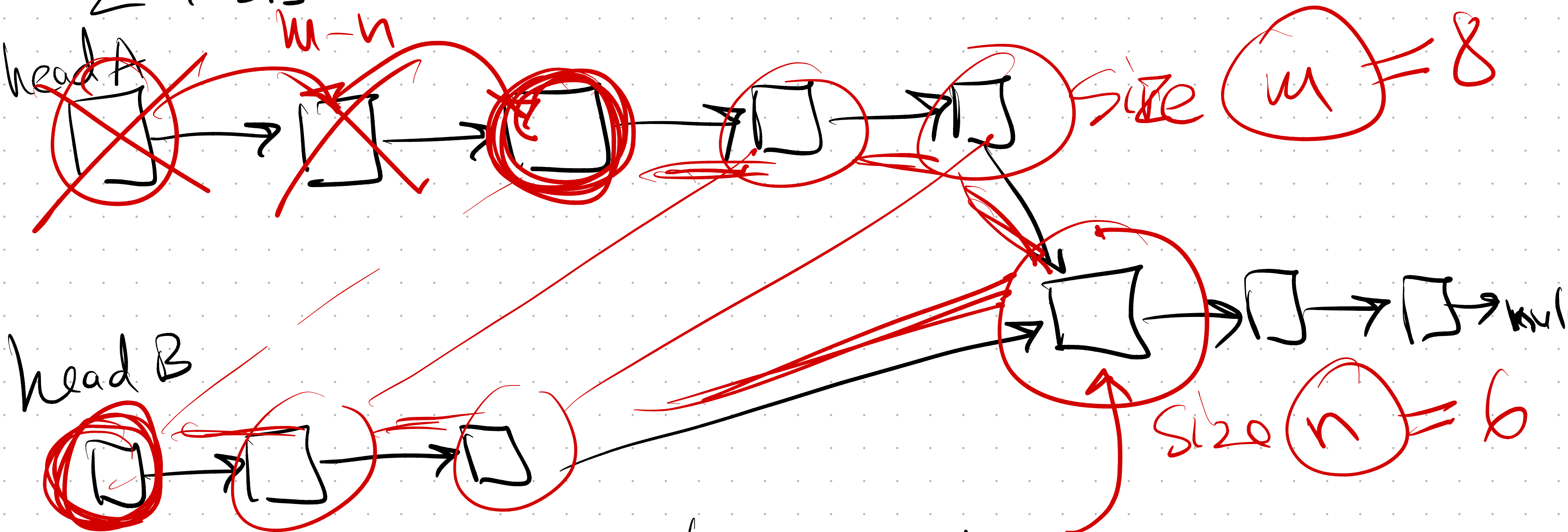
Lecture 2 5/12/22

2 lists that intersect



Find the intersection address

$m \geq n$

naive $\Theta(mn)$

find m, n $\Theta(m)$

shift head A $\Theta(m)$

check $\Theta(n)$

better: $\Delta = m - n$

head A advance Δ steps

Order of growth

c_1, c_2 constants

$c_2 > c_1 > 0$

f, g monotonic functions

f "growths same as" g (f same as g)

$f = \Theta(g) \qquad g = \Theta(f)$

$$\boxed{c_1 g(n) \leq f(n) \leq c_2 \cdot g(n)}$$

upper bound

$f = O(g)$

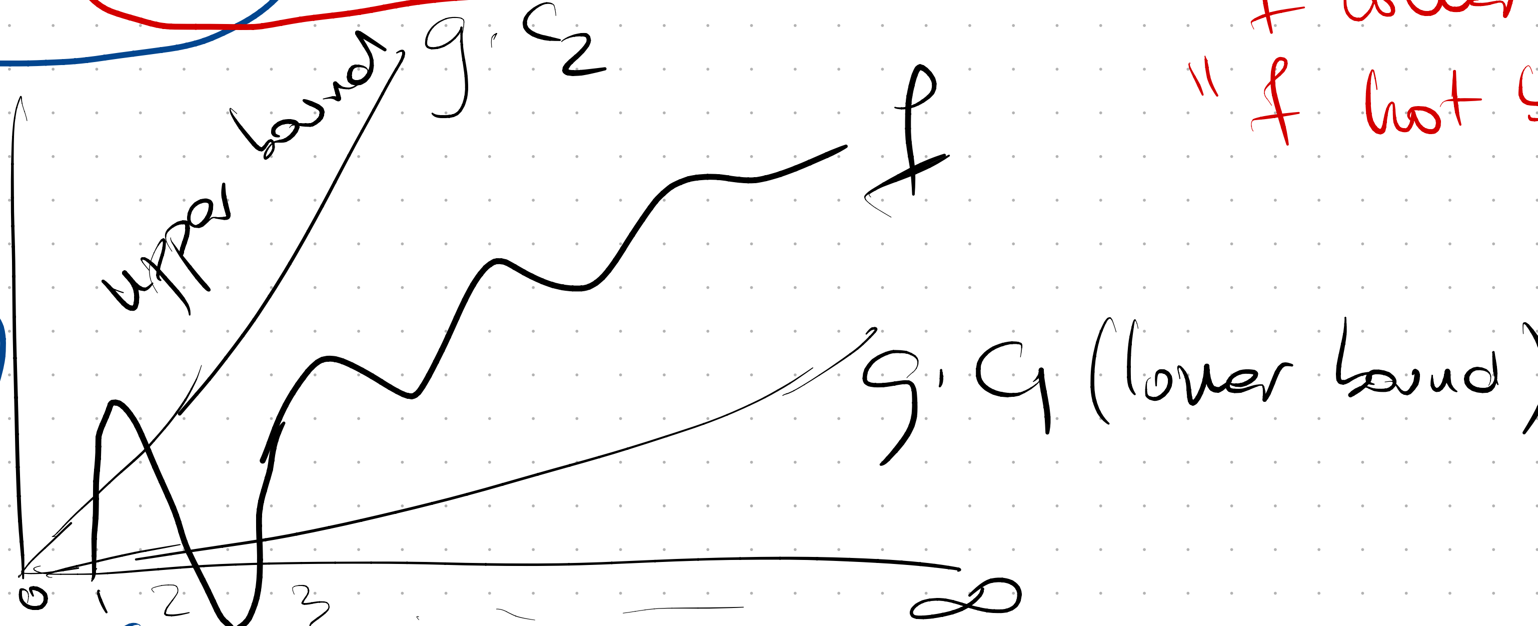
"f lower than g"

"f not bigger than g"

LB

$f = \Omega(g)$

f at least as big as g



Examples

$f(n)$ $2n^2 + 3n + 2$ $\qquad \Theta(n^2)$ $\qquad c_1 n^2 \le 2n^2+3n+2 \le c_2 \cdot n^2$

$c_1 = 2, \quad c_1 = 1$

$c_2 = 2.00000001$

$c_2 = 7$

$2n^2 + 3n + 2 \overset{?}{<} 2.00000001 \cdot n^2$

$\frac{1}{4} n^2$

$2 + \frac{3}{n} + \frac{2}{n^2} < 2.00000001$

$\frac{3}{n} + \frac{2}{n^2} < 0.00000001$

true $\because \lim_{n \to \infty} \frac{3}{n} + \frac{2}{n^2} = 0$

$n^2 \log n + n^3 \qquad \Theta(n^3)$

$2^n + 5n^2 - 3 \qquad \Theta(2^n)$

$3^n + 2^n \qquad \Theta(3^n)$

$c_1 \cdot 3^n \le 3^n + 2^n \le c_2 \cdot 3^n$

$c_1 = 1 \qquad c_2 = 2$

$$\log_a x = b \iff a^b = x$$

$$\log(x \cdot y) = \log x + \log(y)$$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

$$\boxed{\log_3 x \overset{?}{=} \log_2 x \cdot \log_3 2}$$

constant

$$x = 3^{\log_3 x} \overset{?}{=} 3^{\log_2 x \cdot \log_3 2}$$

$$\left(3^{\log_3 2}\right)^{\cdot \log_2 x}$$

$$2^{\log_2 x}$$

$$x = x \quad ✓$$

$$\log_3(x) = \Theta(\log_2 x)$$

$$c_1 \log_2 x \le \log_3 x \le c_2 \log_2 x$$

const

$$c_1 \log_2 x \le \log_2 x \boxed{\log_3 2} \le c_2 \log_2 x$$

$$c_1 < \log_3 2 < c_2$$

$f = 2n^2 - 3n + 1$ $\qquad f = \Theta(g)$

$g = 2f = 4n^2 - 6n + 2$

$3^n \stackrel{?}{=} \Theta(2^n)$ $\qquad c_1 \cdot 2^n \leq \boxed{3^n \leq c_2 \cdot 2^n}$?

$c_1 = 1$

$\Updownarrow$

$\log(f(n)) = O(\overset{\text{polynom}}{f(n)})$ $\qquad \left(\frac{3}{2}\right)^n \leq c_2$

$\forall n \geq$ start point

$\log \leq$ polynomials $\leq$ exp

(same argument)

$\lim_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty$

no c_2 exists

example

$$2^{\ln^2 n} \quad ? \quad n!$$

$$??\ 2^{\ln n} \cdot \ln n \quad ? \quad n! \qquad \underbrace{n \cdot n \cdots n}_{n \text{ time}}$$

$$n^{\ln n} \quad ? \quad n! \qquad \cancel{\geq}\ n! \qquad \underbrace{(n/2) \cdot (n/2) \cdots (n/2)}_{n/2 \text{ time}}$$

$$\underbrace{n \cdot n \cdot n \cdots n}_{\ln n} \quad ? \quad 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$$

$$\leq$$

$$(ab)^x = a^x \cdot b^x$$

$$n^{\log n} \overset{?}{\leq} \left(\frac{n}{2}\right)^{n/2} \leq n!$$

$$\left(\frac{n}{2}\right)^{\log n} \cdot 2^{\log n} \ \text{want} \leq \left(\frac{n}{2}\right)^{n/2}$$

$$n \leq \left(\frac{n}{2}\right)^{n/2 - \log n} \quad ?$$

$$n \leq \left(\frac{n}{2}\right)^{2} \leq \left(\frac{n}{2}\right)^{n/2 - \log n}$$

$$2 \leq n/2 - \log n$$

$$\frac{n}{2} - \log n - 2 \geq 0$$

Recurrences = recurrence - definition functions

def $T(n) = 2 \cdot T(n/2)$ $\quad T = \Theta(n)$

base unknown $T(0), T(1), T(2)$

$T(n) = 1 + T(n/2)$ $\quad T = \Theta(\log n)$

$T(2n) = 1 + T(n)$

$T(2^n) = 1 + T(2^{n-1})$

$T(n) = n + 2T(n/2)$ $\quad \boxed{T(n \cdot \log n)}$

$T(n)$ = Alg RunTime (RT) as function of n = input size

Binary Search   Sorted array

m

A = | -3 | -1 | 0 | 5 | 12 | 17 | 21 | 75 | 92 | 103 |

    1   2   3   4   5   6   7   8   9   10

BinSearch (val v, A, begin, end) → indexes in array

1   10

Find value v in array A[begin : end]

- $m = \dfrac{begin + end}{2}$ $\Theta(1)$
- if $v == A[m] \Rightarrow$ done $\Theta(1)$

  if $v < A[m]$ // search left side

  BS(v, A, begin, m−1)

  if $v > A[m]$ // search right side

  BS(v, A, m+1, end)

Fix termination

R.T. $T(n) = \Theta(1) + T(n/2)$

$T(n) = 1 + T(n/2)$

grinding / iterations

$k=1$ $T(n) = 1 + T(n/2) =$

$k=2$ $= 1 + [1 + T(n/4)] = 2 + T(n/4)$

$k=3$ $= 2 + [1 + T(n/8)] = 3 + T(n/8)$

$k=4$ $= 3 + [1 + T(n/16)] = 4 + T(n/16)$

general by k iter step $= \boxed{k + T(n/2^k)}$

last k $\quad T\left(n/2^k\right) \simeq T(1) \iff k \simeq \log n$

$$n \simeq 2^k$$

$$T(n) = k + T\left(\frac{n}{2^k}\right) = \log n + T(1)$$

$$= \Theta(\log n)$$

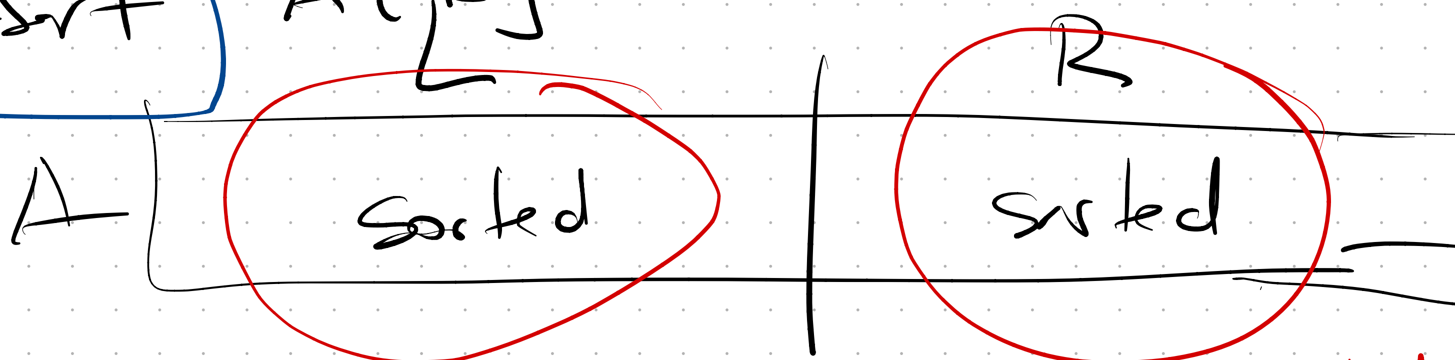
Merge Sort

$$T(n) = n + 2T(n/2)$$

$\Theta(1)$ • split L vs R

$T(n/2)$ • MergeSort A[L]

$T(n/2)$ • MergeSort A[R]



$\Theta(n)$ • Combine / MERGE A[L], A[R] $\Longrightarrow$ sorted output

not rec

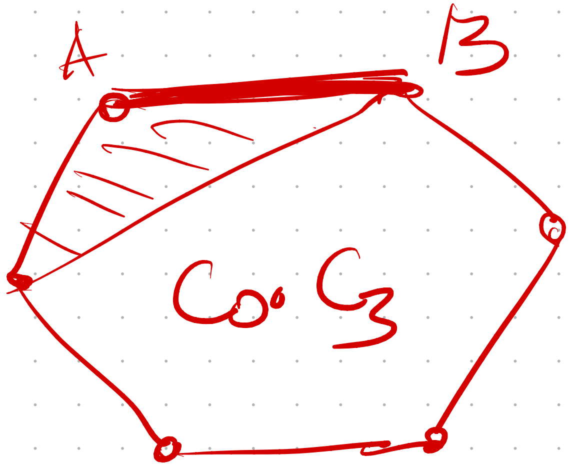
n Δ in n+2 sides polygon   How many triangulations?

$n=4$ $n+2=6$

$C_0=1$

$C_1=1$

$C_2=2$

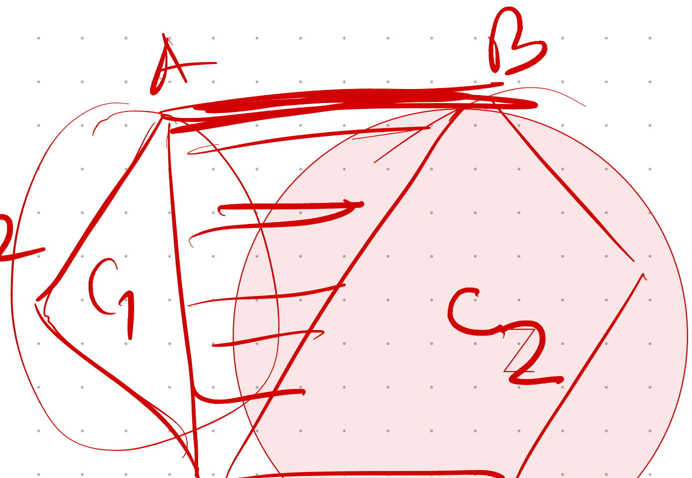


$C_4 = 14$

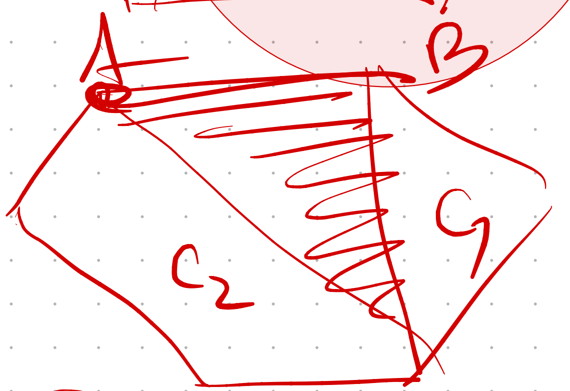
AB side in Δ

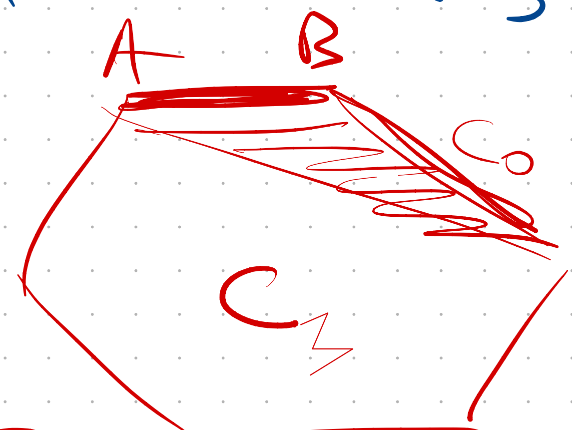
- any triangulation is in one of these 4 cases
- no double counting: any Δ is in exactly one of these cases

$C_1 \cdot C_2$



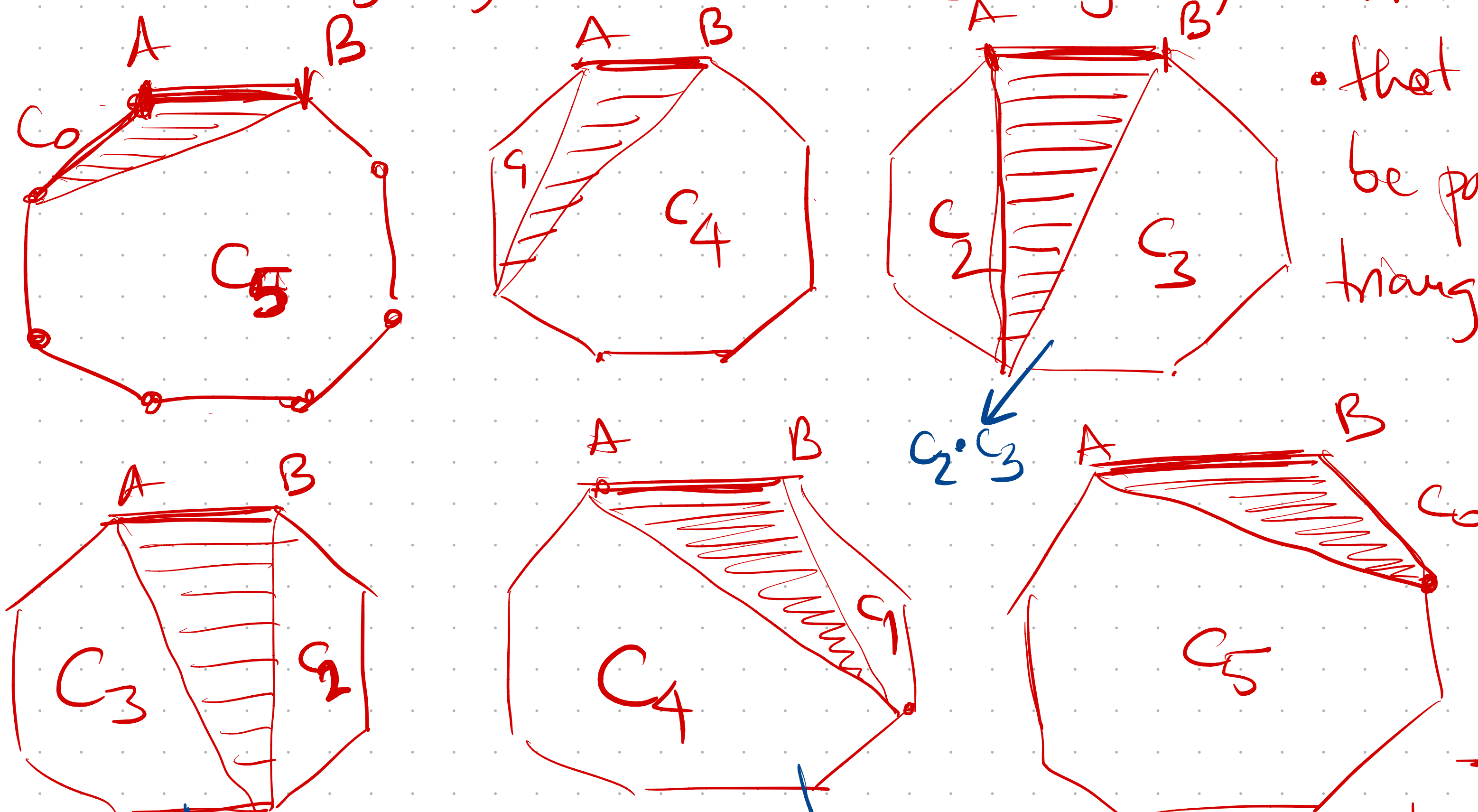
$C_2 \cdot C_1$





$$C_4 = C_0 \cdot C_3 + C_1 \cdot C_2 + C_2 \cdot C_1 + C_3 \cdot C_0$$

$n = 6$ triangles ; $n+2 = 8$ sides (octogon)



- Fix reference side AB
- that side AB can be part of 6 possible triangles, so 6 cases
  - disjoint cases so we can sum up for total
  - each case is counting possibilities for Δ on left/right side of ABX triangle

triangulations left side C_3 on $\times$ on right side C_2

$C_4 \cdot C_1$

$$C_6 = C_0 \cdot C_5 + C_1 \cdot C_4 + C_2 \cdot C_3 + C_3 \cdot C_2 + C_4 \cdot C_1 + C_5 \cdot C_0$$

$$C_n = C_{n-1}C_0 + C_{n-2}C_1 \cdots + C_0 C_{n-1}$$

$$= \sum_{k=0}^{n-1} C_{n-1-k} \cdot C_k$$

Triangulations (max)

$C_0 = 1 \quad C_1 = 1$

For $n = 2 : \max$

$\quad C_n = 0$

$\quad$ For $k = 0 : n-1$

$\quad\quad C_n = C_n + C_k \cdot C_{n-1-k}$

$\mathcal{O}(n^2)$

$$C_n = \text{Catalan}\#(n) = \binom{2n}{n} - \binom{2n}{n+1}$$