8.2 Solving Linear Recurrence Relations

Recall from Section 8.1 that solving a recurrence relation means to find explicit solutions for the recurrence relation.

Definition 1. *A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \tag{*}$$

where $c_1, c_2, \ldots, c_k \in \mathbb{R}$ and $c_k \neq 0$.

Linear refers to the fact that $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ appear in separate terms and to the first power.

Homogeneous refers to the fact that the total degree of each term is the same (thus there is no constant term)

Constant Coefficients refers to the fact that $c_1, c_2, \ldots, c_k$ are fixed real numbers that do not depend on n.

Degree k refers to the fact that the expression for a_n contains the previous k terms $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$.

A consequence of the second principle of mathematical induction is that a sequence satisfying the recurrence relation in the definition (*) is uniquely determined once we know the values of a_j in the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

Example 1. *The recurrence relation $A_n = (1.04)A_{n-1}$ is a linear homogeneous recurrence relation of degree one. The recurrence relation $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree two. The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of degree five.*

Example 2 (Non-examples)**.** *The recurrence relation $a_n = a_{n-1}a_{n-2}$ is not linear. The recurrence relation $m_n = 2m_{n-1} + 1$ is not homogeneous. The recurrence relation $B_n = nB_{n-1}$ does not have constant coefficients.*

Linear homogeneous recurrence relations are studied for two reasons. First, they often occur in modeling of problems. Second, they can be systematically solved. The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the form $a_n = r^n$, where r is a constant.

Remark 1. *Note that $a_n = r^n$ is a solution of the recurrence relation (*) if and only if*

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}.$$

Divide both sides of the above equation by r^{n-k} and subtract the right-hand side from the left to obtain

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_k = 0. \tag{**}$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ is a solution of (*) if and only if r is a solution of (**).

Definition 2. *We call the equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots c_k = 0. \tag{**}$$

the characteristic equation *of the recurrence relation (*). The solutions of this equation are called the* characteristic roots *of the recurrence relation (*).*

As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation.

We first consider the case of degree two.

The Distinct-Roots Case

Consider a second-order linear homogeneous recurrence relation with constant coefficients:

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

where A and B are fixed real numbers. Relation (1) is satisfied when all the $a_i = 0$, but it has nonzero solutions as well. *Suppose* that for some number t with $t \neq 0$, the sequence

$$1, t, t^2, \ldots, t^n, \ldots$$

satisfies relation (1). This means that each term of the sequence equals A times the previous term plus B times the term before that. So for all integers $k \geq 2$,

$$t^k = At^{k-1} + Bt^{k-2}.$$

In particular, when $k = 2$, the equation becomes

$$t^2 = At + B,$$

or equivalently,

$$t^2 - At - B = 0. \tag{2}$$

This is a quadratic equation, and the values of t that make it true can be found either by factoring or by using the quadratic formula.

Now work backward. *Suppose* t is any number that satisfies equation (2). Does the sequence $1, t, t^2, t^3, \ldots, t^n, \ldots$ satisfy relation (1)? To answer this question, multiply equation (2) by t^{k-2} to obtain

$$t^{k-2} \cdot t^2 - t^{k-2} \cdot At - t^{k-2} \cdot B = 0.$$

This is equivalent to

$$t^k - At^{k-1} - Bt^{k-2} = 0$$

or

$$t^k = At^{k-1} + Bt^{k-2}.$$

Hence the answer is yes: $1, t, t^2, t^3, \ldots, t^n, \ldots$ satisfies relation (1).

This discussion proves the following lemma.

Lemma 1. *Let A and B be real numbers. A recurrence relation of the form*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

is satisfied by the sequence

$$1, t, t^2, t^3, \ldots, t^n, \ldots,$$

where t is a nonzero real number, if, and only if, t satisfies the equation

$$t^2 - At - B = 0. \tag{2}$$

Lemma 2. *If $r_0, r_1, r_2, \ldots$ and $s_0, s_1, s_2, \ldots$ are sequences that satisfy the same second-order linear homogeneous recurrence relation with constant coefficients, and if C and D are any numbers, then the sequence $a_0, a_1, a_2, \ldots$ defined by the formula*

$$a_n = Cr^n + Ds^n \quad \text{for all integers } n \geq 0$$

also satisfies the same recurrence relation.

Given a second-order linear homogeneous recurrence relation with constant coefficients, if the characteristic equation has two distinct roots, then Lemmas 1 and 2 can be used to find an explicit formula for *any* sequence that satisfies a second-order linear homogeneous recurrence relation with constant coefficients for which the characteristic equation has distinct roots, provided that the first two terms of the sequence are known. This is made precise in the next theorem.

Theorem 1 (Distinct Roots Theorem)**.** *Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

for some real numbers A and B with $B \neq 0$. If the characteristic equation

$$t^2 - At - B = 0 \tag{2}$$

has two distinct roots r and s, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Ds^n,$$

where C and D are the numbers whose values are determined by the values a_0 and a_1.

Remark 2. *To say "C and D are determined by the values of a_0 and a_1" means that C and D are the solutions to the system of simultaneous equations*

$$a_0 = Cr^0 + Ds^0 \textit{ and } a_1 = Cr^1 + Ds^1,$$

or, equivalently,

$$a_0 = C + D \textit{ and } a_1 = Cr + Ds.$$

This system always has a solution when $r \neq s$.

Proof. Suppose that for some real numbers A and B, a sequence $a_0, a_1, a_2, \ldots$satisfies the recurrence relation $a_k = Aa_{k-1} + Ba_{k-2}$, for all integers $k \geq 2$, and suppose the characteristic equation $t^2 - At - B = 0$ has two distinct roots r and s. We will show that

$$\text{for all integers } n \geq 0, \quad a_n = Cr^n + Ds^n,$$

where C and D are numbers such that

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1.$$

Let $P(n)$ be the equation

$$a_n = Cr^n + Ds^n.$$

We use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 0$. In the basis step, we prove that $P(0)$ and $P(1)$ are true. We do this because in the inductive step we need the equation to hold for $n = 0$ and $n = 1$ in order to prove that it holds for $n = 2$.

Show that $P(0)$ and $P(1)$ are true: The truth of $P(0)$ and $P(1)$ is automatic because C and D are exactly those numbers that make the following equations true:

$$a_0 = Cr^0 + Ds^0 \text{ and } a_1 = Cr^1 + Ds^1.$$

Show that for all integers $k \geq 1$, if $P(i)$ is true for all integers i from 0 through k, then $P(k+1)$ is also true: Suppose that $k \geq 1$ and for all integers i from 0 through k,

$$a_i = Cr^i + Ds^i.$$

We must show that $P(k+1)$:

$$a_{k+1} = Cr^{k+1} + Ds^{k+1}.$$

Now by the inductive hypothesis,

$$a_k = Cr^k + Ds^k \text{ and } a_{k-1} = Cr^{k-1} + Ds^{k-1},$$

so

$$\begin{aligned}
a_{k+1} &= Aa_k + Ba_{k-1} \\
&= A(Cr^k + Ds^k) + B(Cr^{k-1} + Ds^{k-1}) \\
&= C(Ar^k + Br^{k-1}) + D(As^k + Bs^{k-1}) \\
&= Cr^{k+1} + Ds^{k+1}.
\end{aligned}$$

This is what was to be shown. [The reason the last equality follows from Lemma 1 is that since r and s satisfy the characteristic equation (2), the sequences $r^0, r^1, r^2, \ldots$ and $s^0, s^1, s^2, \ldots$ satisfy the recurrence relation (1).] □

Example 3. *The Fibonacci sequence $F_0, F_1, F_2, \ldots$ satisfies the recurrence relation*

$$F_k = F_{k-1} + F_{k-2} \quad \textit{for all integersk} \geq 2$$

with initial conditions

$$F_0 = F_1 = 1.$$

Find an explicit formula for this sequence.

Solution. The Fibonacci sequence satisfies part of the hypothesis of the distinct-roots theorem since the Fibonacci relation is a second-order linear homogeneous recurrence relation with constant coefficients ($A = 1$ and $B = 1$). Is the second part of the hypothesis also satisfied? Does the characteristic equation

$$t^2 - t - 1 = 0$$

have distinct roots? By the quadratic formula, the roots are

$$t = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \begin{cases} \frac{1+\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \end{cases}$$

and so the answer is yes. It follows from the distinct-roots theorem that the Fibonacci sequence is given by the explicit formula

$$F_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for all integers } n \geq 0, \tag{3}$$

where C and D are the numbers whose values are determined by the fact that $F_0 = F_1 = 1$. To find C and D, write

$$F_0 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^0 + D\left(\frac{1-\sqrt{5}}{2}\right)^0 = C \cdot 1 + D \cdot 1 = C + D$$

and

$$F_1 = 1 = C\left(\frac{1+\sqrt{5}}{2}\right)^1 + D\left(\frac{1-\sqrt{5}}{2}\right)^1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right).$$

Thus the problem is to find numbers C and D such that

$$C + D = 1$$

and

$$C\left(\frac{1+\sqrt{5}}{2}\right)+D\left(\frac{1-\sqrt{5}}{2}\right)=1.$$

This may look complicated, but in fact it is just a system of two equations in two unknowns. The solutions are

$$C=\frac{1+\sqrt{5}}{2\sqrt{5}} \text{ and } D=\frac{-(1-\sqrt{5})}{2\sqrt{5}}.$$

Substituting these values for C and D into formula (3) gives

$$F_n=\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

or, simplifying,

$$F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \tag{4}$$

for all integers $n \geq 0$. Remarkably, even though the formula for F_n involves $\sqrt{5}$, all of the values of the Fibonacci sequence are integers. □

Theorem 1 does not work when characteristic equation has double root. In this case, ...

The Single-Root Case

Consider again the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \text{for all integers } k \geq 2, \tag{1}$$

where A and B are real numbers, but suppose now that the characteristic equation

$$t^2 - At - B = 0. \tag{2}$$

has a single real root r. By Lemma 1, one sequence that satisfies the recurrence relation is

$$1, r, r^2, r^3, \ldots, r^n, \ldots$$

But another sequence that also satisfies the relation is

$$0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$$

To see why this is so, observe that since r is the unique root of $t^2 - At - B = 0$, the left-hand side of the equation can be factored as $(t-r)^2$, and so

$$t^2 - At - B = (t-r)^2 = t^2 - 2rt + r^2. \tag{5}$$

Equating coefficients in equation (5) gives

$$A = 2r \text{ and } B = -r^2. \tag{6}$$

Let $s_0, s_1, s_2, \ldots$ be the sequence defined by the formula

$$s_n = nr^n \quad \text{for all integers } n \geq 0.$$

Then

$$\begin{aligned}
As_{k-1} + Bs_{k-2} &= A(k-1)r^{k-1} + B(k-2)r^{k-2} \\
&= 2r(k-1)r^{k-1} - r^2(k-2)r^{k-2} \\
&= 2(k-1)r^k - (k-2)r^k \\
&= (2k-2-k+2)r^k \\
&= kr^k \\
&= s_k.
\end{aligned}$$

Thus $s_0, s_1, s_2, \ldots$ satisfies the recurrence relation. This argument proves the following lemma.

Lemma 3. *Let A and B be real numbers and suppose the characteristic equation*

$$t^2 - At - B = 0. \tag{2}$$

has a single root r. Then the sequences $1, r, r^2, r^3, \ldots, r^n, \ldots$ and $0, r, 2r^2, 3r^3, \ldots, nr^n, \ldots$ both satisfy the recurrence relation

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

Lemmas 2 and 3 can be used to establish the single-root theorem, which shows how to find an explicit formula for any recursively defined sequence satisfying a second-order linear homogeneous recurrence relation with constant coefficients for which the characteristic equation has just one root. Taken together, the distinct-roots and single-root theorems cover all second-order linear homogeneous recurrence relations with constant coefficients. The proof of the single-root theorem is very similar to that of the distinct-roots theorem.

Theorem 2 (Single-Root Theorem)**.** *Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2} \quad \textit{for all integers } k \geq 2, \tag{1}$$

for some real numbers A and B with $B \neq 0$. If the characteristic equation

$$t^2 - At - B = 0 \tag{2}$$

has a single (real) root r, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Dnr^n,$$

where C and D are the numbers whose values are determined by the values a_0 and any other known value of the sequence.

Example 4. *Suppose a sequence $b_0, b_1, b_2, \ldots$ satisfies the recurrence relation*

$$b_k = 4b_{k-1} - 4b_{k-2} \quad \textit{for all integers } k \geq 2, \tag{7}$$

with initial conditions

$$b_0 = 1 \textit{ and } b_1 = 3.$$

Find an explicit formula for $b_0, b_1, b_2, \ldots$.

Solution. This sequence satisfies part of the hypothesis of the single-root theorem because it satisfies a second-order linear homogeneous recurrence relation with constant coefficients ($A = 4$ and $B = -4$). The single-root condition is also met because the characteristic equation

$$t^2 - 4t + 4 = 0$$

has the unique root $r = 2$ [since $t^2 - 4t + 4 = (t-2)^2$].

It follows from the single-root theorem that $b_0, b_1, b_2, \ldots$ is given by the explicit formula

$$b_n = C \cdot 2^n + Dn2^n \quad \text{for all integers } n \geq 0, \tag{8}$$

where C and D are the real numbers whose values are determined by the fact that $b_0 = 1$ and $b_1 = 3$. To find C and D, write

$$b_0 = C \cdot 2^0 + D \cdot 0 \cdot 2^0 = C$$

and

$$b_1 = C \cdot 2^1 + D \cdot 1 \cdot 2^1 = 2C + 2D.$$

Hence the problem is to find numbers C and D such that

$$C = 1$$

and

$$2C + 2D = 3.$$

Substitute $C = 1$ into the second equation to obtain

$$2 + 2D = 3,$$

and so

$$D = \frac{1}{2}.$$

Now substitute $C = 1$ and $D = \frac{1}{2}$ into formula (8) to conclude that

$$b_n = 2^n + \frac{1}{2}n2^n = 2^n\left(1 + \frac{n}{2}\right) \quad \text{for all integers } n \geq 0. \qquad \square$$

Theorem 3. *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = A_1 r_1^n + A_2 r_2^n + \cdots + A_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $A_1, A_2, \ldots$ are constants.

Example 5. *Find the solution to the recurrence relation*

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2, a_1 = 5,$ and $a_2 = 15$.

Solution. The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6$$

By the rational root test, the possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$. We find that $r = 1$ is a root. We find the other roots by dividing $r - 1$ into $r^3 - 6r^2 + 11r - 6$. The characteristic roots are $r_1 = 1, r_2 = 2$, and $r_3 = 3$. Hence, the solutions to this recurrence relation are of the form

$$a_n = A \cdot 1^n + B \cdot 2^n + C \cdot 3^n.$$

To find the constants A, B, and C, use the initial conditions. This gives

$$\begin{aligned} a_0 = 2 &= A + B + C \\ a_1 = 5 &= A + 2B + 3C \\ a_2 = 15 &= A + 4B + 9C \end{aligned}$$

When these three simultaneous equations are solved for A, B, and C, we find that $A = 1, B = -1$, and $C = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n. \qquad \square$$

Theorem 4 gives an analogue of Theorem 3 where roots can have multiplicity.

Theorem 4. *Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation*

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &+ (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &+ \cdots \\ &+ (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.