CS1800-Midterm practice super-problems ★

These problems are definitely harder than the midterm (even the ones without ★), so if you solve them you should have no problem at all with the exam. However be aware that not all concepts are covered in these problems, so make sure to recap:

- binary/hex/octal representations and tranformations, powers of 2, fast multiplication
- logic operators, truth tables, dnf/cnf and logic rules, logic gates, circuits
- modulo/number theory: all items in the summary from notes
- sets operations, power set, cartesian products, inclusion-exclusion
- product rule, permutations, combinations, balls in bins, binomial theorem, Pascal triangle

EC 1 . n=143=11*13
compute 12^{2403} mod n
compute 10^{2403} mod n

EC 2 . Show that any natural number whose binary representation has the 1 bits organized in pairs (011, 0110000, 11011000, 110001100000110, etc) is a multiple of 3.

EC 3 . How many natural numbers smaller than 100 are the sum of 4 different powers of 2 ? (i.e. like $83 = 2^0 + 2^1 + 2^4 + 2^6$)

EC 4 . 3 sets are given
A={naturals multiples of 2, no more than 300}
B={naturals multiples of 3, no more than 300}
C={naturals multiples of 5, no more than 300}
Compute the size of the union of these three sets
Compute the size of the intersection of these three sets

EC 5 (difficulty ★). We have 100 red balls and 150 blue balls, and 20 bins; each bin must contain at least as many blue balls as red ones. In how many ways we can arrange all 250 balls into these 20 bins?

EC 6. In how many ways 20 husband-wife couples can sit at a round table with 40 seats unnumbered, such that every husband sits next to his wife?

EC 7 (difficulty ★★★). What is the probability of 20 husband-wife couples sitting randomly at a round table with 40 seats unnumbered, that no husband sits next to his wife?

EC 8 (difficulty ★). Show that picking a random positive integer (say int 4 bytes), the chance of getting a prime is smaller than 5%.

EC 9 (difficulty ★★). Show that one cannot cut three 3x3 squares and six 2x3 rectangles from a sheet of paper 8x8 square.

EC 10 . A is a set with 21 elements. How many subsets of A have size multiple of 4?

EC 11 (difficulty ★). Given any 7 integers, show that there are 2 of them with either sum or difference or product = multiple of 15.

EC 12 . Given any 5 integers a,b,c,d,e show that there is a nonempty subset of them that with +,- operands gives a multiple of 31 (i.e. at least one of these is multiple of 31: +a; +a+b ; -a+c-d ; -b-d; -c ; c-b-d+e, etc)

EC 13 (difficulty ★★). At bridge card game, the 52 cards in the deck (4 suits each with numerals 2-10 and JQKA) are evaluated as numerals=0points; J=1 point; Q=2points; K=3points; A=4points. A "bridge hand" is a subset of 13 cards. How many different hands have at least 12 points?

EC 14. Linear cipher modulo 26, "cipher = a*message + b mod 26" uses unknown a, b.
You intercept two of your own messages encrypted, so for these you know both the message and the cipher:
message =10 $\Rightarrow$ cipher =5
message =4 $\Rightarrow$ cipher =1
Find a and b.

EC 15 . Two sorted sequences lengths 20 and 7 are given: (1,2,3,...20) and (a,b,c,d,e,f,g). We want to interleave them into a sequence of length 27 such that numbers 1-20 remain in relative order, and also literals a-g remain in relative order. How many ways to do so ?

EC 16 (difficulty ★). Valid passwords of length 6 can use the 10 digits and the 26 capital letters in any order, with the condition that two digits cannot be next to each other. How many passwords ?

EC 17 (difficulty ★★★). In how many ways can we parenthesize (orders of operations) the multiplication of 10 values $abcdefghij$? To be clear, there are 5 ways to multiply 4 numbers $abcd$:

$((ab)c)d$; $(a(bc))d$; $(ab)(cd)$; $a((bc)d)$; $a(b(cd))$

EC 18 (difficulty ★). Show that $\binom{22}{11}$ is a multiple of 12

EC 19 . Show that the number obtained by concatenating "337" with itself 5 times 337337337337337 is not prime.

EC 20 . Is the GCD transitive in the following way ?
Hypothesis: $gcd(a, b) = gcd(b, c) = d \Rightarrow gcd(a, c) = d$

EC 21. A group of 11 students is such that everyone has exactly 10 friends in the group (including himself/herself; friends are reciprocal). Thats impossible.

EC 22. Show that $\sqrt{5}$ is not a ratio of two integers

EC 23. Two positive integers a, b have both the sum and the difference a power of 2. Show that neither of them is a power of two, but their gcd is.

EC 24 (difficulty ★★★). Evil Search Engine finds for a query 15 relevant URL results, and while keeping them ranked in a given score-relevance order, it is mixing them together with other 15 random ads; then serves them as a ranked list of 30. How many arrangements of 30 are there (order among ads doesnt matter) so that when served as 3 pages each with 10 links, each page contains at least 3 relevant and at least 3 ads ?

EC 25 (difficulty ★★). A = {1,2,3,4,5,6,7}. What is the max number of 4-element-subsets we can select, such that intersection of any 3 of them is empty ?

EC 26 (difficulty ★★). A = $\{1,2,3,4,5,6,7\}$. How many sequences of length 20 with elements from A are never decreasing ?