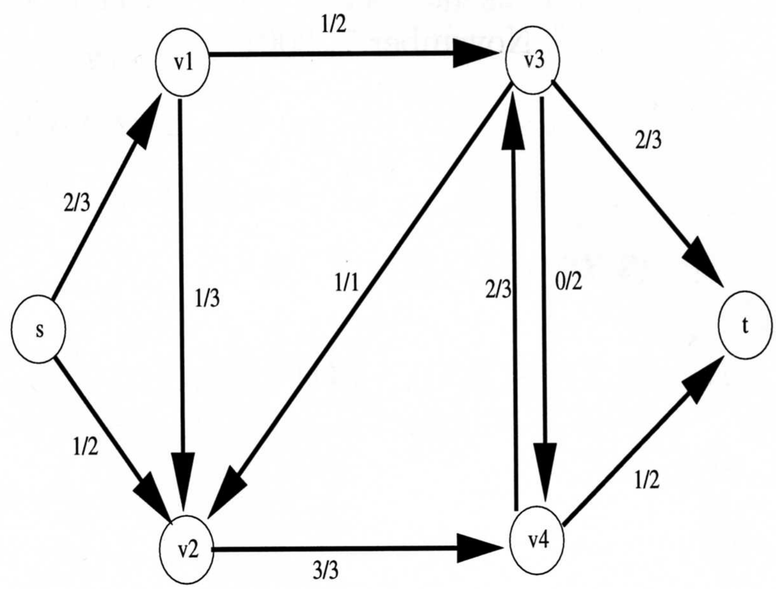
1 Maximum Flow

Definition 1.1 *a* **flow network** *is a directed graph $G = (V, E)$ in which*

- *each edge (u, v) has a capacity $c(u, v) \geq 0$. If $(u, v) \notin E$ then $c(u, v) = 0$.*
- *there is a* **source** *vertex s.*
- *there is a* **sink** *vertex t.*
- *Assume every veretx lies on some path from the source to the sink so $|E| \geq |V| - 1$*



Definition 1.2 *A* **flow** *in G is a function $f : V \times V \to \mathbf{R}$ such that the following hold*

- **Capacity constraint**: *for all $u, v \in V$, $f(u, v) \leq c(u, v)$.*
- **Skew symmetry**: *for all $u, v \in V$, $f(u, v) = -f(v, u)$.*
- **Flow conservation**: *for all $u \in V - \{s, t\}$,*

$$\sum_{v \in V} f(u, v) = 0$$

Definition 1.3 *$f(u, v)$ is called the* **net flow** *from u to v.*

Definition 1.4 *The* **value** *of a flow f is*

$$|f| = \sum_{v \in V} f(s, v)$$

i.e. the total net flow out of the source.

Definition 1.5 (Maximum flow problem) • *Given a flow network G*

- *Find a flow for G with maximum value.*

Proposition 1.6 *1. By skew symmetry, $f(u,u) = 0$ for all $u \in V$.*

2. By skew symmetry, $\sum_{v \in V} f(v,u) = 0$ for all $u \in V - x\{s,t\}$ i.e. the total net flow into a vertex is 0.

3. there can be not net flow between u and v if $(u,v) \notin E$ $(c(u,v) = c(v,u) = 0 \Rightarrow f(u,v) \leq 0$ and $f(v,u) \leq 0$. But $f(v,u) = -f(u,v)$.)

Definition 1.7 • *The* **positive net flow** *entering v is:*

$$\sum_{u \in V f(u,v) > 0} f(u,v)$$

- *The* **positive net flow** *leaving v is:*

$$\sum_{u \in V f(v,u) > 0} f(v,u)$$

Proposition 1.8 *Flow conservation says the positive net flow leaving a vertex must equal the positive net flow entering a vertex.*

Proof.

$$\sum_{v \in V} f(u,v) = 0$$

$$\Longleftrightarrow \sum_{v \in V, f(u,v) > 0} f(u,v) + \sum_{v \in V, f(u,v) < 0} f(u,v) = 0$$

$$\Longleftrightarrow \sum_{v \in V, f(u,v) > 0} f(u,v) = -\sum_{v \in V, f(u,v) < 0} f(u,v) = \sum_{v \in V, f(v,u) > 0} f(v,u)$$

Note the leftmost sum, of this last line in the positive net flow leaving u and the rightmost sum is the positive net flow entering u ∎

Explain an example in detail drawing the graph and the flow using only the positive flow: Is this a valid flow?

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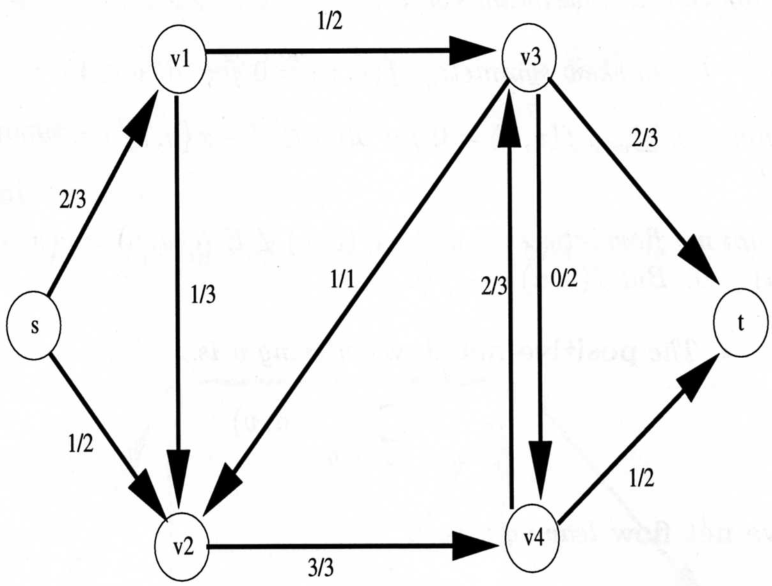
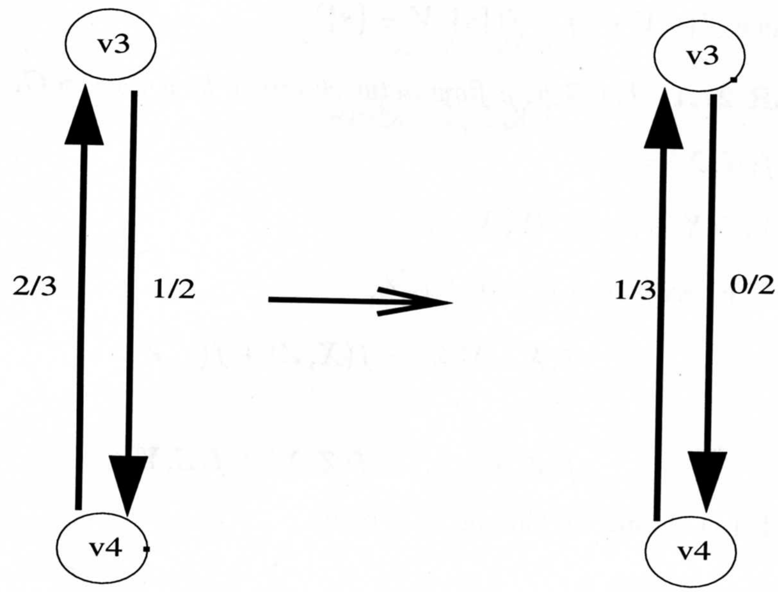


FLow

Cancellation

- We can transform any situation in which shipments are made both from u to v and v to u into a situation in which there is positive flow going only from u to v
- WLOG we can say that positive net flow goes from u to v or from v to u but not both. (If not true, we can transform by cancellation.)

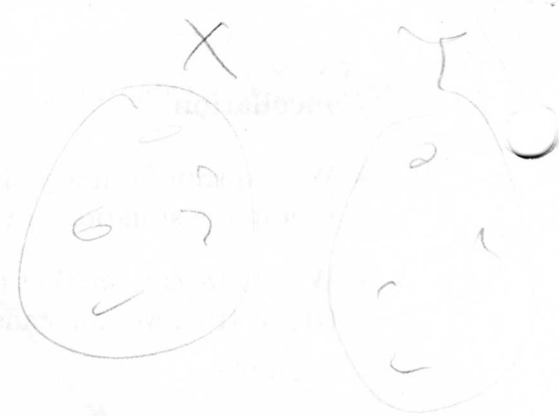
  Example:

  - capacity constraint are still satisfied since flows only decrease.
  - flow conservation still satisfied because flow in and out both reduced by the same amount.

- In other words we are only concerned with the "net" flow between vertices.

Implicit Summation



$$f(X,Y) = \sum_{x \in X} \sum_{y \in Y} f(x,y)$$
$$c(X,Y) = \sum_{x \in X} \sum_{y \in Y} c(x,y)$$

- flow conservation: $f(u,V) = 0 \forall u \in V - \{s,t\}$
- for convenience, $f(s, V - s) = f(\{s\}, V - \{s\})$

Lemma 1.9 (CLR 27.1) *Let G be a flow network and f be a flow in G.*

- *For $X \subseteq V, f(X,X) = 0$*
- *For $X, Y \subseteq V, f(X,Y) = -f(Y,X)$*
- *For $X, Y, Z \subseteq V$, with $X \cap Y = \emptyset$*

$$f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$$

and

$$f(Z, X \cup Y) = f(Z,X) + f(Z,Y)$$

Proof. Exer: 27.1-4 (I should do this proof – Nah). ∎

Proposition 1.10 $|f| = f(V,t)$

Proof.

$$\begin{aligned}
|f| &= \sum_{v \in V} f(s,v) \\
&= f(s,V) \qquad \text{by definition} \\
&= f(V,V) - f(V-s,V) \qquad \text{by Lemma 27.1} \\
&= f(V, V-s) \qquad \text{by Lemma 27.1} \\
&= f(V,t) + f(V, V-s-t) \qquad \text{by Lemma 27.1} \\
&= f(V,t) \qquad \text{flow conservation}
\end{aligned}$$

- Line 3 follows since $f(V,V) = f(V-s,V) + f(s,V)$
- Line 4 follows since $f(V,V) = 0$ and $f(V,V-s) = -f(V-s,V)$ by skew symmetry.
- Line 5 follows since $f(V,u) = 0$ for all $u \in V - \{s,t\}$

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