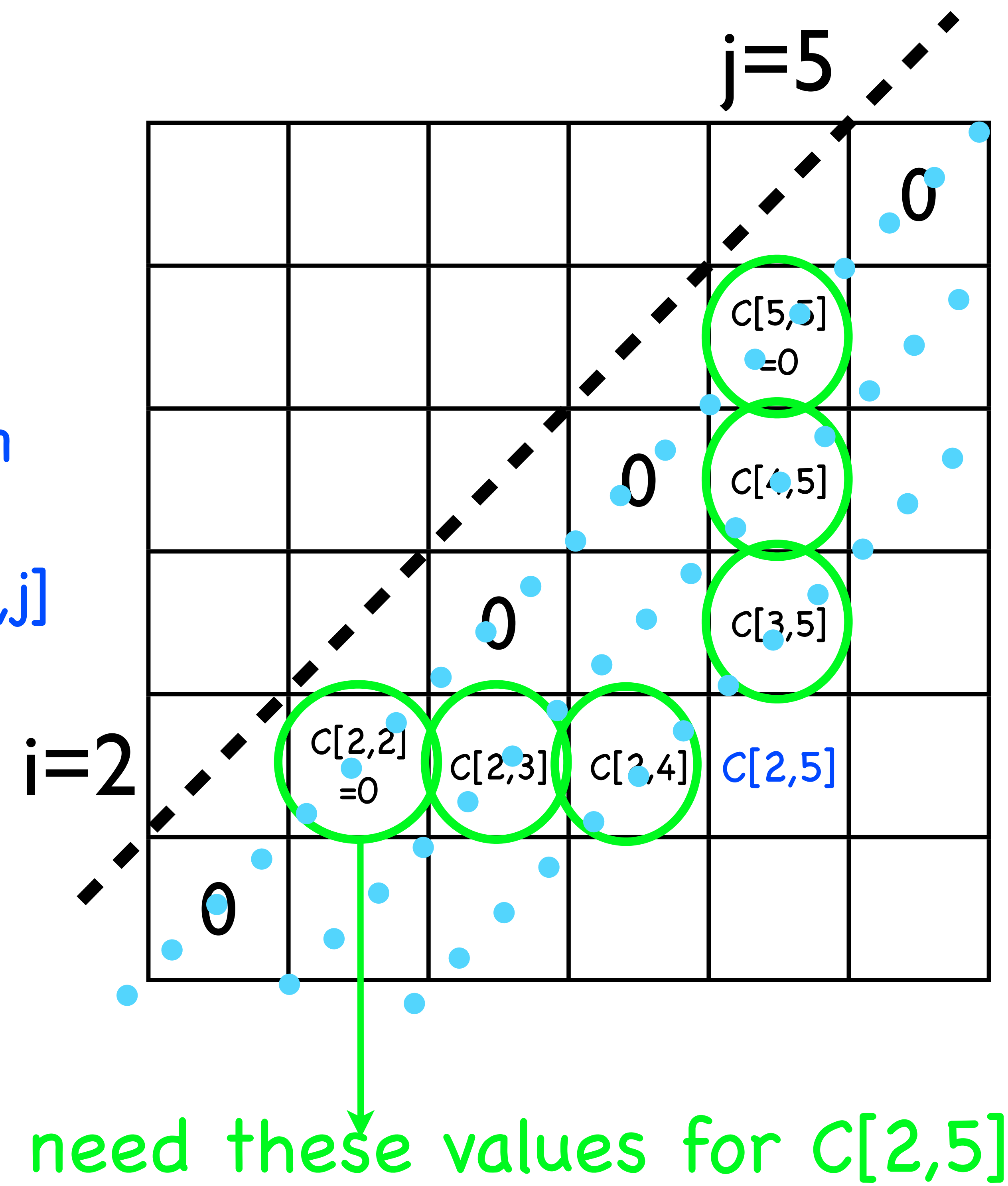

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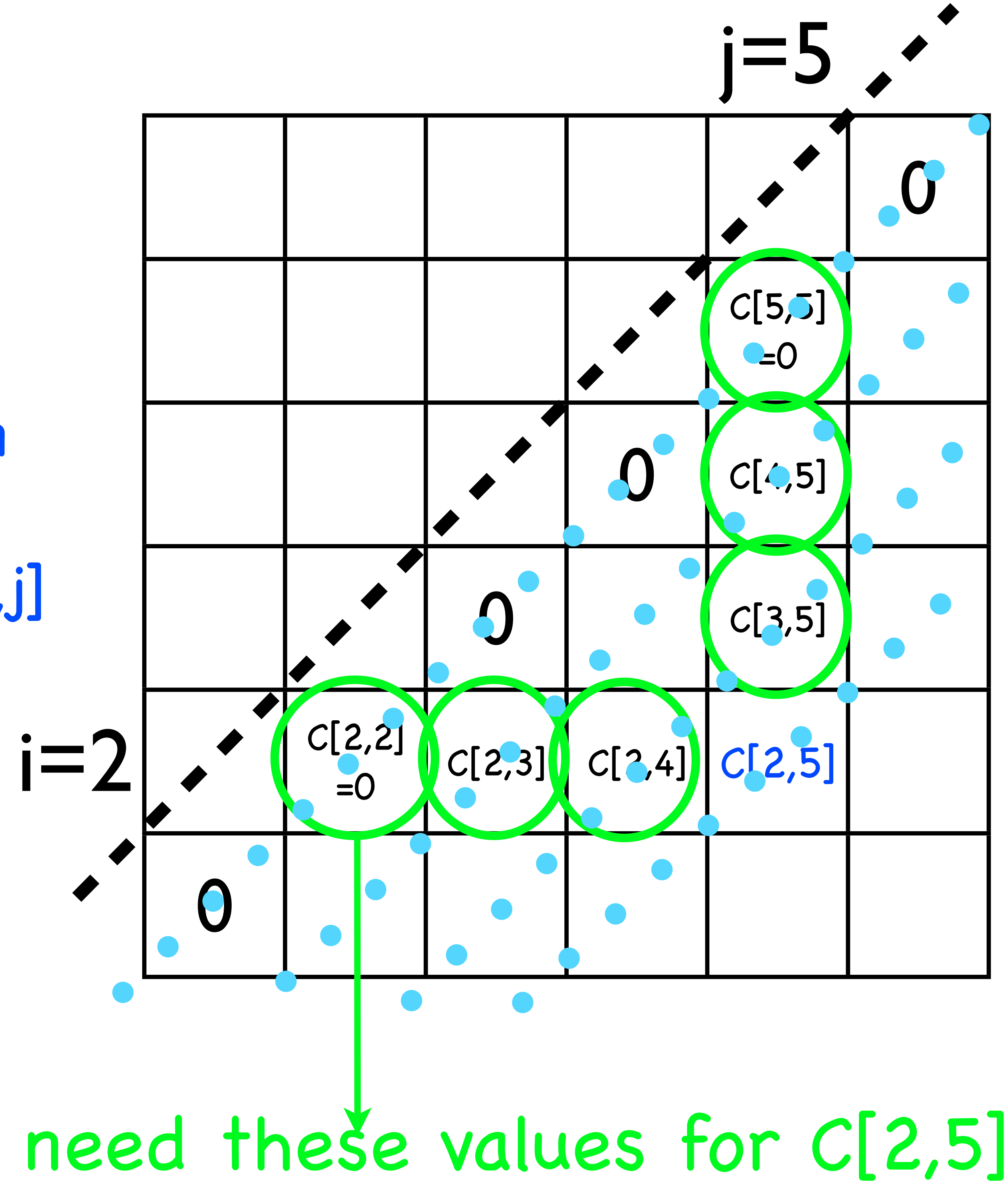

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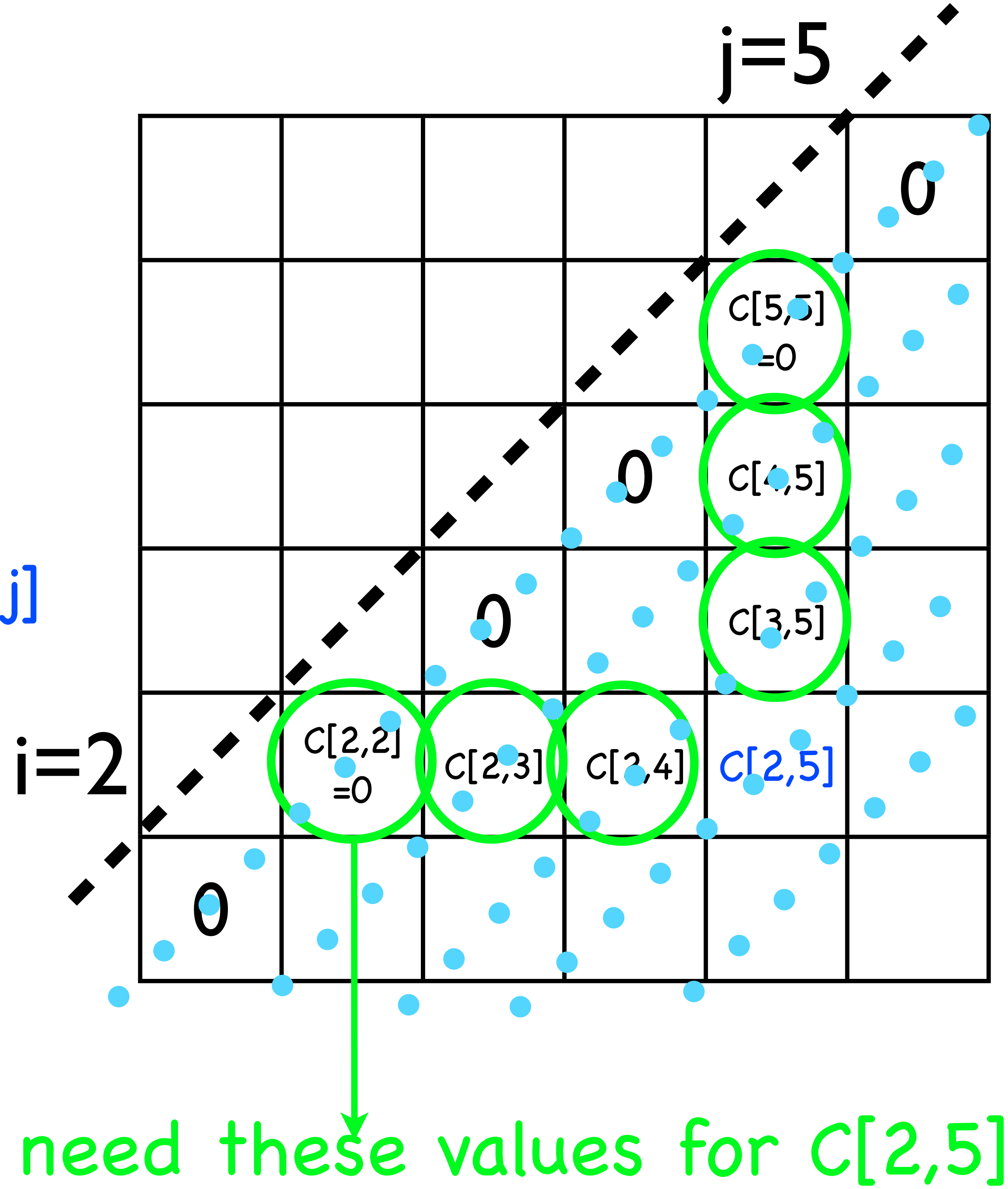

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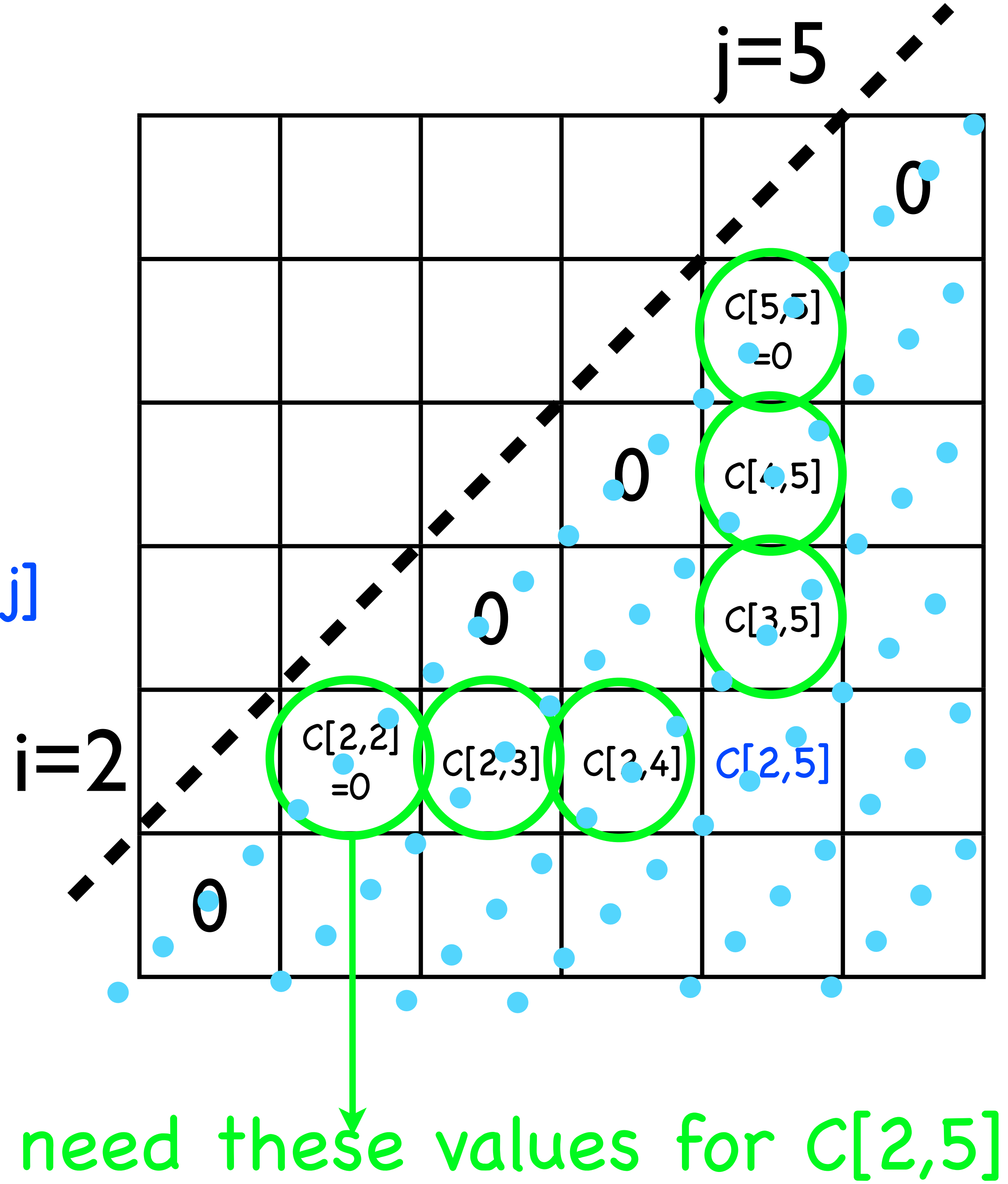


Matrix Multiplication (Parenthesis)

- 3) Bottom-up computation of C[]
  - by diagonal from short length, to long length
- keep track of split at k, for sequence [i...j]: S[i,j]=k
  - $A_i * A_2 * \ldots A_j$ multiplied best as $(A_i * A_{i+1} * \ldots * A_k)(A_{k+1} * \ldots * A_j)$

```
MATRIX-CHAIN-ORDER(p)
n = p.length - 1
let C[1..n, 1..n] and S[1..n - 1, 2..n] be new tables
for i = 1 to n
    C[i, i] = 0
for l = 2 to n  // l is the chain length
    for i = 1 to n - l + 1
        j = i + l - 1
        C[i, j] = 0
        for k = i to j - 1
            q = C[i, k] + C[k + 1, j] + p_{i-1} p_k p_j
            if q < C[i, j]
                C[i, j] = q
                S[i, j] = k
return C and S
```

Matrix Multiplication (Parenthesis)

- 4) Trace the solution - Exercise
  - use S[i,j] to determine the main split
  - run recursion on both sides of the split
- also calculate the running time of the trace

Matrix Multiplication (Parenthesis)

- Running time
  - C[] table fills about 1/2 * n * n cells – $\Theta(n^2)$ cells
  - each cell C[i,j] tries all k ; $1 \leq k < j$ – $\Theta(n)$ steps
- Total $\Theta(n^3)$ time for bottom up computation
- Trace solution: certainly lower than $\Theta(n^3)$, so it doesnt add to the running time asymptote.

Top-down computation instead of bottom up

- Suppose we want to do the computation top down
- Recursively follow the recursion

```
Rec-Matrix-Chain(p,i,j) //bad running time
    if(i==j) return 0;
    m[i,j]=∞
    for k=i:j-1
        q=Rec-Matrix-Chain(p,i,k) + Rec-Matrix-Chain(p,k+1,j) + p_{i-1}p_k p_j;
        if (q<m[i,j]) m[i,j]=q;
    return m[i,j]
```

- Exponential number of calls VS bottom up which is only $\Theta(n^2)$ for this section of the code

Top-down with memoization

- memoization: "store, dont recompute" the computed results; each actual computation only happen once
- init all m[i,j]=∞; call MEMOIZATION-top-down(p,1,n)

```
MEMOIZATION-top-down(p,i,j)
    if (m[i,j]<∞) return m[i,j] // look up previous computed values
    if(i==j) m[i,j] = 0;
    else for k=i:j-1
        q=Rec-Matrix-Chain(p,i,k) + Rec-Matrix-Chain(p,k+1,j) + p_{i-1}p_kp_j;
        if (q<m[i,j]) m[i,j]=q; //store value for future look up
    return m[i,j]
```

rec calls

Memoization

- now same running time as bottom-up : $\Theta(n^3)$ overall
- bottom-up (DP) VS top-down (Memoization):
  - DP advantage: no overhead (stack of calls, recursion), efficient when the whole table has to be computed anyway
  - DP requires a certain fill-order of the table
  - Memoization: better when not all values must be computed
  - Memoization follow literally the recursionl; easier to implement

Longest Common Subsequence
(LCS)

Longest Common Subsequence

- Given two $X=(x_1, x_2, \ldots, x_m)$ and $Y=(y_1, y_2, \ldots, y_n)$ find the longest common subsequence
  - it doesnt have to be continuos in either X or Y
  - not unique: possible that several common sequences have maximum length
- example
  - X=(absscddegt) Y=(xasbsdcggg)
  - LCS=Z=(absdg)

Longest Common Subsequence

- 1) Characterize optimal solution structure - (add general army- needs more cannons story)
  - notation: $X_{m-1} = (x_1, x_2, \ldots, x_{m-1})$; $Y_{n-1} = (y_1, y_2, \ldots, y_{n-1})$ etc
- if $X=(x_1, x_2, \ldots, x_m)$ and $Y=(y_1, y_2, \ldots, y_n)$ have an LCS $Z=(z_1, z_2, \ldots, z_k)$ then
  - if $x_m = y_n$; then $z_k = x_m = y_n$ and $Z_{k-1} = LCS(X_{m-1}, Y_{n-1})$
  - if $x_m \neq y_n$ and $z_k \neq x_m$ then $Z = LCS(X_{m-1}, Y)$ $= (x_1, x_2, \ldots x_{m-1})$
  - if $x_m \neq y_n$ and $z_k \neq y_n$ then $Z = LCS(X_m, Y_{n-1})$ $\to (y_1, y_2, \ldots y_{n-1})$

$c[i,j]$ = longest (size) subseq $(x_1, x_2, \ldots x_i) = X_i$ and $(y_1, y_2, \ldots y_j) = Y_j$

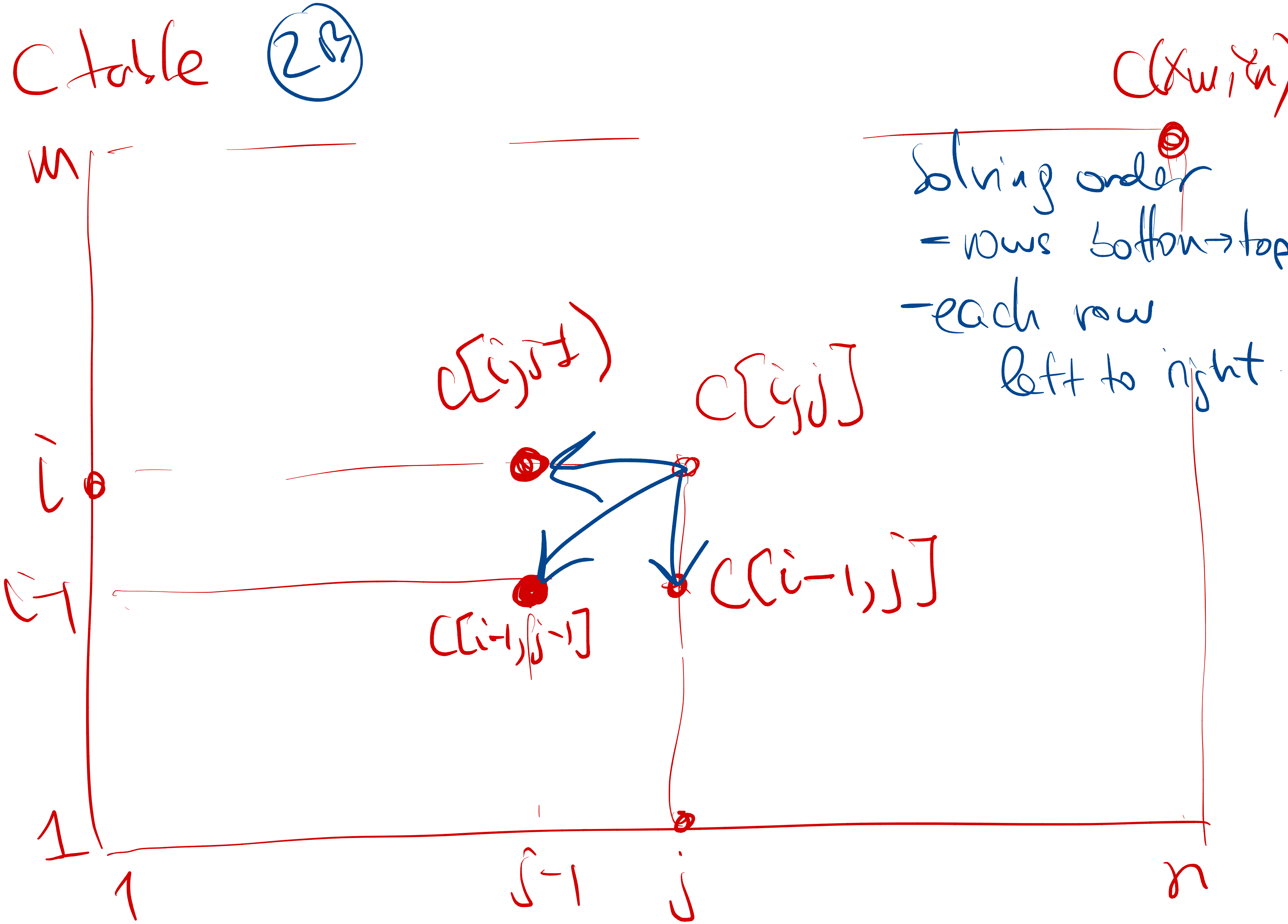
$= \{ \cdot\ 1 + c[i-1, j-1]$ if $x_i = y_j$

Longest Common Subsequence

- 2) dynamic recursion
- $C[i,j]$ = LCS (X_i, Y_j) where $X_i=(x_1, x_2, \ldots x_i)$ $Y_j=(y_1, y_2, \ldots y_j)$
- $C[i,j]$ is
  - 0 ; for base case i=0 or j=0
  - $C[i-1,j-1]+1$; for i,j>0 and xi=yj
  - max $\{C[i-1,j], C[i,j-1]\}$; for i,j>0 and xi≠yj

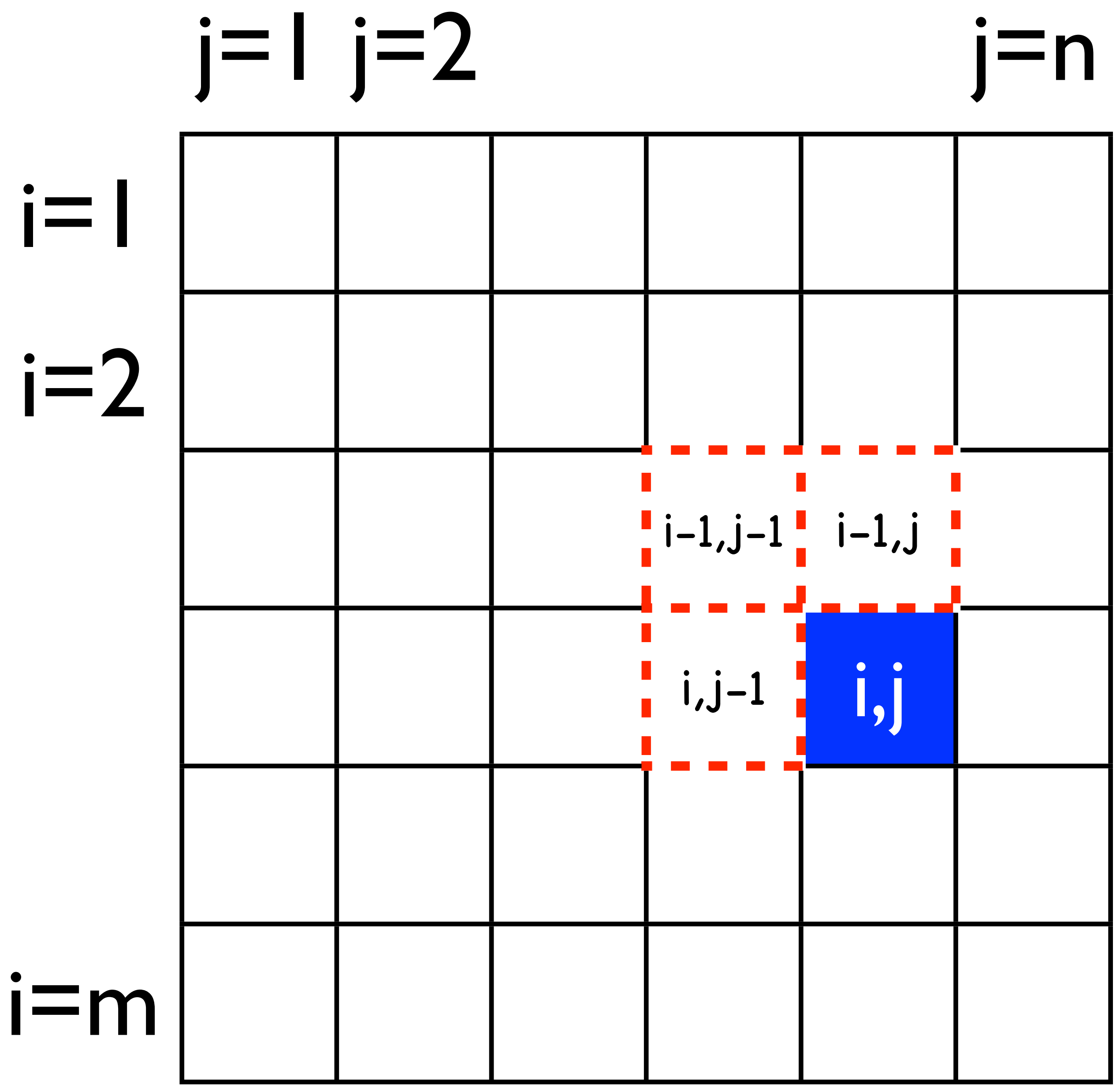
LCS[$(X_1, \ldots X_{i-1}), (Y_1, \ldots Y_j)$]

LSC[$(X_1, \ldots X_i), (Y_1, \ldots Y_{j-1})$]


C table
2B
C(Xm, Yn)
m
i
i-1
1
1
j-1
j
n
c[i, j-1]
c[i, j]
c[i-1, j-1]
c[i-1, j]
Solving order
- rows bottom → top
- each row
left to right

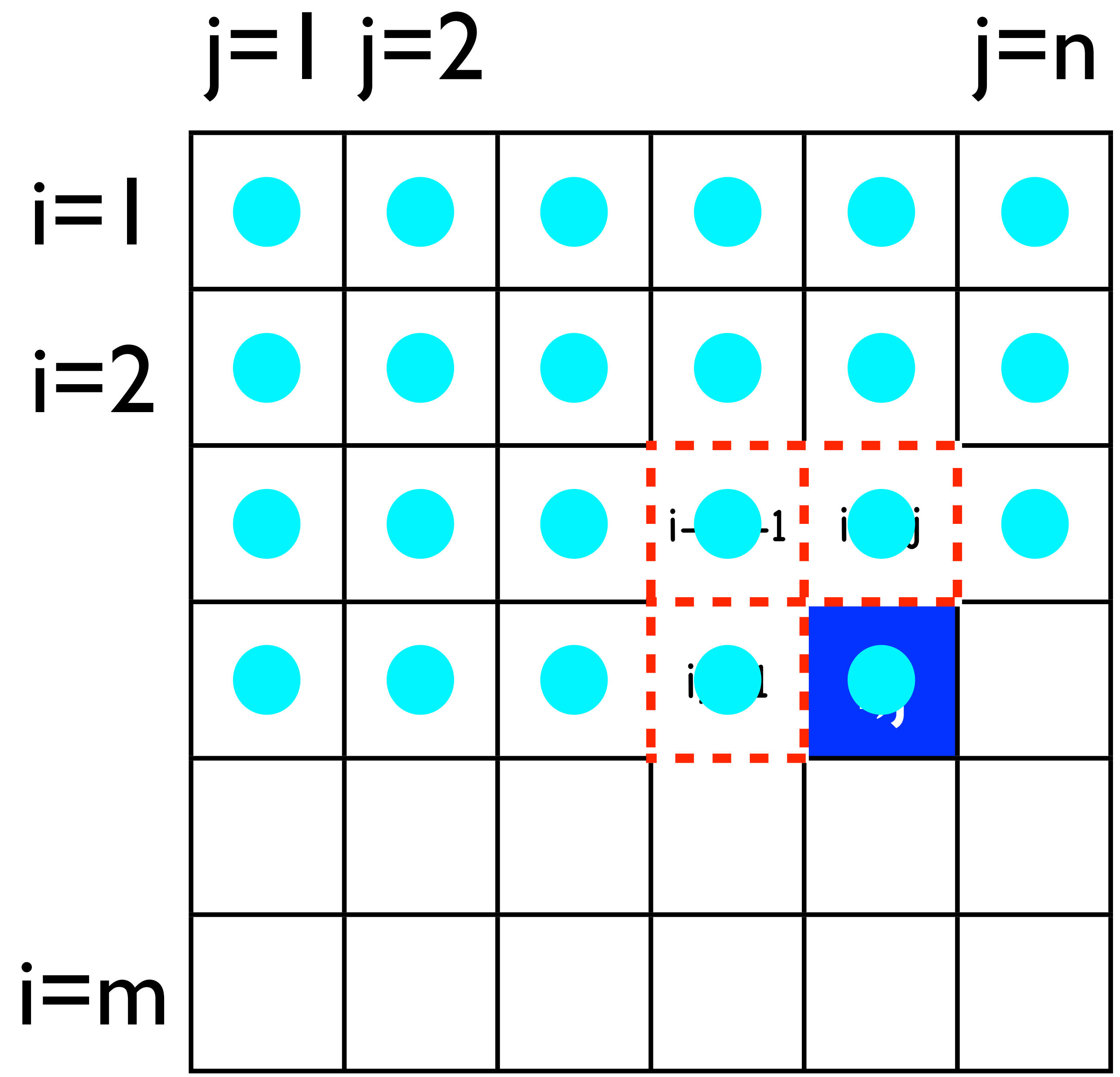
Longest Common Subsequence

- 3) bottom up computation
- in order to compute C[i,j] we need to have already computed the following three values:
  - C[i-1,j-1]
  - C[i,j-1]
  - C[i-1,j]



Longest Common Subsequence

- 3) bottom up computation
- in order to compute C[i,j] we need to have already computed the following three values:
  - C[i–1,j–1]
  - C[i,j–1]
  - C[i–1,j]
- fill row by row, each row from left to right



Longest Common Subsequence

- 3) bottom up computation
- keep track of the solution: S[i,j] remembers which one of the three possibilities we used:
  - C[i–1,j–1] + 1 ; S[i,j] ="↖"
  - C[i,j–1] ; S[i,j] =" ↑ " ;
  - C[i–1,j] ; S[i,j]="←"

```
LCS-LENGTH(X, Y)
m = X.length
n = Y.length
let S[1..m, 1..n] and C[0..m, 0..n] be
for i = 1 to m
    C[i, 0] = 0
for j = 0 to n
    C[0, j] = 0
for i = 1 to m                      rows
    for j = 1 to n                  inside a row
        if x_i == y_j
            C[i, j] = C[i − 1, j − 1] + 1
            S[i, j] = "↖"
        elseif C[i − 1, j] ≥ C[i, j − 1]
            C[i, j] = C[i − 1, j]
            S[i, j] = "↑"
        else C[i, j] = C[i, j − 1]
            S[i, j] = "←"
return C and S
```

Longest Common Subsequence

- 3) bottom up computation
  - illustrated are C[] and S[] tables on the same grid
  - C[i,j] is the size of LCS(X_i,Y_j)
- S[i,j] is the arrow pointing to the subproblem
  - "↖" indicates a common item, part of LCS; subproblem decreases both i and j
  - "↑" indicates discarding last vale of X_i; decrease i
  - "←" indicates discarding last value of Y_{j}; decrease j

i \ j		0	1	2	3	4	5	6
	y_j		(B)	D	(C)	A	(B)	(A)
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	(B)	0	↖ (1)	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ (2)	← 2	↑ 2	↑ 2
4	(B)	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ (3)	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	(A)	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ (4)
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

Longest Common Subsequence

- 4) trace solution
- start at S[m,n], follow arrows:
- every “↖”r means a common item is found by LCS

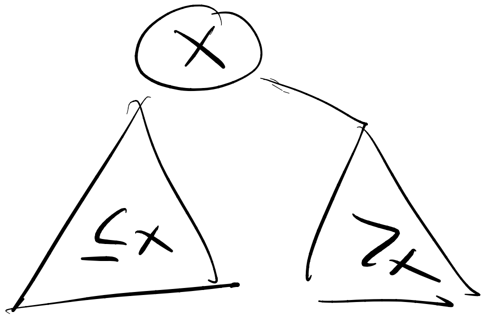
```
PRINT-LCS(S, X, i, j)
  if i == 0 or j == 0
      return
  if S[i, j] == "↖"
      PRINT-LCS(S, X, i − 1, j − 1)
      print x_i
  elseif S[i, j] == "↑"
      PRINT-LCS(S, X, i − 1, j)
  else PRINT-LCS(S, X, i, j − 1)
```

j		0	1	2	3	4	5	6
i		y_i	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

Longest Common Subsequence

- Running time
  - bottom up computation fills a table of m x n cells
  - each cell takes constant time
- overall $\Theta(mn)$
- Trace solution $O(m+n)$
  - we "walk" on the table towards the [0,0] cell either vertical or horizontal or diagonal.

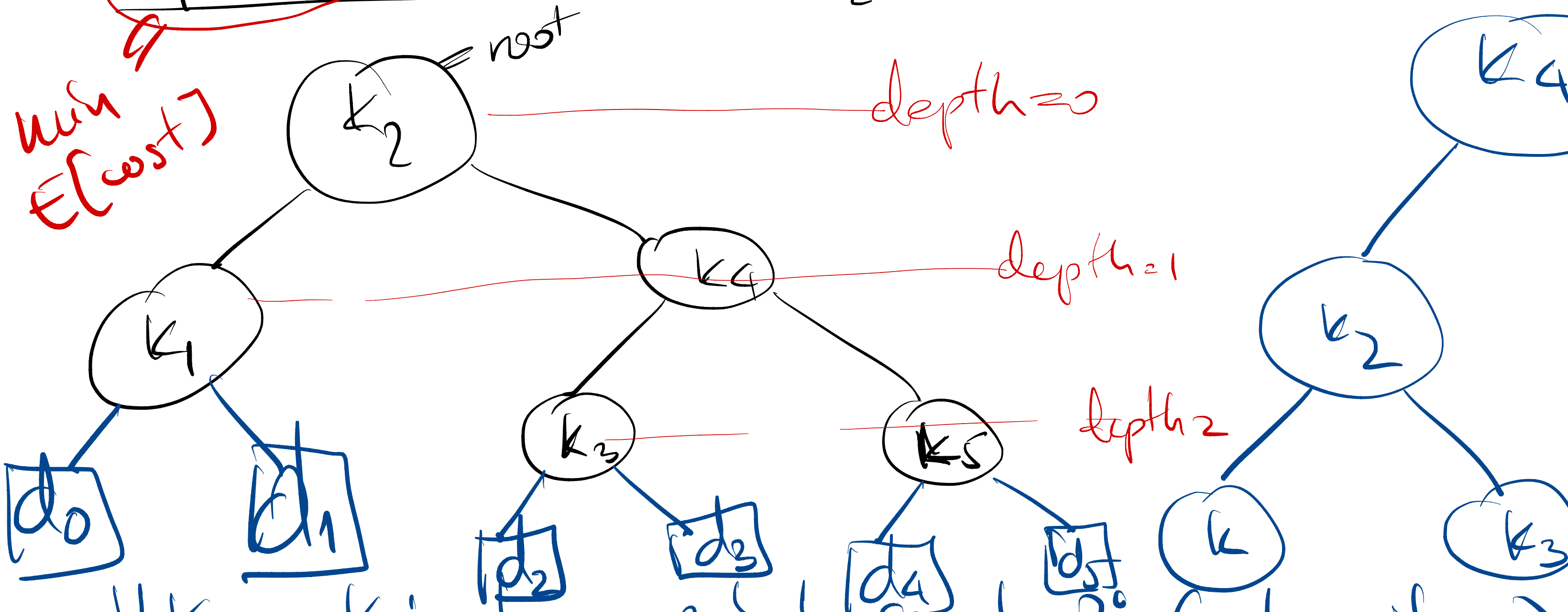
BST: left subtree ≤ node val < right subtree



Optimal BST

$k_1 \le k_2 \le k_3 \le k_4 \le k_5$

min E[cost]



$\forall$ key k_i has probab_search p_i (not uniform)

K = non leaf nodes
d = leaf nodes

dummy keys (not in the tree): d_i = every search for val $k_i <$ val $< k_{i+1}$

$$\sum p_i + \sum q_i = 1$$

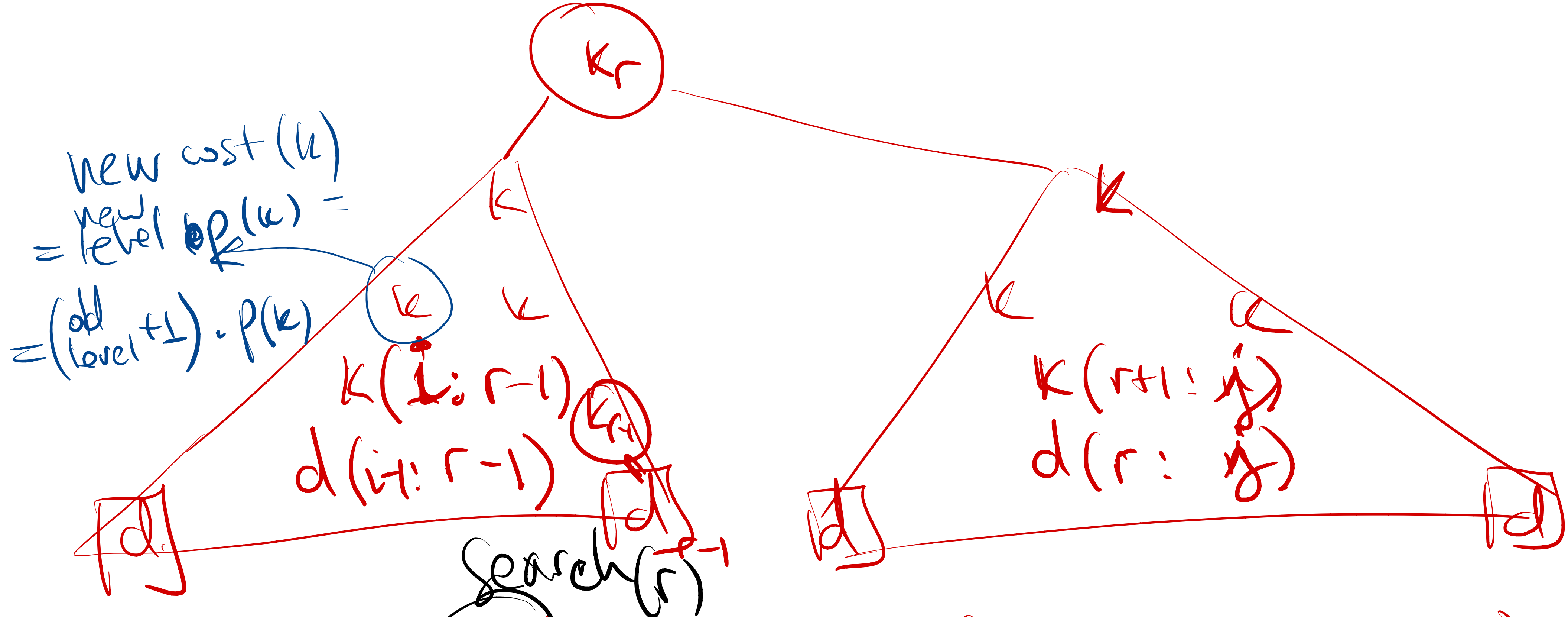
d_i has prob q_i (not uniform)

avg search cost $= E[\text{search cost}] = \sum_{i=1}^{n} (\text{depth}(k_i)+1) \cdot p_i + \sum_{i=0}^{n} (\text{depth}(d_i)+1) \cdot q_i$

(alg steps / cost; prob; alg steps; prob)

$E[X] = \sum_{x_i} \text{val}(x_i) \cdot \text{prob}(x_i)$

OPT SOL = OPT TREE $(k_i, k_{i+1}, \ldots, k_j)$



$C[i, j]$ = best cost for a tree (keys $k_i, k_{i+1}, \ldots, k_j$)

$$1 \cdot p_r + C[i, r-1] + w[i, r-1] + C[r+1 : j] + w[r+1 : j]$$

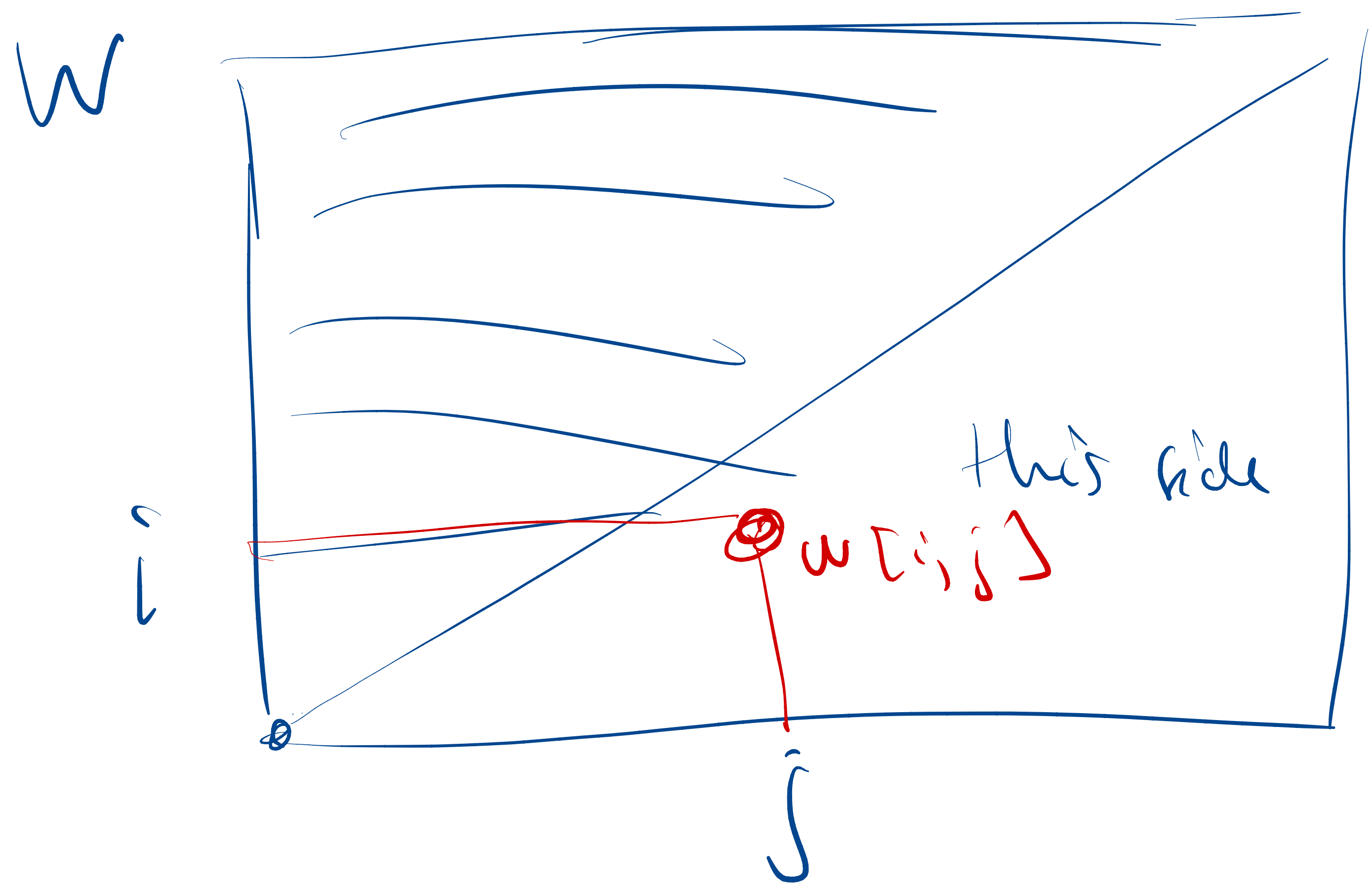
$\sum((\text{level}+1)\ \text{pros}) = \sum \text{level} \cdot \text{prob} + \sum \text{pros}$ (kid = left)

$\sum \text{level} \cdot \text{pros} + \sum \text{pros}$ (right side)

$$\sum_{k,d=\text{left}} \text{probas} = w[i, r-1]$$

english $$w[i,j] = \sum_{t=i}^{j} p_t + \sum_{t=i-1}^{j} q_t =$$

= sum of all probas events from i to j
k,d



15-10 Investment strategy.

T = \$ sum money to invest. 10 years (each Jan 1st reinvest/move)

n stocks

$i \in$

n stocks $\{1, 2, \ldots, n\}$

r_{ij} = percentage return of stock i in year j (given)

put \$d

get \$ $d \cdot r_{ij}$

allocation for year j,

$d_1^j + d_2^j + \ldots d_n^j = T^j$

$T^1 = T$

a) greedy: pick best stock (return) each year?
put all money on it.

b) greedy or opt ⇒ OPT SOL structure (subpb)
⇒ D&C

c) Alg + RT

d) variant any money allocated per stock
per year $\leq \$L$.
still greedy?