Solving Linear Recurrence Relations

Niloufar Shafiei

Review

A recursive definition of a sequence specifies

- Initial conditions
- Recurrence relation

Example:

$a_0=0$ and $a_1=3$ → Initial conditions

$a_n = 2a_{n-1} - a_{n-2}$ → Recurrence relation

$a_n = 3n$ → Solution

Linear recurrences

Linear recurrence:

Each term of a sequence is a linear function of earlier terms in the sequence.

For example:

$a_0 = 1$ $\quad$ $a_1 = 6$ $\quad$ $a_2 = 10$

$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$

$a_3 = a_0 + 2a_1 + 3a_2$

$= 1 + 2(6) + 3(10) = 43$

Linear recurrences

Linear recurrences

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences

Linear homogeneous recurrences

A **linear homogenous recurrence relation of degree k** with constant coefficients is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k},$$

where $c_1, c_2, \dots, c_k$ are real numbers, and $c_k \neq 0$.

a_n is expressed in terms of the previous k terms of the sequence, so its degree is k.

This recurrence includes k initial conditions.

$a_0 = C_0 \qquad a_1 = C_1 \qquad \dots \qquad a_k = C_k$

Example

Determine if the following recurrence relations are linear homogeneous recurrence relations with constant coefficients.

- $P_n = (1.11)P_{n-1}$
  a linear homogeneous recurrence relation of degree one
- $a_n = a_{n-1} + a^2_{n-2}$
  not linear
- $f_n = f_{n-1} + f_{n-2}$
  a linear homogeneous recurrence relation of degree two
- $H_n = 2H_{n-1}+1$
  not homogeneous
- $a_n = a_{n-6}$
  a linear homogeneous recurrence relation of degree six
- $B_n = nB_{n-1}$
  does not have constant coefficient

Solving linear homogeneous recurrences

Proposition 1:

- Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence a_n satisfies the recurrence.
- Assume the sequence a'_n also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n=\alpha a_n$ are also sequences that satisfy the recurrence.

  (α is any constant)

Proof:

$$\begin{aligned} b_n &= a_n + a'_n \\ &= (c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}) + (c_1a'_{n-1} + c_2a'_{n-2} + \ldots + c_ka'_{n-k}) \\ &= c_1(a_{n-1}+ a'_{n-1}) + c_2(a_{n-2}+ a'_{n-2}) + \ldots + c_k(a_{n-k} + a'_{n-k}) \\ &= c_1b_{n-1} + c_2b_{n-2} + \ldots + c_kb_{n-k} \end{aligned}$$

So, b_n is a solution of the recurrence.

Solving linear homogeneous recurrences

Proposition 1:

- Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence a_n satisfies the recurrence.
- Assume the sequence a'_n also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n=\alpha a_n$ are also sequences that satisfy the recurrence.

  (α is any constant)

Proof:

$$
\begin{aligned}
d_n &= \alpha a_n \\
&= \alpha\,(c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}) \\
&= c_1\,(\alpha a_{n-1}) + c_2\,(\alpha a_{n-2}) + \ldots + c_k\,(\alpha a_{n-k}) \\
&= c_1d_{n-1} + c_2d_{n-2} + \ldots + c_kd_{n-k}
\end{aligned}
$$

So, d_n is a solution of the recurrence.

Solving linear homogeneous recurrences

It follows from the previous proposition, if we find some solutions to a linear homogeneous recurrence, then **any linear combination** of them will also be a solution to the linear homogeneous recurrence.

Solving linear homogeneous recurrences

Geometric sequences come up a lot when solving linear homogeneous recurrences.

So, try to find any solution of the form $a^n = r^n$ that satisfies the recurrence relation.

Solving linear homogeneous recurrences

- Recurrence relation

  $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$

- Try to find a solution of form r^n

  $r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_kr^{n-k}$

  $r^n - c_1r^{n-1} - c_2r^{n-2} - \ldots - c_kr^{n-k} = 0$

  $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$ (dividing both sides by r^{n-k})

This equation is called the **characteristic equation**.

Example

Example:

The Fibonacci recurrence is

$$F_n = F_{n-1} + F_{n-2}$$

Its characteristic equation is

$$r^2 - r - 1 = 0$$

Solving linear homogeneous recurrences

Proposition 2:

r is a solution of $r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0$ if and only if r^n is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$.

Example:

consider the characteristic equation $r^2 - 4r + 4 = 0$.

$r^2 - 4r + 4 = (r - 2)^2 = 0$

So, $r=2$.

So, 2^n satisfies the recurrence $F_n = 4F_{n-1} - 4F_{n-2}$.

$2^n = 4 \cdot 2^{n-1} - 4 \cdot 2^{n-2}$

$2^{n-2} (4 - 8 + 4) = 0$

Solving linear homogeneous recurrences

Theorem 1:

- Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.
- Assume $r_1, r_2, \ldots$and r_m all satisfy the equation.
- Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any constants.
- So, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n$ satisfies the recurrence.

Proof:

By Proposition 2, $\forall i\ r_i^n$ satisfies the recurrence.

So, by Proposition 1, $\forall i\ \alpha_i r_i^n$ satisfies the recurrence.

Applying Proposition 1 again, the sequence $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n$ satisfies the recurrence.

Example

What is the solution of the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$

with $a_0=2$ and $a_1=7$?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

  $r^2 - r - 2 = 0$

  $(r+1)(r-2) = 0$ $\quad$ $r_1 = 2$ and $r_2 = -1$
- So, by theorem $a_n = \alpha_1 2^n + \alpha_2(-1)^n$ is a solution.
- Now we should find α_1 and α_2 using initial conditions.

  $a_0= \alpha_1 + \alpha_2 = 2$

  $a_1= \alpha_1 2 + \alpha_2(-1) = 7$
- So, $\alpha_1= 3$ and $\alpha_2 = -1$.
- $a_n = 3 . 2^n - (-1)^n$ is a solution.

Example

What is the solution of the recurrence relation

$f_n = f_{n-1} + f_{n-2}$

with $f_0=0$ and $f_1=1$?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

  $r^2 - r - 1 = 0$

  $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$
- So, by theorem $f_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$ is a solution.
- Now we should find α_1 and α_2 using initial conditions.

  $f_0 = \alpha_1 + \alpha_2 = 0$

  $f_1 = \alpha_1(1+\sqrt{5})/2 + \alpha_2 (1-\sqrt{5})/2 = 1$
- So, $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$.
- $a_n = 1/\sqrt{5}\ .\ ((1+\sqrt{5})/2)^n - 1/\sqrt{5}((1-\sqrt{5})/2)^n$ is a solution.

Example

What is the solution of the recurrence relation

$$a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$$

with $a_0=8$, $a_1=6$ and $a_2=26$?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

  $r^3 + r^2 - 4r - 4 = 0$

  $(r+1)(r+2)(r-2) = 0$ $\quad r_1 = -1$, $r_2 = -2$ and $r_3 = 2$
- So, by theorem $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3 2^n$ is a solution.
- Now we should find α_1, α_2 and α_3 using initial conditions.

  $a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$

  $a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$

  $a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$
- So, $\alpha_1 = 2$, $\alpha_2 = 1$ and $\alpha_3 = 5$.
- $a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n$ is a solution.

Solving linear homogeneous recurrences

If the characteristic equation has k distinct solutions $r_1, r_2, \ldots, r_k$, it can be written as

$$(r - r_1)(r - r_2)\ldots(r - r_k) = 0.$$

If, after factoring, the equation has m+1 factors of $(r - r_1)$, for example, r_1 is called a solution of the characteristic equation with multiplicity m+1.

When this happens, not only r_1^n is a solution, but also nr_1^n, $n^2r_1^n$, … and $n^m r_1^n$ are solutions of the recurrence.

Solving linear homogeneous recurrences

Proposition 3:

- Assume r_0 is a solution of the characteristic equation with multiplicity at least m+1.
- So, $n^m r_0^n$ is a solution to the recurrence.

Solving linear homogeneous recurrences

When a characteristic equation has fewer than k distinct solutions:

- We obtain sequences of the form described in Proposition 3.
- By Proposition 1, we know any combination of these solutions is also a solution to the recurrence.
- We can find those that satisfies the initial conditions.

Solving linear homogeneous recurrences

Theorem 2:

- Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.
- Assume the characteristic equation has $t \le k$ distinct solutions.
- Let $\forall i$ $(1 \le i \le t)$ r_i with multiplicity m_i be a solution of the equation.
- Let $\forall i,j$ $(1 \le i \le t$ and $0 \le j \le m_i - 1)$ α_{ij} be a constant.
- So, $a_n = (\alpha_{10} + \alpha_{11} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n$
  $+ (\alpha_{20} + \alpha_{21} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$
  $+ \ldots$
  $+ (\alpha_{t0} + \alpha_{t1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$

  satisfies the recurrence.

Example

What is the solution of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$

with $a_0=1$ and $a_1=6$?

Solution:

- First find its characteristic equation

  $r^2 - 6r + 9 = 0$

  $(r - 3)^2 = 0$ $r_1 = 3$ (Its multiplicity is 2.)
- So, by theorem $a_n = (\alpha_{10} + \alpha_{11}n)(3)^n$ is a solution.
- Now we should find constants using initial conditions.

  $a_0 = \alpha_{10} = 1$

  $a_1 = 3\,\alpha_{10} + 3\alpha_{11} = 6$
- So, $\alpha_{11} = 1$ and $\alpha_{10} = 1$.
- $a_n = 3^n + n3^n$ is a solution.

Example

What is the solution of the recurrence relation

$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

with $a_0=1$, $a_1=-2$ and $a_2=-1$?

Solution:

- Find its characteristic equation

  $r^3 + 3r^2 + 3r + 1 = 0$

  $(r + 1)^3 = 0$ $\quad$ $r_1 = -1$ $\quad$ (Its multiplicity is 3.)
- So, by theorem $a_n = (\alpha_{10} + \alpha_{11}n + \alpha_{12}n^2)(-1)^n$ is a solution.
- Now we should find constants using initial conditions.

  $a_0= \alpha_{10} = 1$

  $a_1= -\alpha_{10} - \alpha_{11} - \alpha_{12} = -2$

  $a_2= \alpha_{10} + 2\alpha_{11} + 4\alpha_{12} = -1$
- So, $\alpha_{10}= 1$, $\alpha_{11} = 3$ and $\alpha_{12} = -2$.
- $a_n = (1 + 3n - 2n^2) (-1)^n$ is a solution.

Example

What is the solution of the recurrence relation

$a_n = 8a_{n-2} - 16a_{n-4}$, for $n \geq 4$,

with $a_0=1$, $a_1=4$, $a_2=28$ and $a_3=32$?

Solution:

- Find its characteristic equation

  $r^4 - 8r^2 + 16 = 0$

  $(r^2 - 4)^2 = (r-2)^2 (r+2)^2 = 0$

  $r_1 = 2$ $\quad$ $r_2 = -2$ $\quad$ (Their multiplicities are 2.)
- So, by theorem $a_n = (\alpha_{10} + \alpha_{11} n)(2)^n + (\alpha_{20} + \alpha_{21} n)(-2)^n$ is a solution.
- Now we should find constants using initial conditions.

  $a_0 = \alpha_{10} + \alpha_{20} = 1$

  $a_1 = 2\alpha_{10} + 2\alpha_{11} - 2\alpha_{20} - 2\alpha_{21} = 4$

  $a_2 = 4\alpha_{10} + 8\alpha_{11} + 4\alpha_{20} + 8\alpha_{21} = 28$

  $a_3 = 8\alpha_{10} + 24\alpha_{11} - 8\alpha_{20} - 24\alpha_{21} = 32$
- So, $\alpha_{10} = 1$, $\alpha_{11} = 2$, $\alpha_{20} = 0$ and $\alpha_{21} = 1$.
- $a_n = (1 + 2n)\, 2^n + n\, (-2)^n$ is a solution.

Linear non-homogeneous recurrences

A **linear non-homogenous recurrence relation** with constant coefficients is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + f(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $f(n)$ is a function depending only on n.

The recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k},$$

is called the **associated homogeneous recurrence relation**.

This recurrence includes k initial conditions.

$a_0 = C_0$ $\qquad$ $a_1 = C_1$ … $\qquad$ $a_k = C_k$

Example

The following recurrence relations are linear non-homogeneous recurrence relations.

- $a_n = a_{n-1} + 2^n$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$
- $a_n = a_{n-1} + a_{n-4} + n!$
- $a_n = a_{n-6} + n2^n$

Linear non-homogeneous recurrences

Proposition 4:

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$ be a linear non-homogeneous recurrence.
- Assume the sequence b_n satisfies the recurrence.
- Another sequence a_n satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is also a sequence that satisfies the associated homogeneous recurrence.

Linear non-homogeneous recurrences

Proof:

Part1: if h_n satisfies the associated homogeneous recurrence then a_n is satisfies the non-homogeneous recurrence.

- $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + f(n)$
- $h_n = c_1 h_{n-1} + c_2 h_{n-2} + \ldots + c_k h_{n-k}$

$b_n + h_n$

$= c_1 (b_{n-1} + h_{n-1}) + c_2 (b_{n-2} + h_{n-2}) + \ldots + c_k (b_{n-k} + h_{n-k}) + f(n)$

Since $a_n = b_n + h_n$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$.

So, a_n is a solution of the non-homogeneous recurrence.

Linear non-homogeneous recurrences

Proof:

Part2: if a_n satisfies the non-homogeneous recurrence then h_n is satisfies the associated homogeneous recurrence.

- $b_n = c_1b_{n-1} + c_2b_{n-2} + \ldots + c_kb_{n-k} + f(n)$
- $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}+ f(n)$

$a_n - b_n$

$= c_1 (a_{n-1}- b_{n-1}) + c_2 (a_{n-2}- b_{n-2}) + \ldots + c_k (a_{n-k} - b_{n-k})$

Since $h_n = a_n - b_n$, $h_n = c_1h_{n-1} + c_2h_{n-2} + \ldots + c_kh_{n-k}$

So, h_n is a solution of the associated homogeneous recurrence.

Linear non-homogeneous recurrences

Proposition 4:

- Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} + f(n)$ be a linear non-homogeneous recurrence.
- Assume the sequence b_n satisfies the recurrence.
- Another sequence a_n satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is also a sequence that satisfies the associated homogeneous recurrence.

- We already know how to find h_n.
- For many common f(n), a solution b_n to the non-homogeneous recurrence is similar to f(n).
- Then you should find solution $a_n = b_n + h_n$ to the non-homogeneous recurrence that satisfies both recurrence and initial conditions.

Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 3n + 1 \text{ for } n \geq 2,$$

with $a_0=2$ and $a_1=3$?

Solution:

- Since it is linear non-homogeneous recurrence, b_n is similar to f(n)
  Guess: $b_n = cn + d$
  $b_n = b_{n-1} + b_{n-2} + 3n + 1$
  $cn + d = c(n-1) + d + c(n-2) + d + 3n + 1$
  $cn + d = cn - c + d + cn - 2c + d + 3n + 1$
  $0 = (3+c)n + (d-3c+1)$
  $c = -3 \qquad d=-10$
- So, $b_n = -3n - 10$.
  (b_n only satisfies the recurrence, it does not satisfy the initial conditions.)

Example

What is the solution of the recurrence relation

$$a_n = a_{n-1} + a_{n-2} + 3n + 1 \text{ for } n \geq 2,$$

with $a_0=2$ and $a_1=3$?

Solution:

- We are looking for a_n that satisfies both recurrence and initial conditions.
- $a_n = b_n + h_n$ where h_n is a solution for the associated homogeneous recurrence: $h_n = h_{n-1} + h_{n-2}$
- By previous example, we know $h_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$.

  $a_n = b_n + h_n$

  $= - 3n - 10 + \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n$
- Now we should find constants using initial conditions.

  $a_0= -10 + \alpha_1 + \alpha_2 = 2$

  $a_1= -13 + \alpha_1 (1+\sqrt{5})/2 + \alpha_2 (1-\sqrt{5})/2 = 3$

  $\alpha_1 = 6 + 2\sqrt{5}$ $\qquad$ $\alpha_2 = 6 - 2\sqrt{5}$

So, $a_n = -3n - 10 + (6 + 2\sqrt{5})((1+\sqrt{5})/2)^n + (6 - 2\sqrt{5})((1-\sqrt{5})/2)^n$.

Example

What is the solution of the recurrence relation

$a_n = 2a_{n-1} - a_{n-2} + 2^n$ for $n \geq 2$,

with $a_0=1$ and $a_1=2$?

Solution:

- Since it is linear non-homogeneous recurrence, b_n is similar to f(n)

  Guess: $b_n = c2^n + d$

  $b_n = 2b_{n-1} - b_{n-2} + 2^n$

  $c2^n + d = 2(c2^{n-1} + d) - (c2^{n-2} + d) + 2^n$

  $c2^n + d = c2^n + 2d - c2^{n-2} - d + 2^n$

  $0 = (-4c + 4c - c + 4)2^{n-2} + (-d + 2d -d)$

  $c = 4 \qquad d=0$

- So, $b_n = 4 . 2^n$.

  (b_n only satisfies the recurrence, it does not satisfy the initial conditions.)

Example

What is the solution of the recurrence relation

$a_n = 2a_{n-1} - a_{n-2} + 2^n$ for $n \geq 2$,

with $a_0=1$ and $a_1=2$?

Solution:

- We are looking for a_n that satisfies both recurrence and initial conditions.
- $a_n = b_n + h_n$ where h_n is a solution for the associated homogeneous recurrence: $h_n = 2h_{n-1} - h_{n-2}$.
  - Find its characteristic equation

    $r^2 - 2r + 1 = 0$

    $(r - 1)^2 = 0$

    $r_1 = 1$ (Its multiplicity is 2.)
- So, by theorem $h_n = (\alpha_1 + \alpha_2 n)(1)^n = \alpha_1 + \alpha_2 n$ is a solution.

Example

What is the solution of the recurrence relation

$a_n = 2a_{n-1} - a_{n-2} + 2^n$ for $n \geq 2$,

with $a_0=1$ and $a_1=2$?

Solution:

- $a_n = b_n + h_n$
- $a_n = 4 . 2^n + \alpha_1 + \alpha_2 n$ is a solution.
- Now we should find constants using initial conditions.

$a_0 = 4 + \alpha_1 = 1$

$a_1 = 8 - \alpha_1 + \alpha_2 = 2$

$\alpha_1 = -3 \qquad \alpha_2 = -3$

So, $a_n = 4 . 2^n - 3n - 3$.

Recommended exercises

1,3,15,17,19,21,23,25,31,35

Eric Ruppert's Notes about Solving Recurrences

(http://www.cse.yorku.ca/course_archive/2007-08/F/1019/A/recurrence.pdf)