The complexity of distributions

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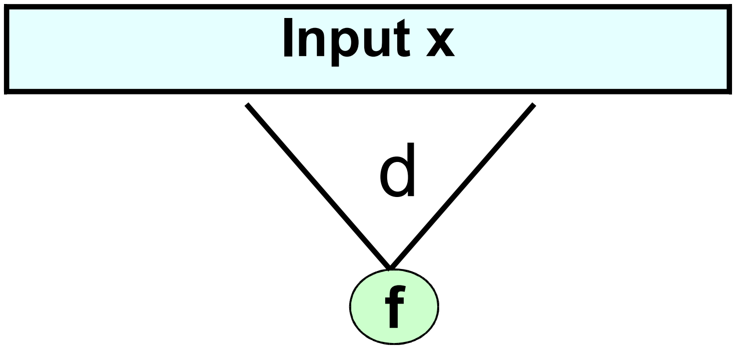
Local functions (NC^0)

- $f : \{0,1\}^n \rightarrow \{0,1\}$ d-local : output depends on d input bits



- Fact: $\text{Parity}(x) = 1 \Leftrightarrow \sum x_i = 1 \bmod 2$ is not n-1 local

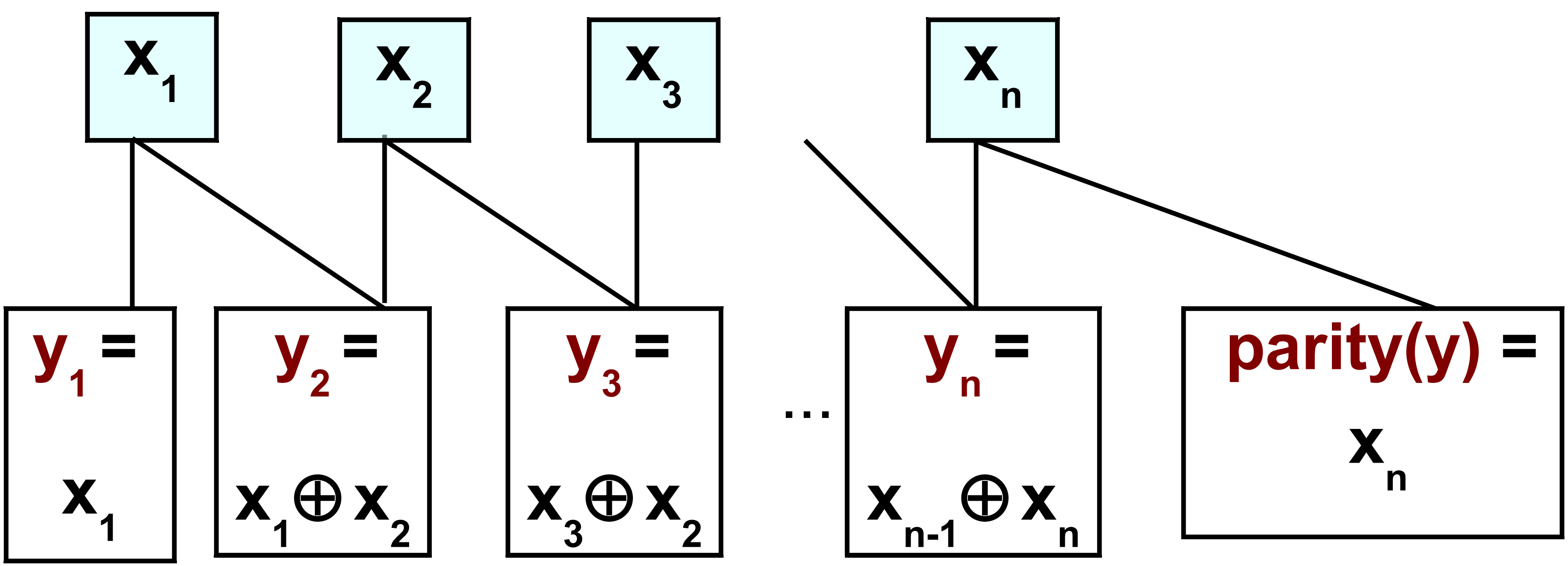
- Proof: Flip any input bit $\Rightarrow$ output flips ♦

Local generation of (Y, parity(Y))

- Theorem [Babai ; Boppana Lagarias '87]

There is $f : \{0,1\}^n \to \{0,1\}^{n+1}$, each bit 2-local

Distribution $f(X) \equiv$ (Y, parity(Y)) (X, Y $\in \{0,1\}^n$ uniform)



Our message

- Complexity theory of distributions (as opposed to functions)

How hard is it to generate (a.k.a. sample)
distribution D given random bits ?

E.g., D = (Y, parity(Y)), D = W_k := uniform n-bit with k 1's

- This work:

DNF

AC^0

NC^0

local

Rest of talk

- Generating W_k := uniform n-bit with k 1's
  - Local (NC^0)
  - Decision tree
- Results for (Y, b(Y))
- Proof of local lower bound for $W_{n/2}$

Our results: local

- Theorem

$f : \{0,1\}^n \to \{0,1\}^n$ 0.1 log n - local

$\Downarrow$

f(X) at Statistical Distance $> 1 - n^{-\Omega(1)}$

from $W_{n/2}$ = uniform w/ weight n/2

- Tight up to $\Omega()$: $f(x) = x$

- Extends to W_k, $k \neq n/2$, tight?

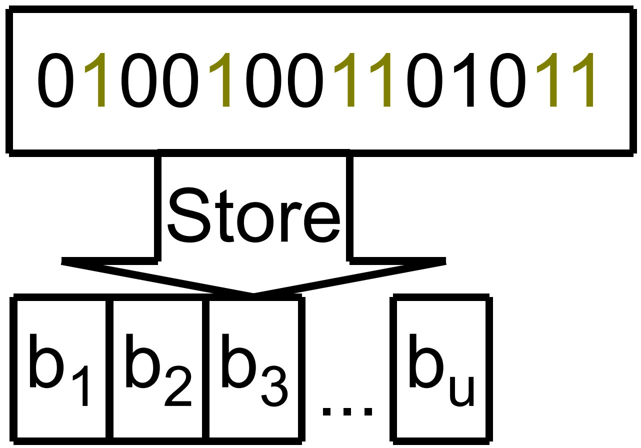
Our results: succinct data structures

- Problem:

  Store k-subset $S \subseteq \{1, 2, \ldots, n\}$

  in u = optimal + r bits,

  answer "$i \in S$?" probing d bits.



- Connection:

  Solution $\Rightarrow$ generate $W_{|S|=k}$ d-local, Stat. Distance $< 1 - 2^{-r}$

- Corollary: Need $r > \Omega(\log n)$ if $d = 0.1 \log n$

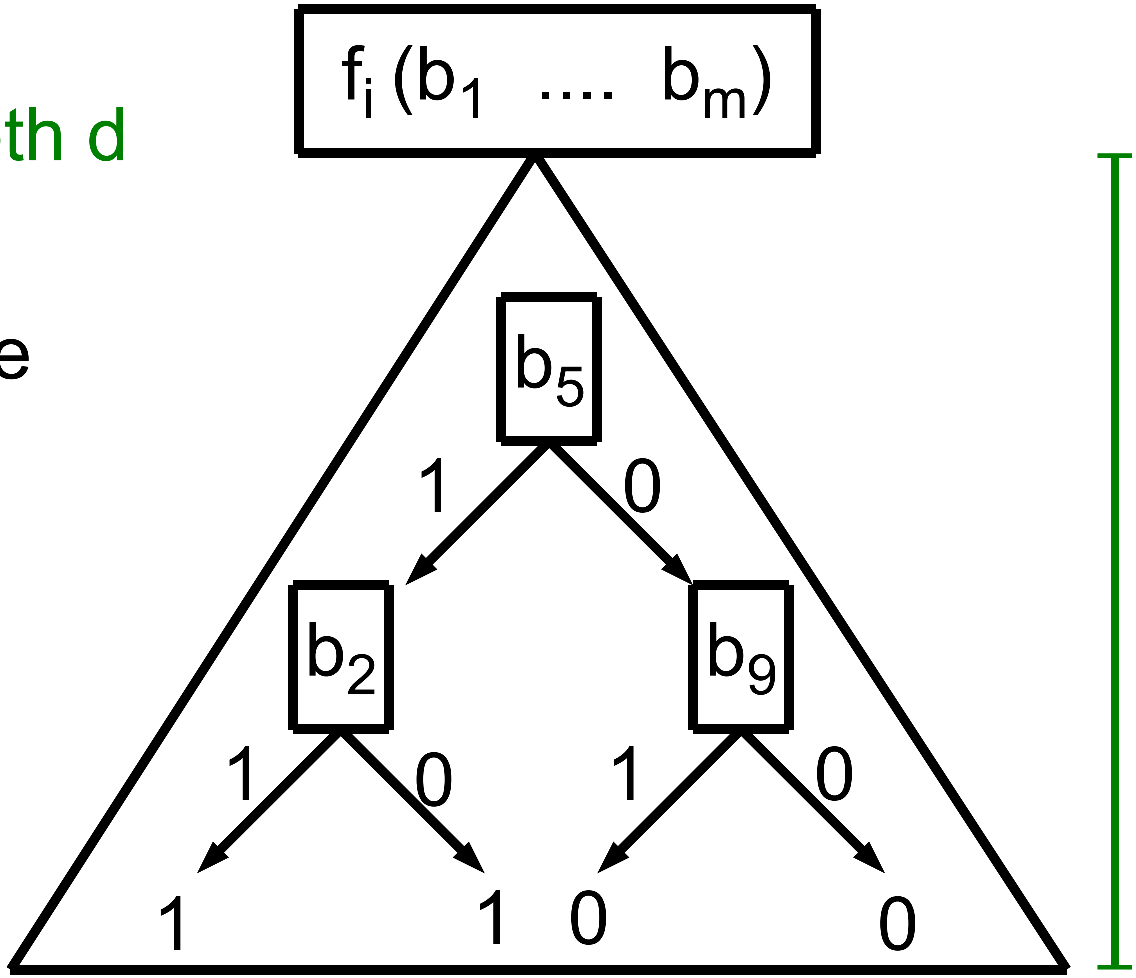
  First lower bound for $|S| = n/2, n/4, \ldots$

Decision tree model

- $f : \{0,1\}^m \to \{0,1\}^n$ depth d

  each output bit f_i

  is depth-d decision tree



- Depth d $\subseteq$ 2^d local

Our results: decision trees

- Theorem $f : \{0,1\}^* \to \{0,1\}^n$ depth $< 0.1 \log n$

  $\Rightarrow$ Distance($f(X)$, $W_{n/2}$) $> 1/n$

- Worse than $1 - n^{-\Omega(1)}$ lower bound for local

- Fact building on [Czumaj Kanarek Lorys Kutyłowski]

  $\exists$ f : depth $O(\log n)$ and Distance($f(X)$, $W_{n/2}$) $< 1/n$

Rest of talk

- Generating W_k := uniform n-bit with k 1's
  - Local (NC^0)
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Our results for (Y, b(Y))

- Theorem: $f : \{0,1\}^n \to \{0,1\}^{n+1}$

  0.1 log n-local $\Rightarrow$ Distance$\Big($ f(X) , (Y, Y mod p >p/2) $\Big)$ >0.49

  0.1 log n-depth $\Rightarrow$ Distance$\Big($f(X) , (Y, majority Y)$\Big)$ > 1/n

- Theorem building on [Matias Vishkin, Hagerup]

  $\exists$ f bounded-depth circuit AC^0 :

  Distance$\Big($ f(X) , (Y, majority Y)$\Big)$ $< 2^{-n}$

- Challenge: explicit boolean b : AC^0 can't generate (Y, b(Y))

Rest of talk

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Local lower bound

- Theorem: Let $f : \{0,1\}^n \to \{0,1\}^n$: d= 0.1 log n-local.

  $\Rightarrow \exists\, T \subseteq \{0,1\}^n$: $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| > 1 - n^{-\Omega(1)}$

- Warm-up scenarios:

- $f(x) = 000111$ Low-entropy $T := \{ 000111 \}$

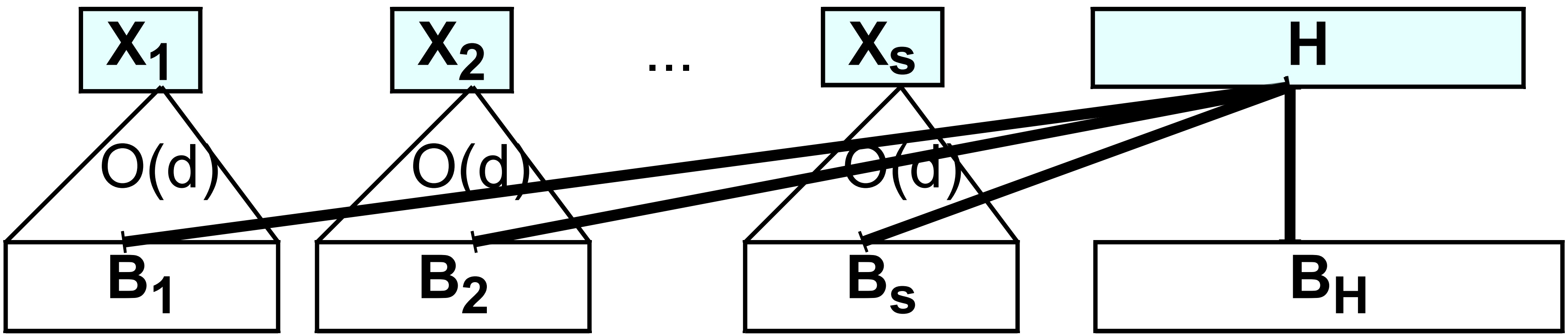
  $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| = \left| 1 - |T| / (n \text{ choose } n/2) \right|$

- $f(x) = x$ "Anti-concentration" $T := \{ z : \sum_i z_i \neq n/2 \}$

  $\left| \Pr[f(x) \in T] - \Pr[W_{n/2} \in T] \right| = \left| 1 - \Theta(1)/\sqrt{n} - 0 \right|$

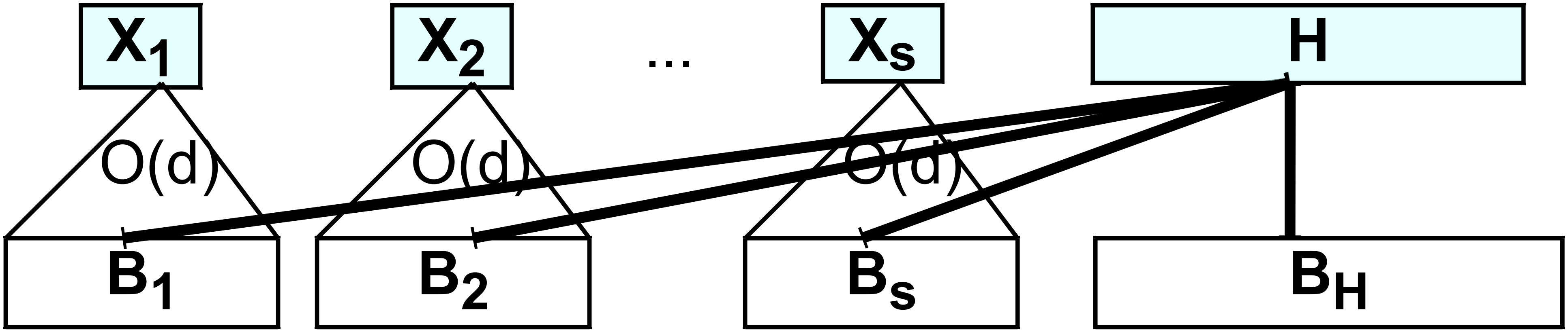
Proof

- Input $X = (X_1, X_2, \ldots, X_s, H)$



- Fix H. Output block B_i depends only on bit X_i
- Many B_i constant ($B_i(0,H) = B_i(1,H)$) $\Rightarrow$ low-entropy
- Many B_i depend on X_i ($B_i(0,H) \neq B_i(1,H)$)

  Idea: Independent $\Rightarrow$ anti-concentration: sum $\neq n/2$ w.h.p.



- If many weight($B_i(0,H)$) $\neq$ weight($B_i(1,H)$), use

> Anti-concentration Lemma [Littlewood Offord]
>
> For $a_1, a_2, \ldots, a_s \neq 0$, any c, $\Pr_{X \in \{0,1\}^s} [\sum_i a_i X_i = c] < 1/\sqrt{n}$

- Problem: $B_i(0,H) = 100$, $B_i(1,H) = 010$
  high entropy but no anti-concentration

- Fix: want many blocks 000 : high entropy $\Rightarrow$ different weight

Conclusion

- Complexity of distributions = uncharted territory

- Lower bounds for W_k := uniform n-bit with k 1's
  - Local $\Rightarrow$ lower bound for storing sets efficiently
  - Decision tree

- Lower bounds for (Y , b(Y)), e.g. (Y, majority Y)

Rest of talk

- Generating W_k := uniform n-bit with k 1's
  - Local (NC^0)
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- Results for (Y, b(Y))
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Our results: decision trees

- Theorem $f : \{0,1\}^* \to \{0,1\}^n$ depth $< 0.1 \log n$

  $\Rightarrow$ Distance($f(X)$, $W_{n/2}$) $> 1/n$

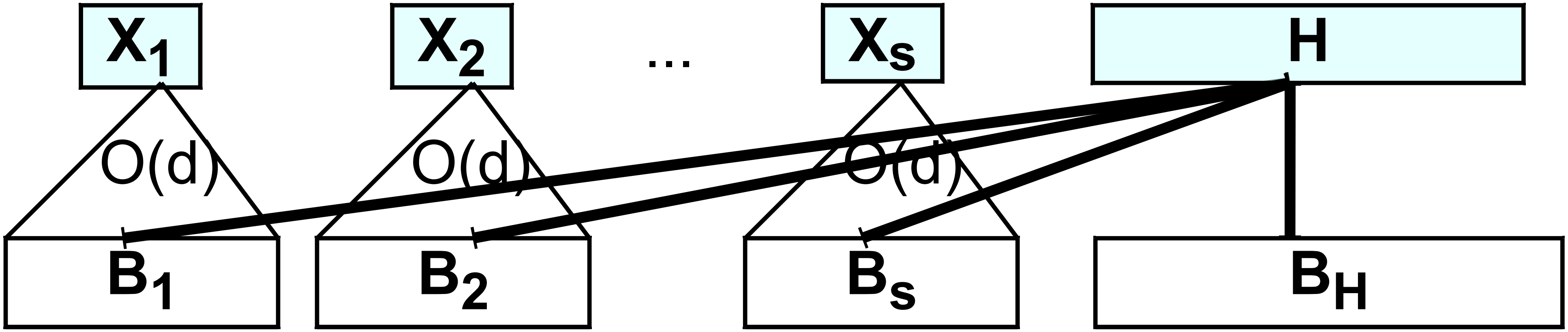
- Proof: Is $f(X)$ 4-wise independent?

  YES: [Paley Zygmund] $\sum f(x)_i$ anti-concentrated, $\neq n/2$ w.h.p.

  NO: Let Q := biased 4 bits of $f(X)$

  Distance ($f(X)\,|_Q$, $W_{n/2}\,|_Q \approx$ uniform) $> 2^{-4\,(0.1 \log n)}$

  by granularity of decision-tree probability ♦



- Test $T \subseteq \{0,1\}^n$: $Pr[f(X_1,\ldots,X_s,H) \in T] \approx 1$; $Pr[W_{n/2} \in T] \approx 0$

$z \in T \Leftrightarrow$

$\exists$ H : $\exists$ $X_1,\ldots,X_s$ w/ many blocks B_i fixed : $f(X_1,\ldots,X_s,H) = z$

OR

Few blocks $z|_{B_i}$ are 000

OR

$\sum_i z_i \neq n/2$

Rest of this talk

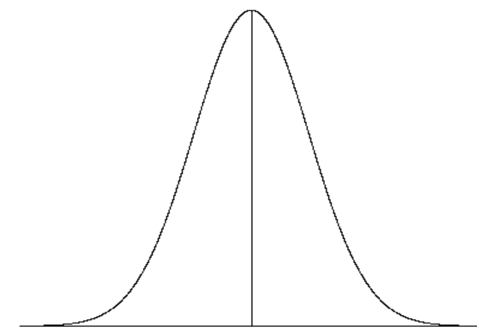
- Connection with succinct data structures

- Lower bound for locally generating $W_{n/2}$ = n-bit with n/2 1's

- Decision tree model

- Bounded-depth circuit model

Tool for lower bound proof



- Central limit theorem:

  $x_1, x_2, \ldots, x_n$ independent $\Rightarrow \sum x_i \approx$ normal

- Bounded-independence central limit theorem [Diakonikolas Gopalan Jaiswal Servedio V.]

  $x_1, x_2, \ldots, x_n$ k-wise independent $\Rightarrow \sum x_i \approx$ normal

- Note: For next result, Paley–Zygmund inequality enough

Proof

- Theorem[V.] $f : \{0,1\}^* \to \{0,1\}^n$: each bit depth $< 0.1 \log n$

  Distance($f(X)$, $W_{n/2}$) $> n^{-\Omega(1)}$

- Proof: Is output distribution $f(X)$ ($k = 10$)-wise independent?

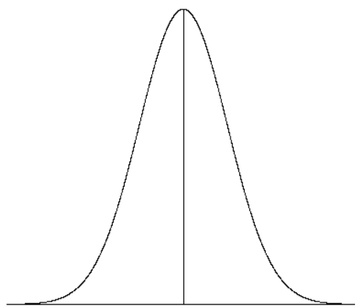
  NO $\Rightarrow W_{n/2} \approx$ k-wise independent

  Distance(those k bits, uniform on $\{0,1\}^k$) $> 2^{-k(0.1 \log n)}$

  (granularity of decision tree probability)

  YES $\Rightarrow$ by prev. theorem $\sum f(X)_i \approx$ normal

  so often $\sum f(X)_i \neq n/2$ ♦



Rest of this talk

- Connection with succinct data structures

- Lower bound for locally generating $W_{n/2}$ = n-bit with n/2 1's

- Decision tree model

- Bounded-depth circuit model

Lower bound for codes

- Code $C \subseteq \{0,1\}^n$ of size $|C| = 2^{k = \Omega(n)}$

  $x \neq y \in C \Rightarrow x, y$ far : hamming distance $\Omega(n)$

- Theorem [Lovett V.] $f : \{0,1\}^* \to \{0,1\}^n$, $f \in AC^0$

  Distance(f(X), uniform over C) $> 1 - n^{-\Omega(1)}$

- Consequences for data structures for codewords, complexity of pseudorand. generators against AC^0 [Nisan]

Warm-up

- Fact: $f : \{0,1\}^k \to \{0,1\}^n$, $f \in AC^0$
  f cannot compute encoding function of C,
  mapping message $m \in \{0,1\}^k$ to codeword

- Proof:

- [Linial Mansour Nisan, Boppana] low sensitivity of AC^0:
  m, m' random at hamming distance 1
  $\Rightarrow$ f(m), f(m') close in hamming distance.

- But $f(m) \neq f(m') \in C \Rightarrow$ far in hamming distance ♦

Lower bound for codes

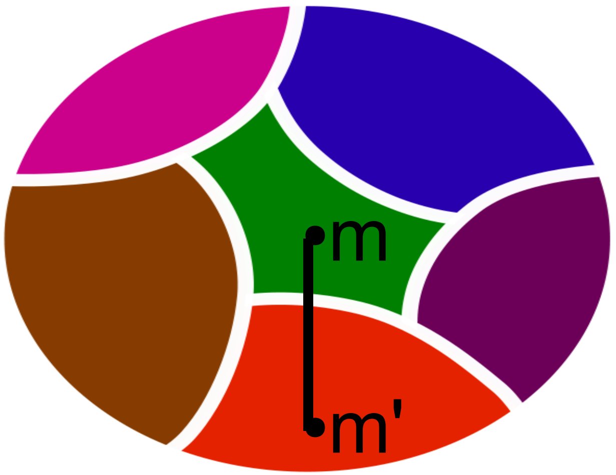
- Theorem [Lovett V.] $f : \{0,1\}^{L >> k} \to \{0,1\}^n$, $f \in AC^0$

  Distance(f(X), uniform over C) $> 1 - n^{-\Omega(1)}$

  Problem: f needs not compute encoding function.

  Input length >> message length

- Idea: Input $\{0,1\}^L$ to f partitioned in |C| sets



- Isoperimetric inequality [Harper, Hart]:

  Random m, m' at distance 1 often in ≠ sets ⇒ low sensitivity

Lower bound for codes

- Theorem [Lovett V.] $f : \{0,1\}^{L >> k} \to \{0,1\}^n$, $f \in AC^0$
  Distance(f(X), uniform over C) $> 1 - n^{-\Omega(1)}$

- Note: to get
  Need isoperimetric inequality for m, m' at distance >> 1

  Fact[thanks to Samorodnitsky] $\forall$ $A \subseteq \{0,1\}^L$ of density α
  random m, m' obtained flipping bits w/ probability p :
  $\alpha^2 \leq \Pr[\text{both } m \in A \text{ and } m' \in A] \leq \alpha^{1/(1-p)}$

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- Recall: edit style changes ALL settings.
- Click on “line” for just the one you highlight

More connections

- More uses of generating W_k := uniform n-bit string with k 1's
- McEliece cryptosystem
- Switching networks, …

Previous results

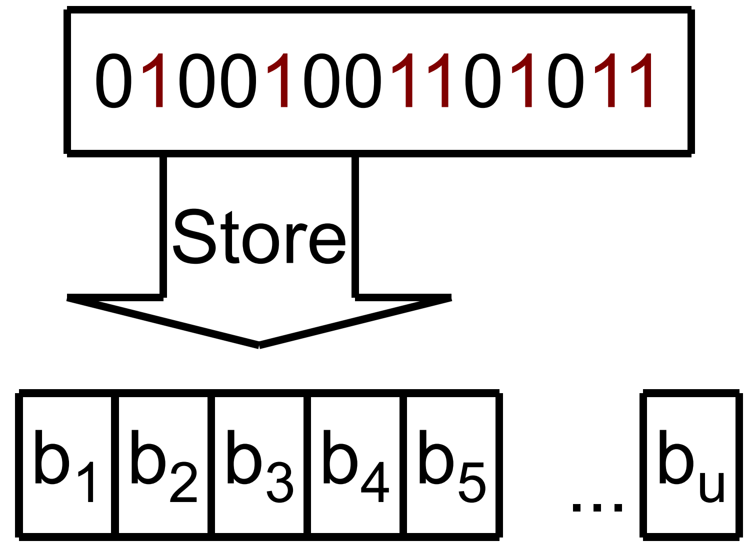
- Store S $\subseteq$ {1, 2, … , n}, |S| = k, in bits, answer "i $\in$ S?"
- [Minsky Papert '69] Average-case study
- [Buhrman Miltersen Radhakrishnan Venkatesh; Pagh '00]

  Space O(optimal), probe O(1) when k = Θ(n)

  Lower bounds for k < $n^{1-\varepsilon}$
- [..., Pagh, Pătraşcu] space = optimal + o(n), probe O(log n)
- [V. '09] lower bounds for k = Ω(n), except k = n / 2^a

Succinct data structures for sets

- Store $S \subseteq \{1, 2, \ldots, n\}$ of size $|S| = k$



In u bits $b_1, \ldots, b_u \in \{0,1\}$

- Want:

  Small space u (optimal = $\lceil \lg_2 (n \text{ choose } k) \rceil$)

  Answer "$i \in S$?" by probing few bits (optimal = 1)

- In combinatorics: Nešetřil Pultr, …, Körner Monti

Previous results

- Store S $\subseteq$ {1, 2, … , n}, |S| = k, in bits, answer "i $\in$ S?"

- [Minsky Papert '69, Buhrman Miltersen Radhakrishnan Venkatesh; Pagh; ...; Pătraşcu; V. '09]

- Surprising upper bounds
  space = optimal + o(n), probe O(log n)

- No lower bounds for $k = n / 2^a$

Rest of this talk

- Local (NC^0)

    Lower bound for $W_{n/2}$ = n-bit with n/2 1's

    Succinct data structures

- Decision tree

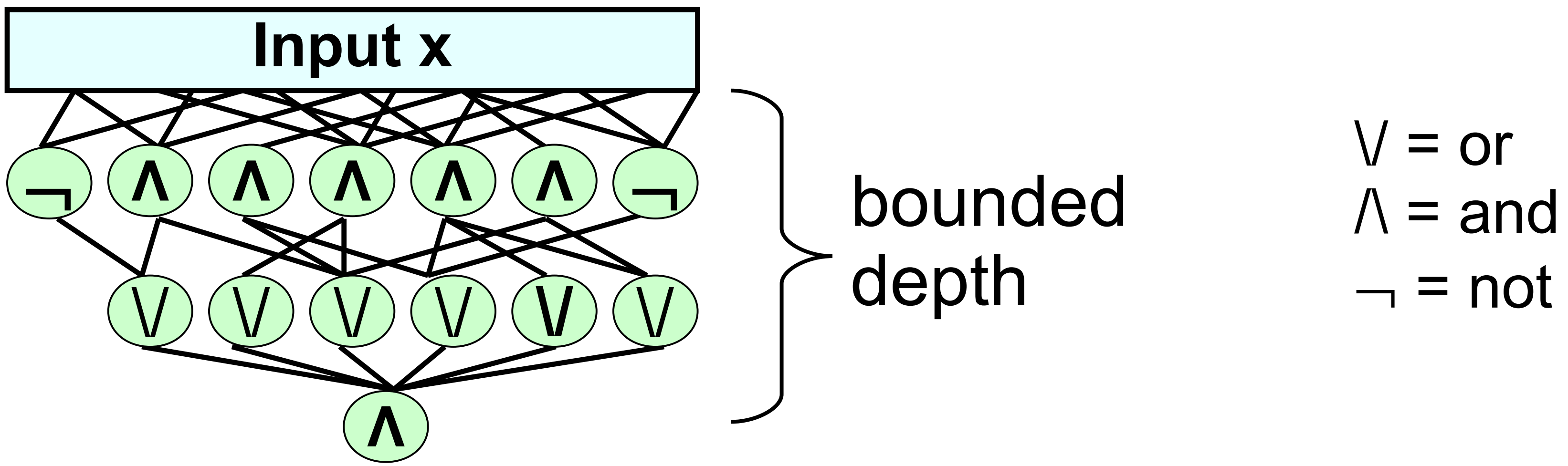
    Lower bound for $W_{n/2}$

- Bounded-depth circuit (AC^0)

- Proof of local lower bound

Bounded-depth circuits (AC^0)

- O(log n)-local $\subseteq$ depth O(log n) $\subseteq$ AC^0



- Theorem [Matias Vishkin, Hagerup, this work]
  Can generate W_k, exp. small error

- Theorem [Lovett V.] Cannot generate error-correcting code

- Challenge: ∃ explicit boolean f : cannot generate (Y, f(Y)) ?

Our results: pseudorandomness for AC^0

- Pseudorandom distribution against circuit of depth d
  (want: reduce randomness w/ minimum overhead)

- Direct implementation of Nisan's generator: depth $\geq$ d
  circuit + generator $\rightarrow$ depth 2d

- Generator in depth 2 circuit + generator $\rightarrow$ depth d+1
  [Braverman] + [Guruswami Umans Vadhan]