Lecture 2: Recurrences → MT

Arithmetic Sum/Series $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Quad Series $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Geometric Serie $x^0 + x^1 + x^2 + \dots + x^n = \frac{x^{n+1} - 1}{x - 1}$

base x $x \neq 1$

Harmonic Series $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n} \simeq \ln(n) + \text{const}$

$\Leftarrow$ def e = base of nat log.

$\log_b x = \log_a x \cdot \log_b a$: $x = b^y$ $y = \log_a b^y \cdot \log_b a$

$y = y \cdot \log_a b \cdot \log_b a$

$a^{\log_b n} = n^{\log_b a}$: $n = b^x$

$a^x = (b^x)^{\log_b a} = (b^{\log_b a})^x$

$\log(n!) \stackrel{?}{=} \Theta(n \log n)$

$\log(abc) = \log a + \log b + \log c$

$c_1 n \log n \le \log(n!) \le c_2 \, n \log n$

n terms

$$c_1 n \log n \le \underbrace{\log 1 + \log 2 + \ldots + \log n}_{} \le c_2 n \log n$$

$$c_2 (\log n + \log n \ldots \log n)$$

$c_2 = 1$

$n \left(\log n + \log n + \ldots \log n \right)$

$\ge$

want $\ge$ $\cancel{n}\,\frac{n}{2} \cdot \log\left(\frac{n}{2}\right)$

$c_1 \cancel{n} \log n$

$\le \frac{1}{2}(\log n - \log 2)$

$c_1 \log n \le \log n - \log 2$

$2c_1 \log n$

$2c_1 \le \dfrac{\log n - \log 2}{\log n}$

$2c_1 < 1$

$c_1 < 1/2$

$f(n) \to \infty$
monotone incr

$$\frac{f(n) - \text{const}}{f(n)} \to 1$$

$$\cancel{\frac{\inf - 1}{\inf}} \to 1$$

$$\frac{3n^2 - 5n + 1}{2n^2 + 7n + 2} \to \frac{3}{2}$$

$$\frac{n}{n^2} \to$$

$$\frac{n^2}{n} \to \infty$$

Last Time

MergeSort $T(n) = 2T(n/2) + \Theta(n)$

Binary Search $T(n) = T(n/2) + \Theta(1)$

$T(n) = 4T(n/2) + n$ guess $O(n^3)$

ind step $T(n/2) \le c\left(\frac{n}{2}\right)^3 \Longrightarrow \boxed{T(n) \le cn^3}$

hyp conc

proof $T(n) = 4T(n/2) + n \le 4c\left(\frac{n}{2}\right)^3 + n$

$= \frac{cn^3}{2} + n \overset{\text{WANT}}{\le} cn^3$

$\frac{c}{2} + \frac{1}{n^2} \overset{\text{WANT}}{\le} c$

lim $C = 1$

$\frac{1}{2} + 0 \le 1$ ✓

Better guess $\boxed{T(n) \le cn^2}$ $O(n^2)$

ind step $T(n/2) \le c\left(\frac{n}{2}\right)^2 \Longrightarrow T(n) \le cn^2$

proof: $T(n) = 4T(n/2) + n \le 4c\left(\frac{n}{2}\right)^2 + n$

$= cn^2 + n \overset{\text{WANT}}{\le} cn^2$ impossible

better proof: include lower-degree term

$T(n/2) \le c\left(\frac{n}{2}\right)^2 - d\,n/2 \Longrightarrow T(n) \le cn^2 - dn$

proof $T(n) = 4T(n/2) + n \le 4\left[c\left(\frac{n}{2}\right)^2 - d\frac{n}{2}\right] + n$

$= cn^2 - 2dn + n \overset{\text{WANT}}{\le} cn^2 - dn$

$$cn^2 - 2\cancel{dn} + n \overset{\text{WANT}}{\leq} \cancel{cn^2} - \cancel{dn}$$

$$-dn + n \leq 0 \qquad d = 2$$

Lower Bound $\Omega(n^2)$?

ind step $\underbrace{T(n/2) \geq c\left(\frac{n}{2}\right)^2}_{\text{hyp}} \Rightarrow T(n) \geq cn^2$

proof $T(n) = 4\underline{T(n/2)} + n \geq 4c\left(\frac{n}{2}\right)^2 + n$

$$= cn^2 + n \overset{\text{WANT}}{\geq} cn^2 \qquad \forall c > 0$$

$$T(n) = \Theta(n^2)$$

wout $T(n) \le c f(n)$

Better/tighter ineq $T(n) \le c f(n) - d g(n)$

g lower than f $g = o(f)$

$n^2 - n = \Theta(n^2)$

$T(n) = 4T(n/2) + n$

$k=2 \quad = 4\left[4T(n/4) + \frac{n}{2}\right] + n \quad = 4^2 T\left(\frac{n}{2^2}\right) + n + 2n$

$k=3 \quad = 4^2\left[4T\left(\frac{n}{2^3}\right) + \frac{n}{4}\right] + n + \frac{n}{2} = 4^3 T\left(\frac{n}{2^3}\right) + n + 2n + 4n$

$k=4 \quad 4^3\left[4T\left(\frac{n}{2^4}\right) + \frac{n}{8}\right] + n + 2n + 4n = 4^4 T\left(\frac{n}{2^4}\right) + n + 2n + 4n + 8n$

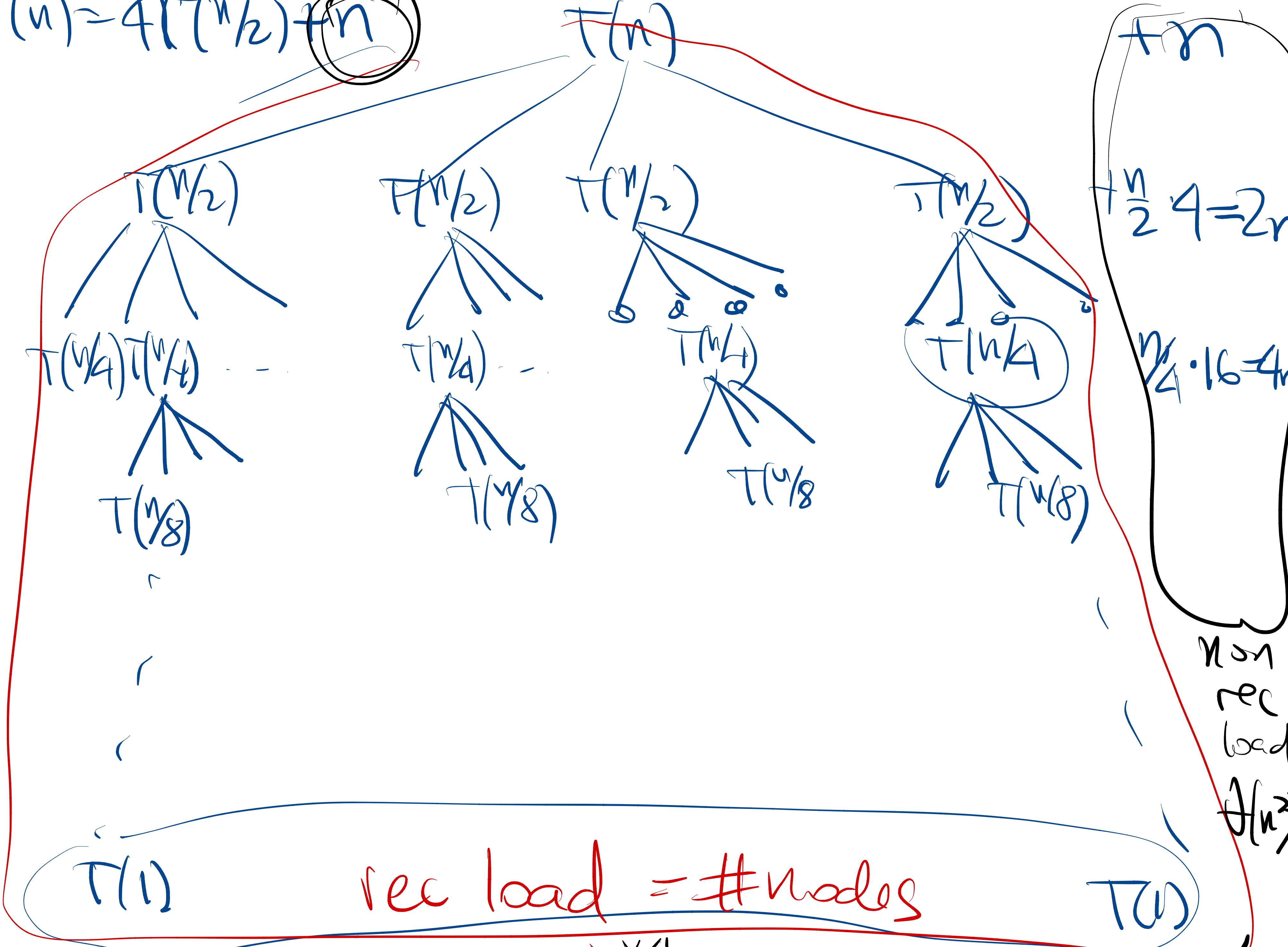
$\boxed{\text{general } k \qquad 4^k T\left(\frac{n}{2^k}\right) + n\underbrace{(1+2+\dots+2^{k-1})}_{2^k-1}}$

$k+1 \qquad 4^{k+1} T\left(\frac{n}{2^{k+1}}\right) + n\left(1+2+\dots+2^k\right)$

last $k \quad T\left(\frac{n}{2^k}\right) \simeq T(1) \iff k \simeq \log_2 n \qquad \Theta(n^2)$

$4^{\log_2 n} T(1) + n \cdot 2^{\log_2 n} = n^{\log_2 4} T(1) + n^2$

$T(n) = 4T(n/2) + n$



$\Theta(n^2)$ $x<1$ $1 + x + x^2 + \dots + x^k + \dots \text{inf} = \frac{1}{1-x}$

$$T(n) = T(n/2) + T(n/4) + n^2$$

$$= n^2 + \left[\left(\frac{n}{2}\right)^2 + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right)\right] + \left[\left(\frac{n}{4}\right)^2 + T\left(\frac{n}{8}\right) + T\left(\frac{n}{16}\right)\right]$$

$$= n^2 + \frac{5}{16}n^2 \quad T\left(\frac{n}{4}\right) + 2T\left(\frac{n}{8}\right) + T\left(\frac{n}{16}\right)$$

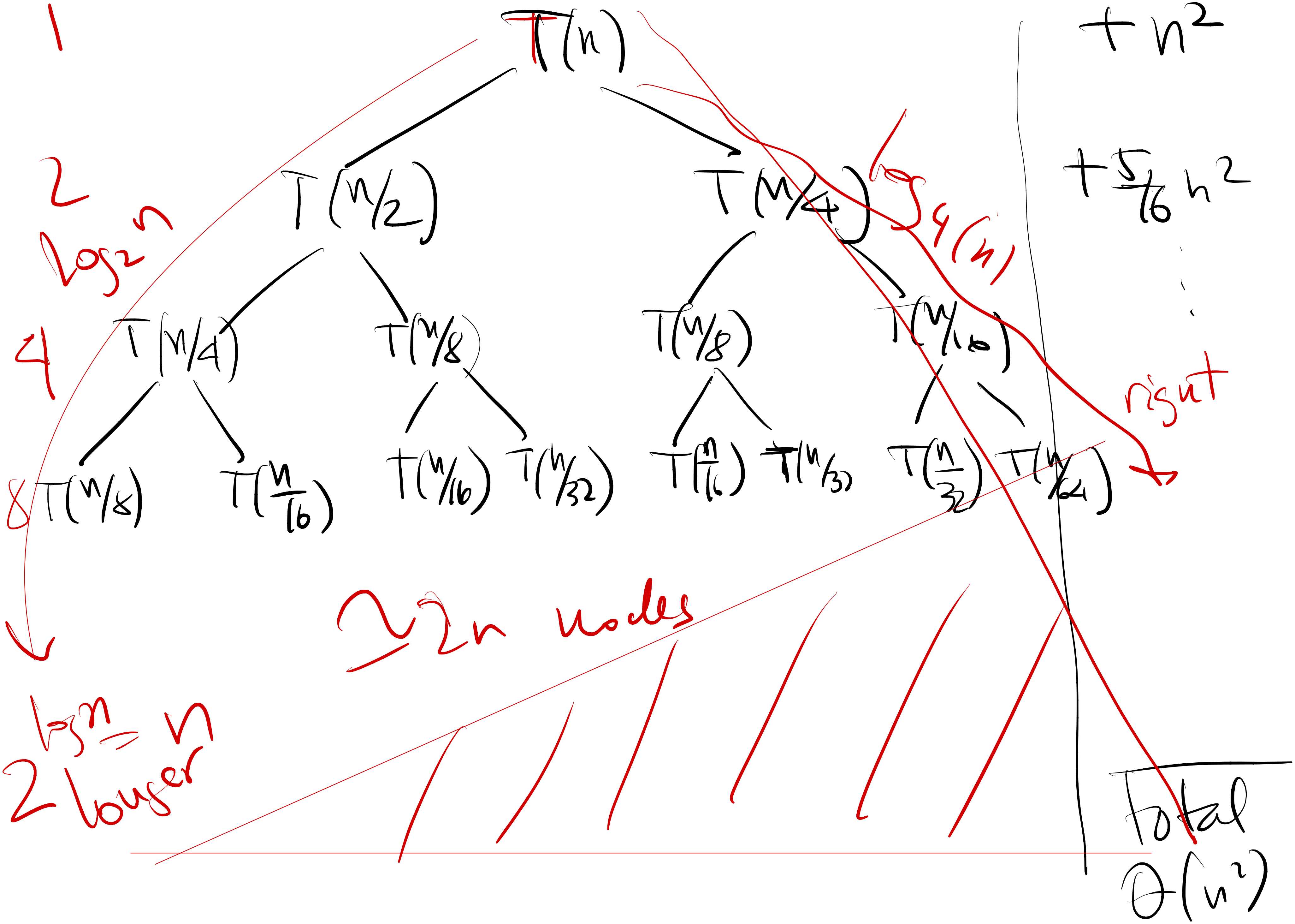
$$= n^2 + \frac{5}{16}n^2 + \left[\left(\frac{n}{4}\right)^2 + T\left(\frac{n}{8}\right) + T\left(\frac{n}{16}\right)\right] + \left[2\left(\frac{n}{8}\right)^2 + 2T\left(\frac{n}{16}\right) + 2T\left(\frac{n}{32}\right)\right] + \left[\left(\frac{n}{16}\right)^2 + T\left(\frac{n}{32}\right) + T\left(\frac{n}{64}\right)\right]$$

$$= n^2 + 5/16\, n^2 + \left(\frac{5}{16}\right)^2 n^2 \quad \underbrace{T\left(\frac{n}{8}\right) + 3T\left(\frac{n}{16}\right) + 3T\left(\frac{n}{32}\right) + 1T\left(\frac{n}{64}\right)}$$

$$1 \quad 3 \quad 3 \quad 6$$

$$x = \frac{5}{16} \qquad + \left(\frac{5}{16}\right)^3 n^2 \qquad 1 \quad 4 \quad 6 \quad 4 \quad 1$$

$$n^2(1 + x + x^2 + x^3 \cdots)$$



1
T(n)
+ n^2
2
T(n/2)
T(n/4)
+ 5/16 n^2
log_2 n
log_4(n)
4
T(n/4)
T(n/8)
T(n/8)
T(n/16)
right
8
T(n/8)
T(n/16)
T(n/16)
T(n/32)
T(n/16)
T(n/32)
T(n/32)
T(n/64)
~2n nodes
2^{log_2 n} = n
2 lower
Total
Θ(n^2)

Guess $T(n) = \Theta(n^2)$ $T(n) = n^2 + T(\frac{n}{2}) + T(\frac{n}{4})$

$\boxed{c_1 n^2 \le T(n)} \le c_2 n^2$

$c_1 = 1$ $n^2 \le n^2 + \dots$ ✓

ind step

$T(\frac{n}{2}) \le c_2 (\frac{n}{2})^2$

$T(n/4) \le c_2 (\frac{n}{4})^2$

$\xrightarrow{?} T(n) \le c_2 n^2$

proof $T(n) = n^2 + T(\frac{n}{2}) + T(\frac{n}{4}) \le n^2 + c_2 (\frac{n}{2})^2 + c_2 (\frac{n}{4})^2$

$= \cancel{n^2} \left(1 + \frac{c_2}{4} + \frac{c_2}{16}\right) \overset{\text{WANT}}{\le} c_2 \cancel{n^2}$

$16 + 4c^2 + c_2 \le 16 c_2$ $c_2 = 2$

$$\boxed{T(n) = aT(n/b) + \Theta(n^c)}$$

$n \cdot \sqrt{n}$

a branches of load ratio $\frac{1}{b}$ + non-rec n^c

$T(n)$ + n^c

say a problems of size $\frac{n}{b}$ each

a

1 2 3 4

$T(n/b)$ $T(n/b)$ $T(n/b)$ $+\left(\frac{n}{b}\right)$ $T(n/b)$ $+a\left(\frac{n}{b}\right)^c$

+

a

a^2

$\log_b n$ layers

$T\left(\frac{n}{b^2}\right)$ $T\left(\frac{n}{b^2}\right)$ $T(n/b^2) + \left(\frac{n}{b^2}\right)$ $a^2\left(\frac{n}{b^2}\right)^c$

+

a^3

$T\left(\frac{n}{b^3}\right)$ $\left(\frac{n}{b^3}\right)^c$ $a^3\left(\frac{n}{b^3}\right)^c$

$T(1)$

$n^c \cdot \left(\frac{a}{b^c}\right)^k = a^k\left(\frac{n}{b^k}\right)^c$

branches at bottom

$$a^{\log_b n} \cdot T(1)$$

$$\Theta\left(n^{\log_b a}\right)$$

$$n^c\left[1+\left(\frac{a}{b^c}\right)+\left(\frac{a}{b^c}\right)^2+\left(\frac{a}{b^c}\right)^3 \cdots k\right]$$

$$n^c\left(1+x+x^2+\quad+x^k\right)$$

$$x = a/b^c$$

$x \neq 1$

$$n^c\left[\sum_{i=0}^{\log_b n - 1}\left(\frac{a}{b^c}\right)^i\right]$$

$$n^c \frac{x^{\log_b n} - 1}{x - 1}$$

3 cases

$x < 1$

$x = 1$

$x > 1$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$b^{\log_b n} = n$$

case 1 $\quad x = \frac{a}{b^c} > 1 \iff c < \log_b a$

tree

$$\Theta\left(n^{\log_b a}\right) + n^c \, \frac{x^{\log_b n} \cancel{-1}}{\cancel{x-1}} \qquad x = \frac{a}{b^c}$$

$$\Theta\left(n^{\log_b a}\right) + \Theta\left(n^c \cdot \frac{a^{\log_b n}}{(b^c)^{\log_b n}}\right) \qquad b^{(\log_b n)c}$$

$$\Theta\left(n^{\log_b a}\right) + \Theta\left(\cancel{n^c} \cdot \frac{n^{\log_b a}}{\cancel{n^c}}\right)$$

$$\Theta\left(n^{\log_b a}\right)$$

Case 2

$c = \log_b a \iff x = \frac{a}{b^c} = 1$

$$n^c \sum_{i=0}^{\log_b n - 1} (1)^i + \Theta(n^{\log_b a})$$

$$n^c \cdot \log_b n + \Theta(n^{\log_b a})$$

$$\boxed{n^{\log_b a} \cdot \log_b n} + \Theta(n^{\log_b a})$$

$$\Theta(n^{\log_b a} \log_b n)$$

$$\Theta(n^c \cdot \log_b n)$$

case 3 $c > \log_b a \iff x = \frac{a}{b^c} < 1$

$$n^c \frac{x^{\text{power}} - 1}{x - 1} + \Theta(n^{\log_b a})$$

Constant

$$\Theta(n^c) + \Theta(n^{\log_b a})$$

$$\Theta(n^c)$$

simpler MT

$c < \log_b a \quad T(n) = \Theta(n^{\log_b a})$

$c = \log_b a \quad T(n) = \Theta(n^c \log n) = \Theta(n^{\log_b a} \log n)$

$c > \log_b a \quad T(n) = \Theta(n^c)$

MT-book $T(n) = aT(n/b) + f(n)$

- $f(n) \leq d \cdot n^{\log_b a - \varepsilon} \Longrightarrow T(n) = \Theta(n^{\log_b a})$

ours: $n^c < n^{\log_b a}$

- $f(n) = \Theta(n^{\log_b a} \log^k n) \Longrightarrow T(n) = \Theta(n^{\log_b a} \cdot \log^{k+1}(n))$

our: $c = \log_b a \quad (k=0)$

- $f(n) \geq d \cdot n^{\log_b a + \varepsilon}$ + Reg Afinity $af(n/b) < f(n)$

our $n^c > n^{\log_b a}$

$\Longrightarrow T(n) = \Theta(f(n))$

$$T(n) = \frac{n^2}{\log n} + 4T(n/2)$$

$$= \frac{n^2}{\log n} + 4\left[\frac{(n/2)^2}{\log n/2} + 4T\left(\frac{n}{4}\right)\right]$$

$$= \frac{n^2}{\log n} + \frac{n^2}{\log n/2} + 4^2 T\left(\frac{n}{2^2}\right)$$

$$= \frac{n^2}{\log n} + \frac{n^2}{\log n/2} + 4^2\left[\frac{(n/4)^2}{\log n/4} + 4T\left(\frac{n}{2^3}\right)\right]$$

$$= \frac{n^2}{\log n} + \frac{n^2}{\log n/2} + \frac{n^2}{\log n/4} + 4^3 T\left(\frac{n}{2^3}\right)$$

$$K \quad \frac{n^2}{\log n} + \frac{n^2}{\log(n/2)} + \frac{n^2}{\log(n/2^2)} \cdots + \frac{n^2}{\log(n/2^k)} + 4^{k+1} T\left(\frac{n}{2^{k+1}}\right)$$

$$\underbrace{\frac{n^2}{\log n} + \frac{n^2}{\log(n/2)} + \frac{n^2}{\log(n/2^2)} - - \frac{n^2}{\log(n/2^k)}} + 4^{k+1}T\left(\frac{n}{2^{k+1}}\right)$$

$$n^2\left[\frac{1}{\log n} + \frac{1}{\log n - 1} + \frac{1}{\log(n)-2} + \frac{1}{\log(n)-3} - - - - + - - \frac{1}{\log(n) - (\log(n)-1)}\right]$$

$$\log(n) - (\log(n)-1) = 1$$

last k $\quad \frac{n}{2^{k+1}} \simeq 1 \iff k \simeq \log n - 1$

$$n^2\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + - - - + \frac{1}{\log n}\right]$$

$$= n^2 H_{(\log n)} \simeq n^2 \log(\log(n))$$