Coalescing-Branching Random Walks on Graphs

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Abstract

We study a distributed randomized information propagation mechanism in networks we call the *coalescing-branching* random walk (cobra walk, for short). A cobra walk is a generalization of the well-studied "standard" random walk, and is useful in modeling and understanding the SIS-type of epidemic processes in networks. It can also be helpful in performing light-weight information dissemination in resource-constrained networks. A cobra walk is parameterized by a *branching factor* k. The process starts from an arbitrary node, which is labeled *active* for step 1. (For instance, this could be a node that has a piece of data, rumor, or a virus.) In each step of a cobra walk, each active node chooses k random neighbors to become active for the next step ("branching"). A node is active for step $t+1$ only if it is chosen by an active node in step t ("coalescing"). This results in a stochastic process in the underlying network with properties that are quite different from both the standard random walk (which is equivalent to the cobra walk with branching factor 1) as well as other gossip-based rumor spreading mechanisms.

We focus on the *cover time* of the cobra walk, which is the the number of steps for the walk to reach all the nodes, and derive almost-tight bounds for various graph classes. Our main technical result is an $O(\log^2 n)$ high probability bound for the cover time of cobra walks on expanders, if either the expansion factor or the branching factor is sufficiently large; we also obtain an $O(\log n)$ high probability bound for the *partial cover time*, which is the number of steps needed for the walk to reach at least a constant fraction of the nodes. We show that the cobra walk takes $O(n \log n)$ steps on any n-node tree for $k \geq 2$, and $O(n^{1/d} \log n)$ steps on a d-dimensional grid for $k \geq d$, with high probability.

Keywords: Random Walks, Networks, Information Spreading, Cover Time, Epidemic Processes.

Regular Presentation

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1 Introduction

We study a distributed propagation mechanism in networks, called the *coalescing-branching random walk* (*cobra walk*, for short). A cobra walk is a variant of the standard random walk, and is parameterized by a *branching factor*, k. The process starts from an arbitrary node, which is initially labeled *active*. For instance, this could be a node that has a piece of data, rumor, or a virus. In a cobra walk, for each discrete time step, each active node chooses k random neighbors (sampled independently with replacement) to become active for the next step; this is the "branching" property, in which each node spawns multiple independent random walks. A node is active for step t if and only if it is chosen by an active node in step $t-1$; this is the "coalescing" property, i.e., if multiple walks meet at a node, they coalesce into one walk.

A cobra walk generalizes the standard random walk [35, 39], which is equivalent to a cobra walk with $k = 1$. Random walks on graphs have a wide variety of applications, including being fundamental primitives in distributed network algorithms for load balancing, routing, information propagation, gossip, and search [15, 16, 8, 45]. Being local and requiring little state information, random walks and their variants are especially well-suited for self-organizing dynamic networks such as Internet overlay, ad hoc wireless, and sensor networks [45]. As a propagation mechanism, one parameter of interest is the *cover time*, the expected time it takes to cover all the nodes in a network. Since the cover time of the standard random walk can be large — $\Theta(n^3)$ in the worst case, $\Theta(n \log n)$ even for expanders [35] — some recent studies have studied simple adaptations of random walks that can speed up cover time [1, 5, 17]. Our analysis of cobra walks continues this line of research, with the aim of studying a lightweight information dissemination process that has the potential to improve cover time significantly.

Our primary motivation for studying cobra walks is their close connection to SIS-type epidemic processes in networks. The SIS (standing for Susceptible-Infected-Susceptible) model (e.g., [19]) is widely used for capturing the spread of diseases in human contact networks or propagation of viruses in computer networks. Three basic properties of an SIS process are: (a) a node can infect one or more of its neighbors ("branching" property); (b) a node can be infected by one or more of its neighbors ("coalescence" property) and (c) an infected node can be cured and then become susceptible to infection at a later stage. Cobra walks satisfy all these properties, while standard random walks and other gossip-based propagation mechanisms violate one or more. Also, while there has been considerable work on the SIS model ([28, 43, 31, 19, 40, 18, 6]), it has been analytically hard to tackle basic *coverage* questions: (1) How long will it take for the epidemic to infect, say, a constant fraction of network? (2) Will every node be infected at some point, and how long will this take? Our analysis of cobra walks in certain special graph classes is a step toward a better understanding of such questions for SIS-type processes.

1.1 Our results and techniques

We derive near-tight bounds on the cover time of cobra walks on trees, grids, and expanders. These special graph classes arise in many distributed network applications, especially in the modeling and construction of peer-to-peer (P2P), overlay, ad hoc, and sensor networks. For example, expanders have been used for modeling and construction of P2P and overlay networks, grids and related graphs have been used as models for ad hoc and sensor networks, and spanning trees are often used as backbones for various information propagation tasks.

We begin with an observation that Matthew's Theorem [37, 35] for random walks extends to cobra walks; that is, the cover time of a cobra walk on an n-node graph is at most $\ln n$ times the maximum hitting time of a node; for many graphs, this is also a tight bound. This enables us to focus on deriving bounds for the hitting time.

We face two technical challenges in our analysis. First, unlike in a standard random walk, cobra walks have multiple "active" nodes at any step, and in almost all graphs, it is difficult to characterize the distribution of the active nodes at any point of time. Second, the combination of the branching and coalescing properties introduces a non-trivial dependence among the active nodes, making it challenging to quantify the probability that a given node is made active during a given time period. Surprisingly, these challenges manifest even in tree networks. We present a result that gives tight bound on the cover time for trees, which we obtain by establishing a recurrence relation for the expected time taken for the cobra walk to cross an edge along a given path of the tree.

- For an arbitrary n-node tree, a cobra walk with $k \geq 2$ covers all nodes in $O(n \log n)$ steps with high probability (w.h.p., for short)[1] (Theorem 5 of Section 3.1).

For a matching lower bound, we note that the cover time of a cobra walk in a star graph is $\Omega(n \log n)$ w.h.p. We conjecture that the cover time for *any n-node graph* is $O(n \log n)$. By exploiting the regular structure of a grid, we establish improved and near-tight bounds for the cover time on d-dimensional grids.

- For a d-dimensional grid, we show that a cobra walk with $k \geq d$ takes $O(n^{1/d} \log n)$ steps, w.h.p. (cf. Theorem 9 of Section 3.2).

Our main technical result is an analysis of cobra walks on expanders, which are graphs in which every set S of nodes of size at most half the number of vertices has at least $\alpha|S|$ neighbors for a constant α, which is referred to as the expansion factor.

- We show that for an n-node constant-degree expander, a cobra walk covers a constant fraction of nodes in $O(\log n)$ steps and all the nodes in $O(\log^2 n)$ steps w.h.p. assuming that either the branching factor or the expansion factor is sufficiently large (cf. Theorems 10 and 11 of Section 4).

Our analysis for expanders proceeds in two phases. We show that in the first phase, which consists of $O(\log n)$ steps, the branching process dominates resulting in an exponential growth in the number of active nodes until a constant fraction of nodes become active, with high probability. In the second phase, though a large fraction of the nodes continues to be active, dependencies caused by the coalescing property prevent us from treating the process as multiple independent random walks, analyzed in [2] (or even d-wise independent walks for a suitably large d). We overcome this hurdle by carefully analyzing these dependencies and bounding relevant conditional probabilities, and define a time-inhomogeneous Markov process that is stochastically dominated by the cobra walk in terms of coverage. We then use the notion of merging conductance and the machinery introduced in [38] to analyze time-inhomogeneous Markov chains, and establish an $O(\log n)$ bound w.h.p. on the maximum hiting time, leading to an $O(\log^2 n)$ bound on the cover time.

1.2 Related work and comparison

Branching and coalescing processes. There is a large body of work on branching processes (without coalescence) on various discrete and non-discrete structures [33, 36, 4]. A study of coalescing random walks (without branching) was performed in [14] with applications to voter models. Others have looked at processes that incorporate branching and coalescing particle systems [3, 41]. However, these studies treat the particle systems as continuous-time systems, with branching, coalescing, and death rates on restricted-topology structures such as integer lattices. To the best of our knowledge, ours is the first work that studies random walks that branch and coalesce in discrete time and on various classes of non-regular finite graphs.

Random walks and parallel random walks. Feige [24, 23] showed that the cover time of a random walk on any undirected n-node connected graph is between $\Theta(n \log n)$ and $\Theta(n^3)$ with both the lower and upper bounds being achieved in certain graphs. With the rapidly increasing interest in information (rumor) spreading processes in large-scale networks and the gossiping paradigm (e.g., see [9] and the references therein), there have been a number of studies on speeding up the cover time of random walks on graphs. One of the earliest studies is due to Adler et al [1], who studied a process on the hypercube in which in each round a node is chosen uniformly at random and covered; if the chosen node was already covered, then an uncovered neighbor of the node is chosen uniformly at random and covered. For any d-regular graph, Dimitrov and Plaxton showed that a similar process achieves a cover time of $O(n + (n \log n)/d)$ [17]. For expander

[1] By the term "with high probability" (w.h.p., for short) we mean with probability $1 - 1/n^c$, for some constant $c > 0$.

graphs, Berenbrink et al showed a simple variant of the standard random walk that achieves a linear (i.e., $O(n)$) cover time [5].

It is instructive to compare cobra walks with other mechanisms to speed up random walks as well as with gossip-based rumor spreading mechanisms. Perhaps the most related mechanism is that of parallel random walks which was first studied in [7] for the special case where the starting nodes are drawn from the stationary distribution, and in [2] for arbitrary starting nodes. Nearly-tight results on the speedup of cover time as a function of the number of parallel walks have been obtained by [21] for several graph classes including the cycle, d-dimensional meshes, hypercube, and expanders. (Also see [20] for results on mixing time.) Though cobra walks are similar to parallel random walks in the sense that at any step multiple nodes may be selecting random neighbors, there are significant differences between the two mechanisms. First the cover times of these walks are not comparable. For instance, while k parallel random walks may have a cover time of $\Omega(n^2/\log k)$ for any $k \in [1, n]$ [21], a 2-branching cobra walk on a line has a cover time of $O(n)$. Second, while the number of active nodes in k parallel random walks is always k, the number of active nodes in any k-branching cobra walk is continually changing and *may not even be monotonic*. Most importantly, the analysis of cover time of cobra walks needs to address several dependencies in the process by which the set of active nodes evolve; we use the machinery of time-inhomogenous Markov chains to obtain the cover time bound for bounded-degree expanders (see Section 4).

The works of [15, 16] presented distributed algorithms for performing a standard random walk in sublinear time, i.e., in time sublinear in the length of the walk. In particular, the algorithm of [16] performs a random walk of length ℓ in $\tilde{O}(\sqrt{\ell D})$ rounds w.h.p. on an undirected network, where D is the diameter of the network. However, this speed up comes with a drawback: the message complexity of the above faster algorithm is much worse compared to the naive sequential walk which takes only ℓ messages. In contrast, we note that the speedup in cover time given by a cobra walk over the standard random walk comes only at the cost of a slightly worse message complexity.

Gossip-based mechanisms. Gossip-based information propagation mechanisms have also been used for information (rumor) spreading in distributed networks. In the most typical rumor spreading models, gossip involves either a push step, in which nodes that are aware of a piece of information (being disseminated) pass it to random neighbors, or a pull step, in which nodes that are unaware of the information attempt to extract the information from one of their randomly chosen neighbors, or some combination of the two. In such models, the knowledgeable nodes or the ignorant nodes participate in the dissemination problem in *every* round (step) of the algorithm. The main parameter of interest in many of these analyses is the number of rounds needed till all the nodes in the network get to know the information.

The rumor spreading mechanism that is most closely related to cobra walks is the basic push protocol, in which in every step every informed node selects a random neighbor and pushes the information to the neighbor, thus making it informed. Feige et al. [22] show that the push process completes in every undirected graph in $O(n \log n)$ steps, with high probability. Since then, the push protocol and its variants have been extensively analyzed both for special graphs, as well as for general graphs in terms of their expansion properties (see e.g., [10, 11, 12, 30, 29, 26, 25]). Again, though cobra walk and push-based rumor spreading share the property that multiple nodes are active in a given step, the two mechanisms differ significantly. While the set of active nodes in rumor spreading is monotonically nondecreasing, this is not so in cobra walks, an aspect that makes the analysis challenging especially with regard to full coverage. Furthermore, the message complexity of the push protocol can be substantially different than that of cobra. A simple example is the star network, which the push protocol covers in $\Theta(n \log n)$ steps with a message complexity of $\Theta(n^2 \log n)$, while the 2-branching cobra walk has both cover time and message complexity $\Theta(n \log n)$. This can be extended to show similar results for star-based networks that have been proposed as models for Internet-scale networks [13].

1.3 Potential applications

As mentioned at the outset, cobra walks are closely related to SIS model in epidemics, but they may be easier to analyze using tools from random walk and Markov chain analyses. While the persistence time and epidemic density of SIS-type epidemic models are well studied [27, 34, 44], to the best of our knowledge the time needed for a SIS-type process to affect a large fraction (or the whole) of the network has not been well-studied. Our results and analyses of cobra walks on more general networks can be useful in predicting the time taken for a real epidemic process following an SIS-type model to spread in a network [27, 34, 44].

Cobra walks can also serve as a lightweight information dissemination protocol in networks, similar to the push protocol. As pointed out earlier, in certain types of networks, the message complexity incurred by a cobra walk to cover a network can be smaller than that for the push protocol. This can be useful, especially in infrastructure-less anonymous networks, where nodes don't have unique identities and and may not even know the number of neighbors. In such networks, it is difficult to detect locally when coverage is completed[2]. If nodes have a good upper bound on n (the network size), however, then nodes can terminate the protocol after a number of steps equal to the estimated cover time. In such a scenario, message complexity is also an important performance criterion.

2 Preliminaries

Let G be a connected graph with vertex set V and edge set E, and let $|V| = n$. We define a coalescing-branching (cobra) random walk on G with branching factor k starting at some arbitrary $v \in V$ as follows: At time $t = 0$ we place a pebble at v. Then in the next and every subsequent time step, every pebble in G clones itself $k-1$ times (so that there are now k pebbles at each vertex that originally had a pebble). Each pebble independently selects a neighbor of its current vertex uniformly at random and moves to it. Once all pebbles make their one-hop moves, if two pebbles are at the same vertex they coalesce into a single pebble, and the next round begins. In a cobra-walk, a vertex can be active an arbitrary number of times.

For a time step t of the process, let S_t be the **active set**, the set of all vertices of G that have a pebble. We will use two different definitions of the neighborhood of S_t: Let $N(S_t)$ be the **inclusive** neighborhood, the union of the set of neighbors of all vertices in S_t (which can include members of S_t itself). Let $\Gamma(S_t)$ be the **non-inclusive neighborhood**, which is the union of the set of neighbors of all vertices of S_t such that $S_t \cap \Gamma(S_t) = \emptyset$.

Let the expected **maximum hitting time** h_{max} of a cobra-walk on G be defined as the $max_{u,v\in V}\mathbb{E}[h_{u,v}]$ where $h_{u,v}$ is the time it takes a cobra-walk starting at vertex u to first reach v with at least one pebble.

We are interested in two different notions of cover time, the time until all vertices of G have been visited by a cobra-walk at least once. Let τ_v be the minimum time t such that, for a cobra-walk starting from v, $\forall u \in V - v$, $u \in S_t$ for some $t \leq \tau_v$. Then we define the **cover time** of a cobra-walk on G to be $\max_{v\in V} \tau_v$. We define the **expected cover time** to be $\max_{v\in V} \mathbb{E}[\tau_v]$. Note that in the literature for simple random walks, cover time usually refers to the expected cover times. In this paper we will show high-probability bounds on the cover time.

In Section 6 we will be proving results for cobra-walks on expanders. In this paper, we will use a spectral definition for expanders and then use Tanner's theorem to translate that to neighborhood and cut-based notions of expanders.

Definition 1. *An ϵ **- expander** graph is a d-regular graph whose adjacency matrix has eigenvalues such that $|\alpha_i| \leq \epsilon d$ for $i \geq 2$.*

We also want to define the notion of an ϵ-approximation:

Definition 2. *G is an ϵ**-approximation** for a graph H if $(1-\epsilon)H \preccurlyeq G \preccurlyeq (1+\epsilon)H$, where $H \preccurlyeq G$ if for all x, $x^T L_H x \leq x^T L_G x$, where L_G and L_H are the Laplacians of G and H, respectively.*

[2]In networks with identities and knowledge of neighbors, a node can locally stop sending messages when all neighbors have the rumor. This reduces the overall message complexity until cover time.

Finally, we will rely on the neighborhood expansion of a set S on G, where we define $N(S)$ as the inclusive neighborhood. For this we will use Tanner's theorem [42], which gives us a lower bound on the size of the neighborhood of S for sufficiently strong expanders.

Theorem 3. *Let G be a d-regular graph that ϵ-approximates $\frac{d}{n}K_n$. Then for all $S \subseteq V$ with $|S| = \delta n$, $|N(S)| \geq \frac{|S|}{\epsilon^2(1-\delta)+\delta}$.*

3 Cover Time for Trees and Grids

A useful tool in bounding the cover time for simple random walks is Matthew's Theorem [37, 35], which bounds the expected cover time of a graph by the maximum expected hitting time $h_{u,v}$ between any two nodes u and v times the harmonic number H_n. Here we show that this result can be extended to cobra walks.

Theorem 4 (Matthew's Theorem for Cobra Walks). *Let G be a connected graph on n nodes. Let w be a cobra walk on G starting at an arbitrary node. Then the cover-time of w on G, $C(G)$, is bounded from above by $h_{max} \ln n$ in expectation and by $O(h_{max} \ln n)$ with high probability.*

Proof. We adopt the language of [37]. Rather than viewing a cobra walk as multiple pebbles moving over a graph, we will view it as a Markov process M' of its own. In this process, the state space consists of 2^n states, each of which corresponds to a particular subset of the nodes that contain pebbles. Transitions between states occur with a probability equal to the probability of one particular pebble configuration in the original graph giving rise to the next state.

We fix an initial position a_0 of M' and a collection of N Borel subsets of M', $\{A_1, \ldots, A_n\}$ to be visited. Here A_i represents the set of all states of M' in which node i in G contains a pebble. For any non-empty collection $\{A_1, \ldots, A_i\}$ of Borel subsets of M' we then define $T(A_j) = \inf\{t \geq 0 : X(t) \in A_j\}$ for $j = 1, \ldots, i$. That is, $T(A_j)$ is the smallest time t such that the walk X on M' visits a member of A_j. We also define $T(A_1, \ldots, A_i) = \max_{j=1,\ldots,i} T(A_j)$. We now define:

$$\mu_+ = \max_{i=1,\ldots,N} \sup_{a \notin A_i} E_a T(A_i).$$

μ_+ is h_{max} in the standard language of random walks. Then from Theorem 2.6 of [37],

$$ET(A_1, \ldots, A_N) \leq \mu_+ \sum_{i=1}^{N} \frac{1}{i}$$

thus proving the lemma. □

Matthew's theorem for cobra walks is used in proving the cover time for trees and grids.

3.1 Trees

Theorem 5. *For any tree, the cover time of a cobra walk starting from any node is $O(n \ln n)$ w.h.p.*

We will prove our main result by calculating the maximum hitting time of a cobra walk on a tree T and then applying Matthew's theorem. Cobra walks on trees are especially tractable because they follow two nice properties. Since a tree has a unique path between any two nodes, we only need keep track of the pebble closes to the target. In addition, the fact that there is one simple path between any two nodes limits the number of collisions we need to keep track of, a property which is not true for general graphs and makes cobra walk harder to analyze on them. For this section, we fix the branching factor $k = 2$. For $k > 2$ but still constant, the cover time would not be asymptotically better.

The general idea behind the proof is as follows. We take the longest path w.r.t. hitting time in the tree. Along each node in this path, except for the first and last, there will be a subtree rooted at that node. If a cobra walk's closest pebble to the endpoint is at node l, the walk from this point can either advance with at least one pebble, or it can not advance by either backtracking along the path, going down the subtree rooted at l, or both. We show via a stochastic dominance argument that a biased random walk from l, whose transition probabilities are tuned to be identical to cobra walk's, will next advance to $l+1$ in a time that is dominated primarily by the size of the subtree at l. This is done by analyzing the return times in the non-advancement scenarios listed above. Thus summing up over the entire walk, the hitting time is dominated by a linear function of the size of the entire tree.

In Lemma 6 we bound the return time of a cobra walk to a root of the tree.

Lemma 6. *Let T be a tree of size M. Pick a root, r, and let r have d children. Then a cobra walk on T starting at r will have a return time to r of $O(4M/d)$.*

Proof. To show that the Lemma holds for a cobra walk, we will actually show that it holds for a simple random walk with transition probabilities modified to resemble those of a cobra walk. For this simple random walk, we start at r and in the first step pick one of the children of r, r'. Let $(d'+1)$ be the degree of r'. Then we define transition probabilities as follows: p is the probability of returning to r in the next step, and p is the probability of continuing down the tree. They are given as:

$$p = \left(1 - \left(\frac{d'}{(d'+1)}\right)^2\right), q = \left(\frac{d'}{(d'+1)}\right)^2, \frac{p}{q} = \frac{(d')^2}{(2d'+1)} \tag{1}$$

Note that these are the exact same probabilities that a cobra walk at node r' would have for sending (not sending) at least one (any) pebbles back to the root. .

The rest of the proof follows by mathematical induction. Consider a tree T that has only two levels. Starting from r, the return time, 2, is constant, the relationship holds. For the inductive case, assume that the hypothesis holds. Then:

$$\begin{aligned} r(T) &\leq 1 + \sum_{r' \in N(r)} p(r') h_{r',r} \leq 1 + \frac{1}{d} \sum_{r' \in N(r)} h_{r',r} && (2) \\ &\leq 1 + \frac{1}{d} \sum_{r' \in N(r)} \left(1 + \frac{d'^2}{2d'+1} c \frac{|T'|}{d'}\right) \leq 2 + \frac{c|T|}{2d} && (3) \end{aligned}$$

Setting $c = 4$ gives us the result of the lemma for the biased random walk, and it is easy to see that by stochastic dominance this holds also for the cobra walk. □

Finally, we show a key lemma for the hitting time of a single step of a path along a tree.

Lemma 7. *Fix a path in a tree T made up of nodes $1, \ldots, l, l+1, \ldots, t$. Then, the expected time it takes for a cobra walk starting at node l to get to $l+1$ with at least one pebble is given by:*

$$h_{l,(l+1)} = \frac{5}{4} + \frac{9}{5} \sum_{i=l}^{2} \left(\frac{1}{5}\right)^{l-i} |T_i| \tag{4}$$

where T_l is the induced subtree formed by taking node l, its neighbors not on the path being traversed, and all of their descendants.

Informally, we prove that the one-step hitting time is bounded by above by the worst case scenario that either both pebbles go back along the path or down the subtree rooted at l and establish a simple recurrence relation.

Proof. Vertex l is viewed through the context of having one edge to the node $l-1$, one edge to node l, and d edges to some other nodes. Thus it can be viewed as the root of a tree, and T_l as the induced subgraph of l and all nodes reached through its d_l not-on-path children. We will need the following probabilities:

- Probability of a pebble going from l to $l+1 = p = \left(1 - \left(\frac{(d_l+1)}{(d_l+2)}\right)^2\right)$
- Probability of a pebble not going from l to $l+1 = 1-p = q$.
- Probability of a cobra walk sending both pebbles from l to $l-1$ conditioned on it not sending any pebbles from l to $l+1 = q_l^{'} = \left(\frac{1}{(d_l+1)^2}\right)$
- Probability of a cobra walk sending at least one pebble to the subtree T_l conditioned on its not sending any pebbles to $l+1 = q_l^{''} = \left(\frac{(d_l)}{(d_l+1)}\right)^2 + 2\left(\frac{d_l}{(d_l+1)^2}\right) = \frac{d_l^2+2d_l}{(d_l+1)^2}$

Note that, conditioned on a pebble not advancing to node $l+1$, we actually have three disjoint events: (A) Both pebbles go to $l-1$, (B) one pebble goes to $l-1$ and one pebble goes into subtree T_l, and (C) both pebbles go into T_l. We define an alternate event B', which is the event that one pebble goes down T_l and nothing else happens (thus, it is not technically in the space of cobra walk actions). If we let R be the time until first return of the cobra walk to l conditioned on no pebble going to $l+1$, we wish to show that $E[R|B] \leq E[R|B']$ and that $E[R|C] \leq E[R|B']$. What is the relationship between B and B'? Consider two random variables, X and Y, and let X be the time until first return of a pebble that travels from l to $l-1$, Y be the time until first return of a pebble that travels into T_l. Then $R|B$ is just another random variable, $U = \min(X, Y)$. Since $U \leq Y$ over the entire space, $E[U] \leq E[Y]$, and clearly $R|B'$ is equivalent to Y. Thus $E[R|B] \leq E[R|B']$ It is also easy to see that $E[R|B'] \geq E[R|C]$. Thus by the law of total expectation we have:

$$\begin{aligned} E[R] &= E[R|A]\Pr(A) + E[R|B]\Pr(B) + E[R|C]\Pr(C) \\ &\leq E[R|A]\Pr(A) + (\Pr(B)+\Pr(C))E[R|B'] \\ &= E[R|A]\Pr(A) + E[R|B'](1-\Pr(A)) \end{aligned}$$

Then the hitting time can be expressed as:

$$\begin{aligned} h_{l,l+1} &\leq p + q(E[R] + h_{l,l+1}) \\ \Rightarrow (1-q)h_{l,l+1} &\leq p + q(E[R]) \\ \Rightarrow h_{l,l+1} &\leq 1 + \frac{q}{p}(q_l^{'}(1+h_{l-1,l}) + q_l^{''} r(T_l)) \end{aligned}$$

Note that $q/p = \frac{(d_l+1)^2}{(2d_l+3)}$. Since $r(T_l) \leq 4|T_l|/d_l$ by Lemma 6, we continue with:

$$\begin{aligned} \text{“}h_{l,l+1} &\leq 1 + \frac{(d_l+1)^2}{(2d_l+3)}\frac{1}{(d_l+1)^2}(1+h_{l-1,l}) + \frac{(d_l+1)^2}{(2d_l+3)}\frac{(d_l^2+2d_l)}{(d_l+1)^2}\frac{4|T_l|}{d_l} \\ &\leq 1 + \frac{1}{5}(1+h_{l-1,l}) + \frac{12}{5}|T_l| w.h.p. \end{aligned}$$

If we expand the relation, we get:

$$\begin{aligned} h_{l,l+1} &\leq \sum_{i=0}^{l}\left(\frac{1}{5}\right)^i + \frac{12}{5}\left(|T_l| + \left(\frac{1}{5}\right)|T_{l-1}| + \left(\frac{1}{5}\right)^2|T_{l-2}| + \cdots + \left(\frac{1}{5}\right)^{l-2}|T_2|\right) \\ h_{l,(l+1)} &\leq \frac{5}{4} + \frac{12}{5}\sum_{i=l}^{2}\left(\frac{1}{5}\right)^{l-i}|T_i| \end{aligned}$$

□

We are finally ready to prove our main results for tree, Theorem 5, that the cobra walk cover time of an arbitrary tree occurs in $O(n \ln n)$ steps.

Proof. By Matthew's Theorem for cobra walks, $C(G) \leq (\ln n + o(1))h_{max}$. We just need to prove that h_{max} occurs in linear time.

Let P be the path for which $h_{u,v}$ is maximized, and let the path consist of the sequence of nodes $1, 2, \ldots, t$. As in the proof of the single-step hitting time, we note that for all but the first and last nodes on P, there is a subtree T_l of size $|T_l|$ rooted at each nodes. Because $h_{1,l} \leq h_{1,2} + h_{2,3} + \ldots h_{t-1,t}$. Then, we obtain the desired result from Lemma 7 as follows:

$$h_{1,t} \leq \frac{5}{4}t + \frac{12}{5}\sum_{j=2}^{t-1}\left[|T_j|\sum_{i=0}^{\infty}\left(\frac{1}{5}\right)^i\right] \leq \frac{5}{4}t + \frac{12}{5}\frac{5}{4}\sum_{j=2}^{t-1}|T_j| \leq 4n. \tag{5}$$

□

We note that for the line network, we can improve the bound we obtain for trees and show that the cover time of a cobra walk is $O(n)$ w.h.p.

3.2 Grids

Lemma 8. *Let G be a finite 2-dimensional grid of size* $(\sqrt{n} \times \sqrt{n})$*. Then the cover-time of a cobra walk with a branching factor 2 on G is* $O(\sqrt{n}\log n) w.h.p.$

Proof. The proof of this lemma makes use of Matthew's bound for cobra walks. The longest path in the grid w.r.t hitting time for a cobra walk is from the point $x = (0, 0)$ to (y = $\sqrt{n}, \sqrt{n}$). We need to show that the hitting time of this walk is $O(n)$. To do this, we only need to keep track of the pebble that is closest to y at each step of the walk, where by closest we mean the Manhattan distance. Let x' be the location of the closest pebble to y at time t. Let A be the event that at least one pebble from x' moves closer to y. Then: $\Pr[A] = 1 - \Pr[\bar{A}] = 1 - (1-p)^k$, where p is the fraction of edges of x' that lead to nodes closer to y and depends on where in the grid x' is. If x' is in the interior of G, then $p = 1/2$, meaning $\Pr[A] > 1/2$ for $k \geq 2$. If x' is on the bottom or left boundary of G, $p = 2/3$, and if on the top or right boundary, $p = 1/3$. In either case, for $k \geq 2$, $\Pr[A] > 1/2$. Hence, we have a biased random walk on a line, which we know has $O(l)$ cover time. In this case, $l = 2\sqrt{n}$, and thus by applying Matthew's bound we get our desired $O(\sqrt{n}\log n)$ bound. □

Theorem 9. *Let G be a finite d-dimensional grid for some constant d. Then the cover time of a cobra walk on G is* $O(n^{1/d}\log n)$ *w.h.p, as long as branching factor* $k \geq d$.

Proof. The proof of this theorem is very similar to that of 8. For G the longest path is $dn^{1/d}$, and the key point is that the probability of a node from any node moving closer to y is even higher than in the 2-d grid, since clearly the worst place to be again is on an "edge" of the grid. When on this edge, the probability of not moving towards y is $(2d - (d))/(2d - (d-1))^k$. Hence the probability of moving towards y will be greater than $1/2$ only when $k \geq d$. □

4 Analysis for Expanders

For expander graphs, we are able to prove a high probability cover time result of $O(\log^2 n)$. We break the proof up into two phases. In the first phase we show that a cobra walk starting from any node will reach a constant fraction of the nodes in logarithmic time w.h.p. In the second phase, we create a process which stochastically dominates the cobra walk and show that this new process, will cover the entire rest of the graph again in polylogarithmic time w.h.p.

The main result of this section can be stated in the following two theorems, which when taken together imply that w.h.p. ϵ-expander G will be covered in $O(\log^2 n)$ time.

Theorem 10. *Let G be an ϵ-expander with ϵ, δ not depending on n (number of nodes in G), with $\delta < \frac{16}{30d^2}$, and ϵ, a sufficiently small constant such that*

$$\frac{1}{\epsilon^2(1-\delta)+\delta} > \frac{d(de^{-k}+(k-1)) - \frac{k^2}{2}}{d(e^{-k}+(k-1)) - \frac{k^2}{2}}, \tag{6}$$

then in time $O(\log n)$, w.h.p. a cobra walk on G with branching factor k, will attain an active set of size δn.

We note that the condition in the above theorem is satisfied if either ϵ is sufficiently small, or k is sufficiently large. For instance when $k = 2$, the above condition holds for strong expanders, such as the Ramanujan graphs, which have $\epsilon \leq 2\sqrt{d-1}/d$, and random d-regular graphs, for d sufficiently large.

Theorem 11. *Let G be as above, and let W be a cobra walk on G that at time T has reached an active set of size δn. Then w.h.p in an additional $O(\log^2 n)$ steps every node of G will have visited by W at least once.*

To prove Theorem 10 we prove that active sets up to a constant fraction of V are growing at each step by a factor greater than one.

Lemma 12. *Let G be an ϵ-expander with ϵ, δ satisfying the conditions of Theorem 10. Then for any time $t \geq 0$, the cobra walk on G with active set S_t such that $|S_t| \leq \delta n$, $\mathbb{E}[|S_{t+1}|] \geq (1+\nu)|S_t|$ for some constant $\nu > 0$.*

Proof. We will instead show that the portion of $N(S_t)$ not selected by the cobra walk is sufficiently small, $\mathbb{E}[|N(S_t) - S_{t+1}|] \leq |N(S_t)| - (1+\nu)|S_t|$, and the result of the lemma will follow immediately.

For each node $u \in N(S_t)$, define X_u as an indicator random variable that takes value 1 if $u \notin S_{t+1}$ and 0 otherwise. Then $\Pr[X_u = 1] = (1-1/d)^{kd_u}$, where d_u is the number of neighbors u has in S_t. Thus:

$$\mathbb{E}[|N(S_t) - S_{t+1}|] = \sum_{u \in N(S_t)} X_u = \sum_{u \in N(S_t)} (1 - \frac{1}{d})^{kd_u} \leq \sum_{u \in N(S_t)} e^{-\frac{kd_u}{d}} \tag{7}$$

Because $\sum_{u \in N(S_t)} d_u = d|S_t|$ and we are working with a convex function, we have that $\sum e^{-\frac{kd_u}{d}}$ is maximized when all the values of d_u are equal to either 1 or d, with the exception of possibly one d_u to act as the remainder. Let R_1 be the number of nodes in $N(S_t)$ where $d_u = 1$, and let R_2 be the number of nodes where $d_u = d$. We have the following system of equations:

$$R_1 + R_2 = |N(S_t)| \tag{8}$$

$$R_1 + dR_2 = d|S_t| \tag{9}$$

solving for R_1 and R_2, we get:

$$R_1 = \frac{d}{d-1}(|N(S_t)| - |S_t|) \tag{10}$$

$$R_2 = \frac{1}{d-1}(d|S_t| - |N(S_t)|). \tag{11}$$

Thus we now want to show

$$\mathbb{E}[|N(S_t) - S_{t+1}|] \leq R_1 e^{-\frac{k}{d}} + R_2 e^{-k} \tag{12}$$

$$= \frac{d}{d-1}(|N(S_t)| - |S_t|)e^{-\frac{k}{d}} + \frac{1}{d-1}(d|S_t| - |N(S_t)|)e^{-k} \tag{13}$$

$$\leq |N(S_t)| - (1+\nu)|S_t| \tag{14}$$

Rearranging, we want to show that

$$|N(S_t)|\left(1-\frac{d}{d-1}e^{-\frac{k}{d}}+\frac{1}{d-1}e^{-k}\right)+|S_t|\left(\frac{d}{d-1}e^{-\frac{k}{d}}-\frac{d}{d-1}e^{-k}-1\right)\geq \nu|S_t| \tag{15}$$

If we let $\alpha=\frac{1}{\epsilon^2(1-\delta)+\delta}$, then $|N(S_t)|\geq \alpha|S_t|$ and we can divide through by $|S_t|$ in 15. Since the first quantity in parenthesis is positive, and we don't care what ν is as long as it's a positive constant, we are down to needing;

$$\alpha\left(1-\frac{d}{d-1}e^{-\frac{k}{d}}+\frac{1}{d-1}e^{-k}\right)+\left(\frac{d}{d-1}e^{-\frac{k}{d}}-\frac{d}{d-1}e^{-k}-1\right)>0 \tag{16}$$

Again rearranging, we want

$$(\alpha-1)\left(1-\frac{d}{d-1}e^{-\dfrac{k}{d}}\right)-\frac{d-\alpha}{d-1}e^{-k}>0 \tag{17}$$

Taking the second-order Taylor approximation $e^{-\frac{k}{d}}\leq 1-\frac{k}{d}+\frac{k^2}{2d^2}$, (17) will be satisfied if

$$(\alpha-1)\left(1-\frac{d}{d-1}\left(1-\frac{k}{d}+\frac{k^2}{2d^2}\right)\right)-\frac{d-\alpha}{d-1}e^{-k}>0 \tag{18}$$

which will be true for

$$\frac{1}{\epsilon^2(1-\delta)+\delta}=\alpha>\frac{d(de^{-k}+(k-1))-\frac{k^2}{2}}{d(e^{-k}+(k-1))-\frac{k^2}{2}} \tag{19}$$

□

Next, we use a standard martingale argument to show that the expected number of nodes in S_t is concentrated around its expectation.

Lemma 13. *For a cobra walk on a d-regular ϵ-expander that satisfies the conditions in Lemma 12, at any time t*

$$\Pr\left[|S_{t+1}|-\mathbb{E}[|S_{t+1}|]\leq -\tau|S_t|\right]\leq e^{-\dfrac{\tau^2|S_t|}{2k}} \tag{20}$$

Proof. Arbitrarily index the the nodes of $S_t, i=\{1,\ldots,|S_t|=m\}$. Let (Z_i^j) be a sequences of random variables ranging over the indices i and also $j=\{1,\ldots,k\}$, where $Z_i^j=v$ indicates the i^{th} element of S_t has chosen node v to place it's j^{th} pebble. Define A as the random variable that is the size of S_{t+1}. Then $X_i^j=\mathbb{E}[A|Z_1^1,\ldots,Z_1^k,\ldots,Z_i^1,\ldots,Z_i^j]$ is the Doob martingale for A, with $X_m^k=|S_{t+1}|$. Since $X_i^j-X_i^{j-1}\leq 1$ and $X_i^1-X_{i-1}^k\leq 1$ for all i,j, Azuma's inequality yields:

$$\Pr\left[|S_{t+1}|-\mathbb{E}[|S_{t+1}|]\leq -\tau|S_t|\right]\leq e^{-\frac{\tau^2m^2}{2km}}=e^{-\frac{\tau^2m}{2k}} \tag{21}$$

□

Finally, using the bound of Lemma 13 we show that with high probability we will cover at least δn of the nodes of G with a cobra walk in logarithmic time by showing that the active set for some $t=O(\log n)$ is of size at least δn.

Lemma 14. *For a cobra walk on d-regular, ϵ-expander G, there exists a time T such that $T=O(\log n)$ and $|S_T|\geq\delta n$.*

Proof. Let $\{Y_t\}_{t=0}^{\infty}$ be a sequence of discrete random variables, where Y_t is the size of the active set of a cobra walk at time t. We next define a Markov process that stochastically dominates $\{Y_t\}$ of the cobra walk as follows. Set $\tau = \nu/2$, where ν is the growth factor of the expected size of the active set (ν and τ are as defined in Lemma 12 and 13). Then let $\{i\}$ be a Markov chain over the state space $\{1, \ldots, n\}$ (i lower bounds the size of the active set). State i has two transitions: 1) Transition to state $(1+\nu/2)i$ with probability $p_i = 1 - e^{-\frac{\nu^2 i}{8k}}$ and 2) Transition to state 1 with probability $1 - p_i$ (p_i is determined by Lemma 13).

To get an upper bound on $1 - p_i$ that does not depend on i, we define a second dominating Markov process on the state space $\{C, \ldots, C(1+\nu/2)^i, \ldots, n\}$ for some suitably large constant C. We then have the following transitions for each state in the chain (which will begin once a value of C is obtained). Setting $r = \nu^2/8k$, at state $(1+\nu/2)^i C$: 1) Transition to state $(1+\nu/2)^{i+1}C$ with probability $p_i' = 1 - e^{-rC(1+\frac{i\nu}{2})}$ 2) Transition to state C with probability $1 - p_i'$. This Markov chain oscillates between failure (going to C) and growing by a factor of $1 + \nu/2$. Note that to get success (i.e., reaching a state of at least δn), we need $\Omega(\log n)$ growing transitions.

The probability that in a walk on this state space that we "fail" and go back to C before hitting δn is bounded by $1/2$, since $\sum_{i=0}^{\infty} e^{-rC(1+i\frac{\nu}{2})} \leq e^{-rC} \sum_{i=0}^{\infty} e^{irC\frac{\nu}{2}} = \frac{e^{-rC}}{1-e^{-rC\frac{\nu}{2}}} \leq \frac{1}{2}$, provided that C is sufficiently large as a function of r (which is itself only a function of the branching factor and the constant ν).

Consider each block of steps that end in a failure (meaning we return to C). Then clearly w.h.p. after $b \log n$ trials, for some constant b, we will have a trial that ends in success (i.e., reaching an active set of size δn nodes). In these $b \log n$ trials, there are exactly that many returns to C. However, looking across all trials that end in failure, there are also only a total of $O(\log n)$ steps that are successful (i.e., involve a growth rather than shrinkage). To see why this is true, note that the probability of a failure after a string of growth steps goes down supralinearly with each step, so that if we know we are in a failing trial it is very likely that we fail after only a few steps. Thus, there cannot be too many successes before each failure. Indeed, the probability that we fail at step i within a trial can be bounded:

$$\Pr[\text{Failure at step i} \mid \text{eventual failure}] = \frac{\Pr[\text{Failure at step i}]}{\Pr[\text{Eventual failure}]} = \frac{e^{-rC(1+i\nu/2)}}{\sum_{i=1}^{\infty}\left(\prod_{j=1}^{l-1}(1-e^{-rC(1+j\nu/2)}\right)e^{-rC(1+l\nu/2)}}$$
$$\geq \frac{1}{\sum_{i=1}^{\infty} e^{-irC\nu/2}} \geq 1 - e^{-rC\nu/2},$$

and thus the probability of advancing is no more than $e^{-rC\nu/2}$, also a quantity that does not depend on i. This is a negative binomial random variable with distribution $w(k,p)$, the number of coin flips needed to obtain k successes with success probability p. Reversing the definition of "success" (i.e., now success means returning to C) and "failure" (now failure means making a growing transition), we have a random variable $w(k,p)$, the number of coin flips needed for k failures with probability of failure $p = 1 - e^{-rC\nu/2}$. It is well known that $\Pr[w(k,p) \leq m] = \Pr[B(m,p) \geq k]$, where $B(m,p)$ is the binomial random variable counting the number of heads within m p-biased coin flips. Thus, $\Pr[w(k,p) > m] = \Pr[B(m,p) < k]$. Setting $k = a \log n$ and $m = b \log n$, we have, $\Pr[B(m,p) \leq \mathbb{E}[B(m,p)] - t] = \Pr[B(m,p) < pm - t] \leq e^{\frac{-2t^2}{m}}$. We let $k = pm - t$, and solving for t we get $t = (pb - a) \log n$. This gives us

$$\Pr[B(m,p) < k)] \leq \frac{1}{n^{\frac{(pb-a)^2}{b}}},$$

establishing there are at most $O(\log n)$ success within $O(\log n)$ trials ending in failure. Via stochastic dominance this bound holds for our original cobra walk process. □

Once the active set has reached size $\Omega(n)$, we need a different method to show that the cobra-walk achieves full coverage in $O(\log^2 n)$ time. We can not simply pick a random pebble and restart the cobra-walk from this point $O(\log n)$ times because we know nothing about the distribution of the δn pebbles after restart, and the restarting method would require the pebbles to be i.i.d. uniform across the nodes of G. As a result, we are unable to establish a straightforward bound on h_{max} and invoke Matthew's Theorem.

Hence, we develop a different process, which we will call W_{alt}, that is stochastically dominated by the cobra walk. In W_{alt}, no more branching or coalescing occurs, and we also modify the transition probabilities of the pebbles on a node-by-node basis, depending on the number of pebbles at a node.

Definition 15. *For any time t and any collection of S pebbles on V (there can be more than 1 pebble at a node), define $W_{alt}(t+1)$ as follows. Let $A \subseteq V$ be the set of all nodes with 1 pebble at time t. Let $B \subseteq V$ be the set of all nodes with exactly 2 pebbles, and let C be the set of all nodes with more than 2 pebbles. Then, (a) for every $v \in A$, the pebble at v uniformly at random selects a node in $N(v)$ and moves to it; (b) for every $v \in B$, each pebble at v uniformly at random selects its own node in $N(v)$ and moves to it; (c) for every $v \in C$, arbitrarily order the pebbles at v, the first two pebbles then pick a neighbor to hop to uniformly at random. The remaining pebbles then pick with probability $1/2$ one of the two neighbors already selected and move to that node.*

If at time t a node during process W_{alt} has two or more pebbles, at each time step it behaves identically to a node running a cobra walk. On the other hand, if there is only one pebble at node running W_{alt} it acts like a simple random walk. Thus the number of active nodes at the next time step in W_{alt} is a (possibly proper) subset of the nodes with pebbles if the graph were running the cobra walk instead. Since this will be true at every time step, W_{alt} stochastically dominates the cobra walk w.r.t cover time τ of G, and it will be enough to prove the following:

Theorem 16. *Let G be a bounded-degree d-regular ϵ-expander graph, with ϵ sufficiently high to satisfy the conditions in Lemma 12. Let there be δn pebbles distributed arbitrarily (with no distribution assumptions) over V, with at most one pebble per node. Let $\delta < \dfrac{16}{30d^2}$. Let λ be the second-largest eigenvalue of the adjacency matrix of G. From our ϵ-expander definition, $\lambda = \epsilon d$. For every ϵ, there is a constant ϵ' that is the node expansion constant of G. Furthermore, let constant $\gamma = \frac{\epsilon'}{\epsilon^2(1-\delta)+\delta}$, and let $s = \frac{5\log n + 6\log d + \log 9}{-\log\left(1-\frac{1}{2}\left(\frac{\gamma}{64d^{10}}\right)^2\right)}$. Starting from this configuration, the cover time of W_{alt} on G is $O(\log^2 n)$, with high probability.*

Proof. Our proof relies on showing that each node in G has a constant probability of being visited by at least one pebble during an epoch of W_{alt} lasting $\Theta(\log n)$ time. Once this has been established, all nodes of G will be covered w.h.p. after $O(\log n)$ epochs lasting $\Theta(\log n)$ steps each.

Define E_i to be the event that pebble i covers an arbitrary node v in s steps. We want to prove that the probability that v is covered by at least one pebble, $\Pr\left[\bigcup_i E_i\right]$, is constant. Using a second-order inclusion-exclusion approximation:

$$\Pr\left[\bigcup_i E_i\right] \geq \sum_i \Pr\left[E_i\right] - \sum_{i \neq j} \Pr\left[E_i \cap E_j\right] = \sum_i \Pr\left[E_i\right] - \sum_{i \neq j} \Pr\left[E_i\right] \Pr\left[E_j | E_i\right]. \tag{22}$$

As a marginal probability, $\Pr[E_i]$ can be viewed as the probability that the random walk of pebble i hits v at time s. Thus, we only need to look at the elements of zA^i, where A is the stochastic matrix of the simple random walk on G and z is a vector with $z(l) = 1$ for the l, the position of pebble i at the beginning of the epoch and 0 in all other positions. In [2] it is proved in Lemma 4.8 that each coordinate of $A^{s'}z$ differs from $1/n$ by at most $\frac{1}{2n}$ for $s' = \frac{\ln 2n}{\ln \epsilon}$. Since $s > s'$, this hold for our case as well. Thus $\Pr[E_i = 1] \geq \frac{1}{2n}$.

Next we establish an upper bound for $\Pr[E_j|E_i]$. Due to the conditioning on the walk of pebble i, we can't use the transition matrix A^i, but we would like to do something similar. The transition matrix governing the walk of pebble j conditioned on a fixed walk of pebble i can be characterized at each step by transition matrix $P_{l(i,t)}$, where $l(i,t)$ is the location of pebble i at time t, can be described as follows. For every row k of $P_{l(i,t)}$ s.t. $k \neq l(i,t)$ we have an exactly copy of the k^{th} row of A, the transition matrix of an independent random walk on G. When $k = l(i,t)$ this represents the walk of j when pebbles i and j are co-located at node k. To establish an upper bound, we assume the worst case, that j is ordered as the 3rd or higher pebble at k. Let τ be the neighbor of node k chosen by pebble i. Then $P[k, \tau] = 1/2 + 1/2d$, and for all other positions of row k where A is non-zero, the corresponding position in $P = 1/2d$. These represent the transition probabilities according to W_{alt} as described earlier.

From an initial probability distribution z chosen over $V(G)$, the probability of pebble j being at node v conditioned on the walk of pebble i is the v^{th} component of $z \prod_{t=1}^{s} P_{l(i,t)}$. In Lemma 17 we show that the largest component of $z \prod_{t=1}^{s} P_{l(i,t)}$ is no more than $\frac{5d^2}{2n}$. With this result, we then have:

$$\Pr\left[\bigcup E_i\right] \geq \sum_i \Pr\left[E_i\right] - \frac{1}{2}\sum_{i \neq j} \Pr\left[E_i\right] \Pr\left[E_j | E_i\right] \geq \delta n \frac{1}{2n} - \frac{1}{2}\binom{\delta n}{2}\frac{3}{2n}\frac{5d^2}{2n} \geq \frac{\delta}{2} - \frac{15}{16}\delta^2 d^2,$$

which will be a constant for the sufficiently small δ (depending only on d) given in the statement of the Theorem. □

Lemma 17. *Let G, γ, ϵ', and s be as stated in Theorem 16. Let i and j be two pebbles walking according to the rules of W_{alt} on G. Fix the walk of i, and let $\{P_{l(i,t)}\}$ be the sequence of perturbed transition matrices for the walk of pebble j depending on i. Then starting i from an arbitrary node, after s steps, the probability that j is at any node is at most $5d^2/2n$.*

Proof. The proof of this lemma relies heavily on Theorem 3.2 in [32], which we review and state here. Let P be an irreducible, ergodic Markov process for which reversibility and strong aperiodicity are not required. Consider the weighted transition from state i to j, $w_{ij} = \pi_i p_{ij}$, where π_i is the stationary distribution of i and p_{ij} is the transition probability from i to j of P. For $A \subset V$, we define the **merging conductance** of set A as

$$\Phi_P^*(A) = \frac{\sum_{j_1 \in A} \sum_{j_2 \in V-A} \sum_i \frac{w_{j_1 i} w_{j_2 i}}{\pi_i}}{\sum_{i \in A} \pi_i} \tag{23}$$

The merging conductance of graph G is thus $\Phi_P^*(G) = \min_{A \subset S : \sum_{i \in A} \pi_i \leq \frac{1}{2}} \Phi_P^*(A)$. Intuitively, the merging conductance can be viewed as a measure of the flow coming into all nodes from both A and $V - A$ for some set A. The higher the merging conductance of a graph, the more well connected it is and evenly distributed the flow is. If we define $\|\vec{x}(t)\| = \sum \frac{(p_i(t) - \pi_i)^2}{\pi_i}$ to be a measure of the distance of a distribution $\vec{p}$ over V from the stationary distribution of P, then [38] gives us the following theorem, which indicates that for a graph with merging conductance bounded away from zero, convergence to the stationary distribution occurs in logarithmic time.

Theorem 18 ([38, Theorem 3.2])**.** *For any initial distribution $\vec{x}(0)$ over V, $\|\vec{x}(t)\| \leq (1 - \frac{1}{2}(\Phi_P^*)^2)^t \|\vec{x}(0)\|$*

We also need the following lemma for bounds on the maximum and minimum of the stationary distribution of the conditional walk of pebble j.

Lemma 19. *For the walk of pebble j as described on W_{alt} for a suitable d-regular ϵ-expander G, conditioned on the walk of pebble i. the stationary distribution of the walk of j has bounds $\pi_{min} \geq \frac{1}{2nd^2}$ and $\pi_{max} \leq \frac{2d^2}{n}$.*

Proof. We first demonstrate the relationship between π_{max} and π_{min}. Let u be the node of $P_{l(i,t)}$ be $l(i,t)$, that is, the node whose transition probabilities are perturbed from the standard random walk. Let v be the neighbor of u that receives a walk from v with probability $\frac{1}{2} + \frac{1}{2d}$, and let $u_1, \ldots, u_{d-1}$ be the remaining neighbors of v, that receive a walk from u with probability $\frac{1}{2d}$. We claim that $\pi_v = \pi_{max}$. To see this, assume on the contrary that node i, not v or a neighbor of v, has $\pi_i = \pi_{max}$. Because $\sum_{j \in N(i)} w_{ij} = \sum_{j \in N(i)} w_{ji}$, $\pi_i = \sum_{j \in N(i)} \pi_j p_{ji}$. But since $p_{ji} = \frac{1}{d}$, it follows that $\pi_{max} = \pi_i = \frac{1}{d} \sum j \in N(i) \pi_j$, that is π_j is equal to the mean of the π_j's. However, since the π_j's are all the same (due to their identical transition probabilities), it follows that $\pi_j = \pi_i = \pi_{max}$ for all $j \in N(i)$. We can continue doing this until we reach one of the

u_k's implying that $\pi_{u_k} = \pi_{max}$. This is clearly a contradiction, since $\pi_{u_k} = \frac{1}{2d}\pi_u + \frac{d-1}{d}\pi_{max} \leq \pi_{max}$. Thus π_{max} must either be π_u or π_v, and w.l.o.g. we assume that it is π_v. A similar argument shows that $u_1, \ldots, u_{d-1}$ all have stationary distributions π_{min}.

To get explicit bounds for π_{max} and π_{min}, we note that $\pi_{min} \geq \frac{1}{2d}\pi_u$ and that $\pi_u = \frac{d-1}{d}\pi_{min} + \frac{1}{d}\pi_{max}$. Thus we have:

$$\pi_{min} = \frac{d}{d-1}\pi_u - \frac{1}{d-1} \geq \frac{1}{2d}\pi_u \tag{24}$$

$$\frac{1}{d-1}\pi_{max} \leq \left(\frac{d}{d-1} - \frac{1}{2d}\right)\pi_u \tag{25}$$

$$\pi_{max} \leq \left(d - \frac{d-1}{2d}\right)\pi_u \tag{26}$$

$$\pi_{max} \leq \left(\frac{2d^2-d+1}{2d}\right)\pi_u \leq \frac{2d^2}{2d}\pi_u \leq d\pi_u \leq 2d^2\pi_{min} \tag{27}$$

Thus we have $\pi_{min} \geq \frac{1}{2d^2}\pi_{max}$. Since $\bar{\pi} = 1/n$ and $\pi_{min} \leq \bar{\pi} \leq \pi_{max}$, this gives $\pi_{min} \geq (\frac{1}{n})(\frac{1}{2d^2})$ and $\pi_{max} \leq (\frac{2d^2}{n})$.

□

Next we establish a lower bound for the number of terms in the sum in the numerator of Equation 23. Let A be the set for which $\Phi^*_P(G)$ is minimized. Furthermore, since G is an ϵ-expander, we also know that its cobra walk expansion is a constant ϵ' and depends only on ϵ. We would like to calculate the number of nodes in G that have at least one neighbor in A and at least one neighbor in $V - A$. First, we lower-bound the size of the set of nodes with at least one edge to A. This set is just $N(A)$, the inclusive neighborhood of A, which from Tanner's theorem can be bounded from below by $\frac{|A|}{\epsilon^2(1-\delta)+\delta}$. Of the node in $N(A)$, we also need to bound the number that also have at least one edge to $V - A$. However, this is just the non-inclusive neighborhood of $N(A)$, $\Gamma(N(A))$, and we can use the node expansion of G to show that $|\Gamma(N(A))| \geq \frac{\epsilon'|A|}{\epsilon^2(1-\delta)+\delta}$. Thus we get:

$$\Phi^*_P(G) \leq \frac{\epsilon'|A|}{\epsilon^2(1-\delta)+\delta} \frac{\frac{\pi^2_{min}(1/2d)^2}{\pi_{max}}}{|A|\pi_{max}} \leq \frac{\epsilon'}{\epsilon^2(1-\delta)+\delta}\left(\frac{1}{2d^2}\right)^2\left(\frac{1}{2d}\right)^2\frac{1}{n^2}\left(\frac{n}{2d^2}\right)^2 \leq \frac{\epsilon'}{\epsilon^2(1-\delta)+\delta}\frac{1}{64d^{10}}.$$

Letting $\gamma = \frac{\epsilon'}{\epsilon^2(1-\delta)+\delta}$, we note that the expression above is a constant as long as $d, \epsilon, \delta, \epsilon'$ are constants, which will be true in a d-regular ϵ-expander.

Starting from a distribution $\vec{x}(0)$ whose norm $\|\vec{x}(0)\|$ will be maximized when the walk is started from node s.t. $\pi_i = \pi_{min}$, we have: $\|\vec{x}(0)\| \leq \frac{(1-\pi_{min})^2}{\pi_{min}} + (n-1)\frac{(\pi_{max})^2}{\pi_{min}} \leq 2d^2n + (n-1)\left(\frac{2d^2}{n}\right)^2(2d^2n)$ $\leq 2d^2n + 8d^6 < 9d^6n$ for $d > 1$. Finally, we want to show that $\|\vec{x}(s)\| < \frac{1}{n^4}$. With this, it is clear to see that the maximum difference $|p_i(t) - \pi_i| < \frac{1}{n^2}$ which implies that the maximum probability $\Pr[E_j|E_i] < \frac{2d^2}{n} + \frac{1}{n^2} < \frac{5d^2}{2n}$ as required in Theorem 16. To do this we need to show that $\left(1 - \frac{1}{2}\left(\frac{\gamma}{64d^{10}}\right)^2\right)^s \leq \frac{1}{9d^9n^5}$, which will be true for the set value of s in the definition of the Theorem.

A final note: because $\Phi^*_P(G) \leq \frac{\gamma}{64d^{10}}$ for every matrix $P_{l(i,t)}$, we can apply Theorem 18 in the exponentiation even though each matrix is different. □

cobra walk

5 Cliques

Finally, we consider the complete graph (with self loops added for ease of analysis). Previously, we have use two strategies to study the cover time of a cobra walk. For trees and grids we were able to show a result for the hitting time on G and then apply Matthew's Theorem for Cobra-walks. For expanders we relied on the expansion to show that the active set grew to a certain size, and then relied on the merging conductance to show that the full graph got covered. Here we consider a third method, in which we map a cobra walk on G to a simple random walk on graph with a node set equal to all $2^n - 1$ non-empty subsets of V. Though this is a very large space, we are able to show that for the complete graph there are so many edges leading to nodes that correspond to larger active sets in the original graph G that with extremely high probability the cobra walk will cover a constant fraction of the nodes in $\log n$ time. Once this happens, since the pebbles are distributed i.i.d. uniformly across V, it is easy to show that full coverage again occurs in $\log n$ time. Intuitively, one would expect full coverage of K_n in logarithmic time. We include this result because the method of proving it may be applicable to a wider class of graphs than just the complete graph.

We state the main result for K_n:

Theorem 20. *Let $G = K_n$, the complete graph on n node. Then a cobra walk starting from any node in K_n will cover the entire graph w.h.p. in $O(\log n)$ time.*

Consider the complete graph K_n. For simplicity, assume that every node in K_n also has a self loop. Consider an active set S_t, and for now assume that $|S_t| << n$. In the next step of the cobra walk, we note than any node can become a member of S_{t+1}, and that $|S_{t+1}|$ can range from 1 to $2|S_t|$. We are going to show that with high probability, the active set will grow by at least a constant factor $(1+\epsilon)$ in each round up until the active set reaches a size δn for some $\delta \in (0,1)$. Once in reaches this active set size, the rest of the graph will be covered in $O(\log n)$ steps.

As indicated in the application of Matthew's Theorem for cobra walks, it is possible to view a cobra walk on G as a simple random walk on a related graph M_G. The state space of M_G are all the possible active sets of $V(G)$ in a cobra walk, which are just all of the subsets of $V(G)$ with the exception of the empty set. Hence M_G has $2^n - 1$ states. An edge between $u, v \in M_G$ exists for every possible way in which the active in G corresponding to state space u in M_G can give rise to the active set corresponding to v in M_G with one step of a cobra walk. For a node $v \in M_G$, denote $act(v)$ to be size of the corresponding active set in the inverse mapping from nodes of M_G to cobra walk configurations of G. Note that the graph formed by M_G and its edges is a multi-graph, as it is possible for one active-set configuration to give rise to another through multiple combinations branching and collisions of pebbles in a nodes. In the past it has has been deemed inadvisable to study such "meta-processes", due to the exponential size of M_G . However, for the case of the complete graph, we are able to show that the size of the active set is growing with extremely high probability in every step, thus providing a simpler interpretation of our analysis.

Lemma 21. *Let a random walk on M_G be at $v \in M_G$ such that $act(v) = s \le \delta n$ for $\delta \le \frac{1}{e^2(1+c)}$ for $c \in (0,1)$. Then with probability $1 - \frac{1}{e^3} e^{-\frac{1.6n}{1.1e^3}}$ the walk's position at time $t+1$ will be at a node u with $act(u) \ge (1+c)s$.*

Proof. Vertex $v \in M_G$ has $act(v) = s$, meaning there are $2s$ pebbles on the corresponding graph G just prior to the next step, each of which chooses a node of G uniformly at random, meaning there are n^{2s} total edges leading out from $v \in M_G$, though we note that M_G is a multigraph and hence many edges from v will have common endpoints. All nodes of M_G with $act(u) = i$ for $i \in [1, 2s]$ are neighbors of v. Pick a particular node u with $act(u) = k$. The number of edges between v and u can be calculated as:

$$\text{edges between v,u } = k! \left\{ {2s \atop k} \right\} \tag{28}$$

where $\left\{ {2s \atop k} \right\}$ is the Stirling number of the second kind, counting the number of ways s items can be partitioned into k non-empty, unlabeled bins. Thus, the total probability of a simple random walk on M_G at v with

$act(v) = s$ walking to any node u with $act(u) = k$ in the next step is:

$$\binom{n}{k}\frac{k!\left\{{2s \atop k}\right\}}{n^{2s}} \tag{29}$$

We can make use of the recursive identity $k!\left\{{n \atop k}\right\} = k^n - \sum_{i=1}^{k-1}\frac{k!}{i!}\left\{{n \atop i}\right\}$, so that the probability of walking from u to v in one step is bounded from above by $\binom{n}{k}\frac{k^{2s}}{n^{2s}}$. Since $p << n/2$, this expression is maximized at p over the range $[1,\ldots,p]$ and we have that:

$$\Pr\left[u \to v\right] \leq \binom{n}{p}\frac{p^{2s}}{n^{2s}} \leq n^p e^p \frac{p^{2s}}{n^{2s}} = n^{p-2s}p^{2s-p}e^p \tag{30}$$

When s and p are small, the n^{p-2s} term dominates (so long as s is still a sufficiently large constant), so the only case we need to examine is when s is of size δn for some constant $\delta < 1$. Then we have:

$$\begin{aligned} n^{p-2s}p^{2s-p}e^p &\leq n^{\epsilon\delta n - 2\delta n}(\epsilon\delta n)^{2\delta n - \epsilon\delta n}e^{\epsilon\delta n} && (31)\\ &= (\epsilon\delta)^{(2-\epsilon)\delta n}e^{\epsilon\delta n} && (32)\\ &\leq (\frac{1}{e})^{(6-4\epsilon)\delta n} && (33) \end{aligned}$$

This comes about because for $\epsilon = 1 + c$, if we set $c = 0.1$, then this holds for δ so that $\delta \leq (1/e)^3(1/1.1)$ which gives us the final result that we do not grow our active set by more than a factor of $(1 + c)$ with probability $\leq (1+c)\delta n(1/e)^{\frac{1.6n}{1.1e^3}}$. Thus w.h.p. the random walk on M_G will visit in one time step a node with $act(u) > (1 + c)s$ for the appropriate constant $c = 1.1$. □

Next we want to prove that w.h.p. in $O(\log n)$ steps the random walk will move from a node u' with $act(u') = C$ to a node v' with $act(v') = \Theta(\delta n)$ for some large constant C and constant δ described as above.

Lemma 22. *Let W be a random walk on M_G. Suppose that at time t, $W(t)$ is a node v with $act(v) = C$ for a constant C. Then for $c = 0.1$ and $\epsilon = 1 + c = 1.1$ w.h.p $W(t + O(\log n))$ will be at a node with $act(v') \geq \delta n$ for $\delta = \frac{1}{1.1e^3}$.*

Proof. As noted in Lemma 21, for a walk at a node with $act(v) = s$, the probability of not advancing to a node in the next time step with $act(v') > \epsilon s$ is no more than:

$$\epsilon s\left(\frac{\epsilon s}{n}\right)^{(2-\epsilon)s}e^{\epsilon s} \tag{34}$$

We want to show that for values of s between C and δn, this quantity is decreasing. The first derivative of the expression w.r.t. s is:

$$f'(n,s,\epsilon) = -\epsilon e^{\epsilon s}\left(\frac{\epsilon s}{n}\right)^{(2-\epsilon)s}\left[(\epsilon-2)s\ln\left(\frac{\epsilon s}{n}\right) - 2s - 1\right] \tag{35}$$

This will be negative when the quantity in brackets is positive:

$$\begin{aligned} (\epsilon-2)s\ln\left(\frac{\epsilon s}{n}\right) - 2s - 1 &> 0\\ (\epsilon-2)s\left[\ln(\epsilon) + \ln(s) - \ln(n)\right] &> 2s + 1\\ (2-\epsilon)\left[\ln(n) - \ln(\epsilon) - \ln(s)\right] &> 2 + 1/s \end{aligned}$$

Clearly this will hold when s is small, and since the quantity in brackets is a concave function and is growing between 1 and δ we only need to show that it also holds for the other end of the range, when $s = \delta n$:

$$
\begin{aligned}
(2-\epsilon)\left[\ln(n) - \ln(\epsilon) - \left(\ln(\frac{n}{\epsilon e^3})\right)\right] &> 2 + \frac{1}{\frac{n}{\epsilon e^3}} \\
(2-\epsilon)\left[\ln(n) - \ln(\epsilon) - \ln(n) + \ln(\epsilon) + 3\right] &> 2 + \frac{\epsilon e^3}{n}
\end{aligned}
$$

which will hold for $\epsilon = 1.1$. Now we need to show that the expression in equation 34 is $\leq n^{-3}$ for $s = C$. But it is just:

$$1.1C\left(\frac{1.1C}{n}\right)^{0.9C} e^{1.1C} \tag{36}$$

which is $O(n^{-0.9C})$, which will be less than n^{-3} for even small $C > 5$.

We are finally ready to prove the statement in the lemma. Consider a failure to be the event that the value of $act()$ does not grow by a factor of $(1+c)$ with a step. Since this probability is decreasing with increasing s, if we take C as our starting value, it is bounded from above by $\frac{1}{n^3}$. Hence the probability of success at any step is at least $1 - \frac{1}{n^3}$. Thus, w.h.p we will have $a \log n$ successes in a row, allowing our active set to grow to size δn in logarithmic time. $\square$

6 Conclusion

We studied a generalization of the random walk, namely the cobra walk, and analyzed its cover time for trees, grids, and expander graphs. The cobra walk is a natural random process, with potential applications to epidemics and gossip-based information spreading. We plan to explore further the connections between cobra walks and the SIS model, and pursue their practical implications. From a theoretical standpoint, there are several interesting open problems regarding cobra walks that remain to be solved. First is to obtain a tight bound for the cover time of cobra walks on expanders. Our upper bound is $O(\log^2 n)$, while the diameter $\Omega(\log n)$ is a basic lower bound. Another pressing open problem is to determine the worst-case bound on the cover time of cobra walks on general graphs. It will also be interesting to establish and compare the message complexity of cobra walk with the standard random walk and other gossip-based rumor spreading processes.

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