Javed Aslam
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Average Case Analysis of Quicksort

1. $$T(n) = \frac{2}{n} \sum_{k=1}^{n-1} T(k) + n$$

Pivot rank	(small) left problem	(large) right problem
n	$n-1$	0
$n-1$	$n-2$	1
$n-2$	$n-3$	2
$\vdots$	$\vdots$	$\vdots$
	1	$n-2$
1	0	$n-1$

So, $$T(n) = \frac{1}{n} \sum_{j=0}^{n-1} \left(T(j) + T(n-j-1) \right) + n$$

$$= \frac{2}{n} \sum_{k=0}^{n-1} T(k) + n$$

$$= \frac{2}{n} \sum_{k=1}^{n-1} T(k) + n$$

(no "recursive" call to 0-sized subproblem)

Consider $T(n-1)$ as well:

2. $$T(n-1) = \frac{2}{n-1} \sum_{k=1}^{n-2} T(k) + n-1$$

Multiplying (1) by n and (2) by $n-1$:

$$n\,T(n) = 2 \sum_{k=1}^{n-1} T(k) + n^2$$

$$-\quad (n-1)\,T(n-1) = 2 \sum_{k=1}^{n-2} T(k) + (n-1)^2$$

$$n\,T(n) - (n-1)\,T(n-1) = 2\,T(n-1) + 2n - 1$$

$$\Rightarrow \quad n\,T(n) = (n+1)\,T(n-1) + 2n - 1$$

$$\Rightarrow \quad T(n) = \frac{n+1}{n} T(n-1) + 2 - 1/n$$

Ignoring the $1/n$ term, we have

$$T(n) \leq \frac{n+1}{n} T(n-1) + 2$$

Solve by iteration:

J. Aslam

$$T(n) \le 2 + \frac{n+1}{n} T(n-1)$$

$$\le 2 + \frac{n+1}{n}\left(2 + \frac{n}{n-1} T(n-2)\right)$$

$$= 2 + 2\frac{n+1}{n} + \frac{n+1}{n-1} T(n-2)$$

$$\le 2 + 2\frac{n+1}{n} + \frac{n+1}{n-1}\left(2 + \frac{n-1}{n-2} T(n-3)\right)$$

$$= 2 + 2\frac{n+1}{n} + 2\frac{n+1}{n-1} + \frac{n+1}{n-2} T(n-3)$$

$$= 2 \cdot \frac{n+1}{n+1} + 2\frac{n+1}{n} + 2\frac{n+1}{n-1} + \frac{n+1}{n-2} T(n-3)$$

$$\vdots$$

$$= 2(n+1)\sum_{i=0}^{k-1} \frac{1}{(n+1)-i} + \frac{n+1}{n-(k-1)} T(n-k)$$

$k = n-1$

$T(n-k) = T(1)$

$$= 2(n+1)\sum_{i=0}^{n-2} \frac{1}{(n+1)-i} + \frac{n+1}{2} T(1)$$

$$= 2(n+1)\sum_{j=3}^{n+1} \frac{1}{j} + \frac{n+1}{2}\Theta(1)$$

$$= 2(n+1)\,\Theta(\ln n) + \Theta(n)$$

$$= \Theta(n \log n)$$

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(2b)

Alternate page (2) :

$$T(n) \le \frac{n+1}{n} T(n-1) + 2$$

Divide by $n+1$:

$$\frac{T(n)}{n+1} \le \frac{T(n-1)}{n} + \frac{2}{n+1}$$

Let $R(n) = \frac{T(n)}{n+1}$ (and thus, $R(n-1) = \frac{T(n-1)}{n}$)

$$\Rightarrow \quad R(n) \le R(n-1) + \frac{2}{n+1}$$

Note that $R(1) = \frac{T(1)}{1+1} = \Theta(1)$. Now, simple iteration :

$$R(n) \le \frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + \cdots + \frac{2}{3} + R(1)$$

$$= 2 \sum_{k=3}^{n+1} \frac{1}{k} + \Theta(1)$$

$$= \Theta(\ln n)$$

But $T(n) = (n+1) R(n)$

$$= (n+1) \cdot \Theta(\ln n)$$

$$= \Theta(n \ln n)$$