

29.1 Standard and slack forms

This section describes two formats, standard form and slack form, that will be useful in specifying and working with linear programs. In standard form, all the constraints are inequalities, whereas in slack form, the constraints are equalities.

Standard form

In **standard form**, we are given n real numbers c_1, c_2, \dots, c_n ; m real numbers b_1, b_2, \dots, b_m ; and mn real numbers a_{ij} for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. We wish to find n real numbers x_1, x_2, \dots, x_n that

$$(29.16) \quad \text{maximize} \quad \sum_{j=1}^n c_j x_j$$

subject to

$$(29.17) \quad \sum_{j=1}^n a_{ij} x_j \leq b_i \quad \text{for } i = 1, 2, \dots, m$$

$$(29.18) \quad x_j \geq 0 \quad \text{for } j = 1, 2, \dots, n.$$

Generalizing the terminology we introduced for the two-variable linear program, we call expression (29.16) the **objective function** and the $n + m$ inequalities in lines (29.17) and (29.18) the **constraints**. The n constraints in line (29.18) are called the **nonnegativity constraints**. An arbitrary linear program need not have nonnegativity constraints, but standard form requires them. Sometimes we find it convenient to express a linear program in a more compact form. If we create an $m \times n$ matrix $A = (a_{ij})$, an m -dimensional vector $b = (b_i)$, an n -dimensional vector $c = (c_j)$, and an n -dimensional vector $x = (x_j)$, then we can rewrite the linear program defined in (29.16)–(29.18) as

$$(29.19) \quad \text{maximize} \quad c^T x$$

subject to

$$(29.20) \quad Ax \leq b$$

$$(29.21) \quad x \geq 0.$$

In line (29.19), $c^T x$ is the inner product of two vectors. In line (29.20), Ax is a matrix-vector product, and in line (29.21), $x \geq 0$ means that each entry of the vector x must be nonnegative. We see that we can specify a linear program in standard form by a tuple (A, b, c) , and we shall adopt the convention that A , b , and c always have the dimensions given above.

We now introduce terminology to describe solutions to linear programs. Some of this terminology was used in the earlier example of a two-variable linear program. We call a setting of the variables \bar{x} that satisfies all the constraints a **feasible solution**, whereas a setting of the variables \bar{x} that fails to satisfy at least one constraint is called an **infeasible solution**. We say that a solution \bar{x} has **objective value** $c^T \bar{x}$. A feasible solution \bar{x} whose objective value is maximum over all feasible solutions is an **optimal solution**, and we call its objective value $c^T \bar{x}$ the **optimal objective value**. If a linear program has no feasible solutions, we say that the linear program is **infeasible**; otherwise it is **feasible**. If a linear program has some feasible solutions but does not have a finite optimal objective value, we say that the linear program is **unbounded**. [Exercise 29.1-9](#) asks you to show that a linear program can have a finite optimal objective value even if the feasible region is not bounded.

Converting linear programs into standard form

It is always possible to convert a linear program, given as the minimization or maximization of a linear function subject to linear constraints, into standard form. A linear program may not be in standard form for one of four possible reasons:

1. The objective function may be a minimization rather than a maximization.
2. There may be variables without nonnegativity constraints.
3. There may be **equality constraints**, which have an equal sign rather than a less-than-or-equal-to sign.
4. There may be **inequality constraints**, but instead of having a less-than-or-equal-to sign, they have a greater-than-or-equal-to sign.

When converting one linear program L into another linear program L' , we would like the property that an optimal solution to L' will yield an optimal solution to L . To capture this idea, we say that two maximization linear programs L and L' are **equivalent** if for each feasible solution \bar{x} to L with objective value z , there is a corresponding feasible solution \bar{x}' to L' with objective value z , and for each feasible solution \bar{x}' to L' with objective value z , there is a corresponding feasible solution \bar{x} to L with objective value z . (This definition does not imply a one-to-one correspondence between feasible solutions.) A minimization linear program L and a maximization linear program L' are equivalent if for each feasible solution \bar{x} to L' with objective value z , there is a corresponding feasible solution \bar{x}' to L' with objective value $-z$, and for each feasible solution \bar{x}' to L' with objective value z , there is a corresponding feasible solution \bar{x} to L with objective value $-z$.

We now show how to remove, one by one, each of the possible problems in the list above. After removing each one, we shall argue that the new linear program is equivalent to the old one.

To convert a minimization linear program L into an equivalent maximization linear program L' , we simply negate the coefficients in the objective function. Since L and L' have identical sets of feasible solutions and, for any feasible solution, the objective value in L is the negative of the objective value in L' , these two linear programs are equivalent. For example, if we have the linear program

$$\begin{array}{ll} \text{minimize} & -2x_1 + 3x_2 \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0, \end{array}$$

and we negate the coefficients of the objective function, we obtain

$$\begin{array}{ll} \text{minimize} & 2x_1 - 3x_2 \\ \text{subject to} & \\ & x_1 + x_2 = 7 \\ & x_1 - 2x_2 \leq 4 \\ & x_1 \geq 0. \end{array}$$

Next, we show how to convert a linear program in which some of the variables do not have nonnegativity constraints into one in which each variable has a non-negativity constraint. Suppose that some variable x_j does not have a nonnegativity constraint. Then we replace each occurrence of x_j by $x'_j - x''_j$, and add the non-negativity constraints $x'_j \geq 0$ and $x''_j \geq 0$. Thus, if the objective function has a term $c_j x_j$, it is replaced by $c_j x'_j - c_j x''_j$, and if constraint i has a term $a_{ij} x_j$, it is replaced by $a_{ij} x'_j - a_{ij} x''_j$. Any feasible solution x to the new linear program corresponds to a feasible solution to the original linear program with $\bar{x}_j = \bar{x}'_j - \bar{x}''_j$ and with the same objective value, and thus the two solutions are equivalent. We apply this conversion scheme to each variable that does not have a nonnegativity constraint to yield an equivalent linear program in which all variables have non-negativity constraints.

Continuing the example, we want to ensure that each variable has a corresponding nonnegativity constraint. Variable x_1 has such a constraint, but variable x_2 does not. Therefore, we replace x_2 by two variables x'_2 and x''_2 , and we modify the linear program to obtain

$$\text{maximize } 2x_1 - 3x'_2 + 3x''_2$$

subject to

$$(29.22) \quad \begin{array}{ll} x_1 + x'_2 - x''_2 & = 7 \\ x_1 - 2x'_2 + 2x''_2 & \leq 4 \\ x_1, x'_2, x''_2 & \geq 0. \end{array}$$

Next, we convert equality constraints into inequality constraints. Suppose that a linear program has an equality constraint $f(x_1, x_2, \dots, x_n) = b$. Since $x = y$ if and only if both $x \geq y$ and $x \leq y$, we can replace this equality constraint by the pair of inequality

constraints $f(x_1, x_2, \dots, x_n) \leq b$ and $f(x_1, x_2, \dots, x_n) \geq b$. Repeating this conversion for each equality constraint yields a linear program in which all constraints are inequalities.

Finally, we can convert the greater-than-or-equal-to constraints to less-than-or-equal-to constraints by multiplying these constraints through by -1 . That is, any inequality of the form

$$\sum_{j=1}^n a_{ij}x_j \geq b_i$$

is equivalent to

$$\sum_{j=1}^n -a_{ij}x_j \leq -b_i .$$

Thus, by replacing each coefficient a_{ij} by $-a_{ij}$ and each value b_i by $-b_i$, we obtain an equivalent less-than-or-equal-to constraint.

Finishing our example, we replace the equality in constraint (29.22) by two inequalities, obtaining

$$\text{maximize } 2x_1 - 3x'_2 + 3x''_2$$

subject to

$$\begin{aligned} (29.23) \quad & x_1 + x'_2 - x''_2 \leq 7 \\ & x_1 + x'_2 - x''_2 \geq 7 \\ & x_1 - 2x'_2 + 2x''_2 \leq 4 \\ & x_1, x'_2, x''_2 \geq 0 . \end{aligned}$$

Finally, we negate constraint (29.23). For consistency in variable names, we rename x'_2 to x_2 and x''_2 to x_3 , obtaining the standard form

$$(29.24) \text{ maximize } 2x_1 - 3x_2 + 3x_3$$

subject to

$$(29.25) \quad x_1 + x_2 - x_3 \leq 7$$

$$(29.26) \quad -x_1 - x_2 + x_3 \leq -7$$

$$(29.27) \quad x_1 - 2x_2 + 2x_3 \leq 4$$

$$(29.28) \quad x_1, x_2, x_3 \geq 0 .$$

Converting linear programs into slack form

To efficiently solve a linear program with the simplex algorithm, we prefer to express it in a form in which some of the constraints are equality constraints. More precisely, we shall convert it into a form in which the nonnegativity constraints are the only inequality constraints, and the remaining constraints are equalities. Let

$$(29.29) \quad \sum_{j=1}^n a_{ij}x_j \leq b_i$$

be an inequality constraint. We introduce a new variable s and rewrite inequality (29.29) as the two constraints

$$(29.30) \quad s = b_i - \sum_{j=1}^n a_{ij}x_j ,$$

$$(29.31) \quad s \geq 0 .$$

We call s a **slack variable** because it measures the **slack**, or difference, between the left-hand and right-hand sides of equation (29.29). Because inequality (29.29) is true if and only if both equation (29.30) and inequality (29.31) are true, we can apply this conversion to each inequality constraint of a linear program, obtaining an equivalent linear program in which the only

inequality constraints are the nonnegativity constraints. When converting from standard to slack form, we shall use x_{n+i} (instead of s) to denote the slack variable associated with the i th inequality. The i th constraint is therefore

$$(29.32) \quad x_{n+i} = b_i - \sum_{j=1}^n a_{ij}x_j,$$

along with the nonnegativity constraint $x_{n+i} \geq 0$.

Applying this conversion to each constraint of a linear program in standard form, we obtain a linear program in a different form. For example, for the linear program described in (29.24)–(29.28), we introduce slack variables x_4 , x_5 , and x_6 , obtaining maximize

$$(29.33) \quad \text{maximize} \quad 2x_1 - 3x_2 + 3x_3$$

subject to

$$(29.34) \quad x_4 = 7 - x_1 - x_2 + x_3$$

$$(29.35) \quad x_5 = -7 + x_1 + x_2 - x_3$$

$$(29.36) \quad x_6 = 4 - x_1 + 2x_2 - 2x_3$$

$$(29.37) \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

In this linear program, all the constraints except for the nonnegativity constraints are equalities, and each variable is subject to a nonnegativity constraint. We write each equality constraint with one of the variables on the left-hand side of the equality and all others on the right-hand side. Furthermore, each equation has the same set of variables on the right-hand side, and these variables are also the only ones that appear in the objective function. The variables on the left-hand side of the equalities are called **basic variables**, and those on the right-hand side are called **nonbasic variables**.

For linear programs that satisfy these conditions, we shall sometimes omit the words "maximize" and "subject to," and we shall omit the explicit nonnegativity constraints. We shall also use the variable z to denote the value of the objective function. We call the resulting format **slack form**. If we write the linear program given in (29.33)–(29.37) in slack form, we obtain

$$(29.38) \quad z = 2x_1 - 3x_2 + 3x_3$$

$$(29.39) \quad x_4 = 7 - x_1 - x_2 + x_3$$

$$(29.40) \quad x_5 = -7 + x_1 + x_2 - x_3$$

$$(29.41) \quad x_6 = 4 - x_1 + 2x_2 - 2x_3.$$

As with standard form, it will be convenient to have a more concise notation for describing a slack form. As we shall see in [Section 29.3](#), the sets of basic and nonbasic variables will change as the simplex algorithm runs. We use N to denote the set of indices of the nonbasic variables and B to denote the set of indices of the basic variables. We will always have that $|N| = n$, $|B| = m$, and $N \cup B = \{1, 2, \dots, n + m\}$. The equations will be indexed by the entries of B , and the variables on the right-hand sides will be indexed by the entries of N . As in standard form, we use b_j , c_j , and a_{ij} to denote constant terms and coefficients. We also use v to denote an optional constant term in the objective function. Thus we can concisely define a slack form by a tuple (N, B, A, b, c, v) , denoting the slack form

$$(29.42) \quad z = v + \sum_{j \in N} c_j x_j$$

$$(29.43) \quad x_i = b_i - \sum_{j \in N} a_{ij} x_j \quad \text{for } i \in B,$$

in which all variables x are constrained to be nonnegative. Because we subtract the sum $\sum_{j \in N} a_{ij} x_j$ in (29.43), the values a_{ij} are actually the negatives of the coefficients as they "appear" in the slack form.

For example, in the slack form

$$\begin{aligned}
 z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
 x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
 x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
 x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2},
 \end{aligned}$$

we have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$,

$$A = \begin{pmatrix} a_{13} & a_{15} & a_{16} \\ a_{23} & a_{25} & a_{26} \\ a_{43} & a_{45} & a_{46} \end{pmatrix} = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix},$$

$c = c_3 \ c_5 \ c_6)^T = (-1/6 \ -1/6 \ -2/3)^T$, and $v = 28$. Note that the indices into A , b , and c are not necessarily sets of contiguous integers; they depend on the index sets B and N . As an example of the entries of A being the negatives of the coefficients as they appear in the slack form, observe that the equation for x_1 includes the term $x_3/6$, yet the coefficient a_{13} is actually $-1/6$ rather than $+1/6$.

Exercises 29.1-1

If we express the linear program in (29.24)–(29.28) in the compact notation of (29.19)–(29.21), what are n , m , A , b , and c ?

Exercises 29.1-2

Give three feasible solutions to the linear program in (29.24)–(29.28). What is the objective value of each one?

Exercises 29.1-3

For the slack form in (29.38)–(29.41), what are N , B , A , b , c , and v ?

Exercises 29.1-4

Convert the following linear program into standard form:

$$\begin{aligned}
 &\text{minimize} && 2x_1 + 7x_2 \\
 &\text{subject to} && \\
 &&& x_1 = 7 \\
 &&& 3x_1 + x_2 \geq 24 \\
 &&& x_2 \geq 0 \\
 &&& x_3 \leq 0.
 \end{aligned}$$

Exercises 29.1-5

Convert the following linear program into slack form:

$$\text{maximize} \quad 2x_1 \quad - \quad 6x_3$$

subject to

$$\begin{aligned}x_1 + x_2 - x_3 &\leq 7 \\3x_1 - x_2 &\geq 8 \\-x_1 + 2x_2 + 2x_3 &\geq 0 \\x_1, x_2, x_3 &\geq 0.\end{aligned}$$

What are the basic and nonbasic variables?

Exercises 29.1-6

Show that the following linear program is infeasible:

$$\begin{aligned}\text{maximize} \quad & 3x_1 - 2x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \leq 2 \\ & -2x_1 - 2x_2 \leq -10 \\ & x_1, x_2 \geq 0.\end{aligned}$$

Exercises 29.1-7

Show that the following linear program is unbounded:

$$\begin{aligned}\text{maximize} \quad & x_1 - x_2 \\ \text{subject to} \quad & \\ & -2x_1 + x_2 \leq -1 \\ & -x_1 - 2x_2 \leq -2 \\ & x_1, x_2 \geq 0.\end{aligned}$$

Exercises 29.1-8

Suppose that we have a general linear program with n variables and m constraints, and suppose that we convert it into standard form. Give an upper bound on the number of variables and constraints in the resulting linear program.

Exercises 29.1-9

Give an example of a linear program for which the feasible region is not bounded, but the optimal objective value is finite.